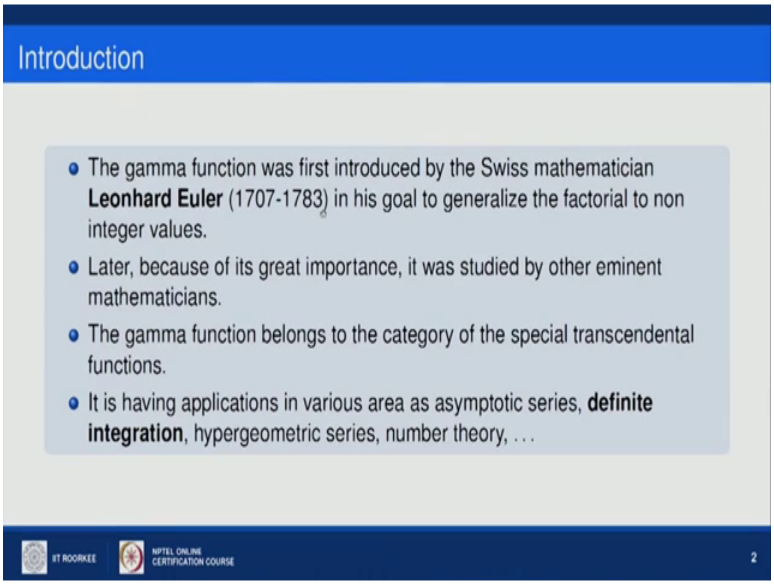
Multivariable Calculus
Dr. Sanjeev Kumar
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 23
Introduction to Gamma Function

Hello friends, so welcome to the third lecture of this unit. And in this lecture, I am going to introduce a special type of function called gamma function this function is related to the improper integral and having plenty of applications in multiple integral, so that is why I put this particular topic here, many multiple integral can be solved directly in terms of gamma function.

(Refer Slide Time: 01:26)

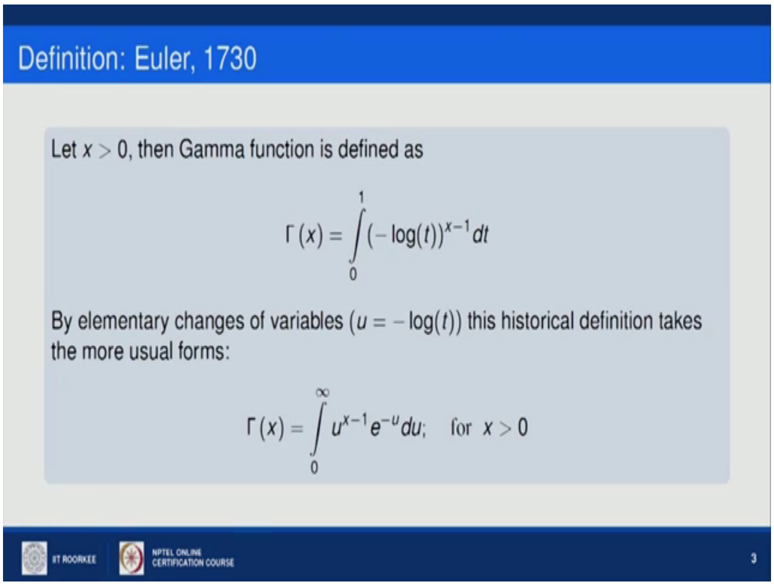


So, let us look at a bit history of this function. So, the gamma function was first introduced by the Swiss mathematician Leonhard Euler, in seventeen 1730. In his goal to generalize the factorial function to the non-integer value. As we know that factorial function is defined only for integers and how to generalize it for non-integer values from that concept the development of gamma function was started.

Since this function was quite important. So, it was later study by other eminent mathematician. Now, gamma function belongs to the category of the special transcendental function due to that involvement of exponential term in its definition.

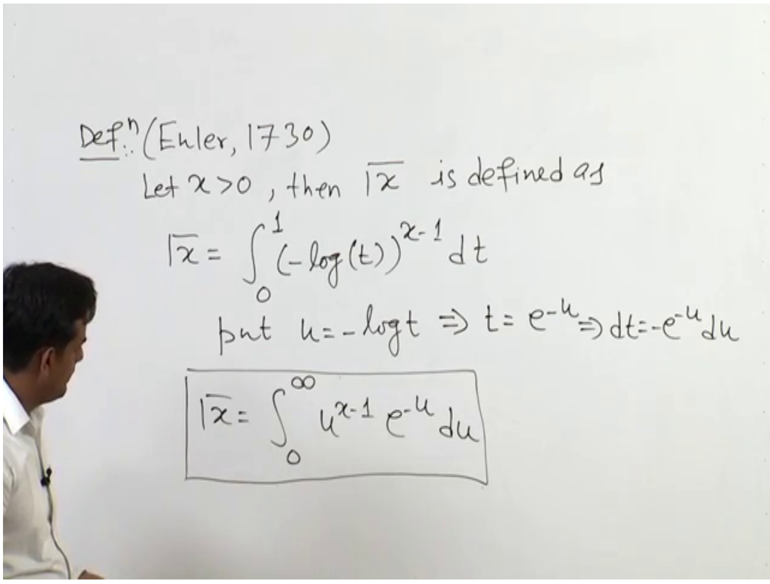
Furthermore, it is having applications in various area as asymptotic series, definite integration, hypergeometric series, number theory and many more.

(Refer Slide Time: 01:59)



So, let us start this lecture with the definition of gamma function.

(Refer Slide Time: 02:14)

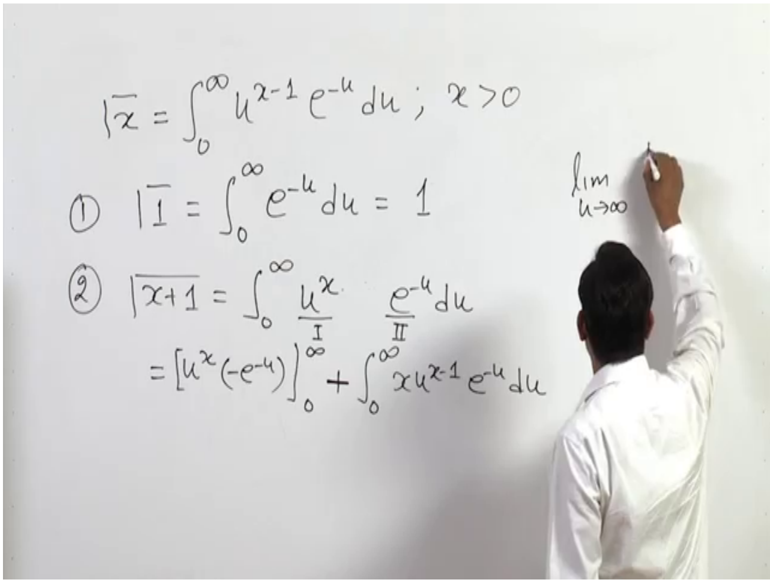


So, in 1730 Euler define the gamma function with the help of an improper integral. And the definition was something like this. Let x be a positive number then gamma of x. So, this symbol stands for gamma function. So, then gamma of x is defined as gamma x equals to integral over 0 to 1 minus log t raise to power x minus 1 d t. So, this was the definition given by Euler in 1730; however, rarely we find this definition in textbook.

So, if I put or change the variable as u equals to minus log t, here log is defined with the base exponential means natural log. So, what I can write t equals to e raise to power minus u. So, I have taken minus this side and then taken the exponential on both side, and this gives d t equals to minus e raise to power minus u d u. So, if I put it here then gamma x will be 0 to infinity and then u raise to power x minus 1 into e raise to power minus u, it is coming from here, limit has been changed from 0 to 1 will become infinity to 0.

And since I am having a minus here, so it will become 0 to infinity and then finally, d u. So, this is the most popular definition of gamma function which we use to see very frequently in text books related to this topic, and this is defined for positive x. Now, with this definition, we will see some important property of gamma function.

(Refer Slide Time: 05:18)

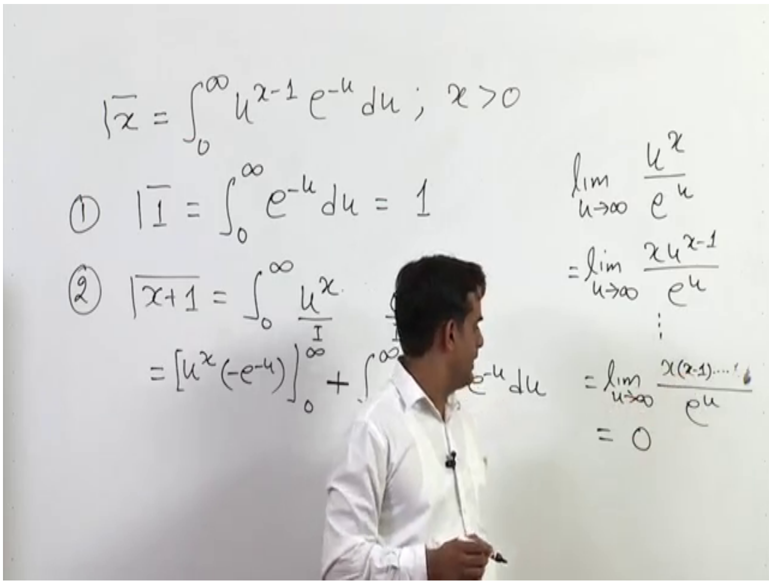


So, the definition which we have just seen it like gamma x equals to 0 to infinity u raise to power x minus 1 into e raise to power minus u d u and it is when x is a positive number. Now, if I want to find out the value of gamma 1, so gamma 1 means here I need to put x equals to 1. So, 0 to infinity u raise to power 1 minus 1 will become 1 0. So, u raise to power 0, will be 1 and then e raise to power minus d u and this equals to 1. So, hence gamma 1 equals to 1.

If I need to calculate gamma x plus 1, so in this definition I will replace x by x plus 1 so, it will become u raise to power x plus 1 minus 1 into e raise to power minus u d u. So, basically one will be cancel out with this one. So, it becomes u raise to power x into e e raise to power minus u d u. Now, if I do this integration by parts, I assume it as my first function this one as my second function. So, this will be first function as such integration of second.

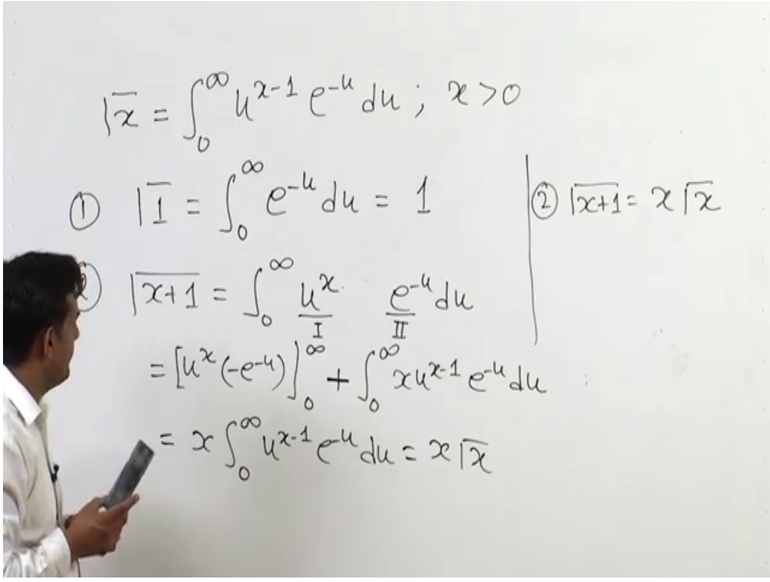
So, it will become minus e raise to power minus u, and this will be having 0 to infinity minus integral over 0 to infinity, the differentiation of the first function. So, u raise to power x can be written as x u raise to power x minus 1 and this will be again integration of second. So, minus e raise to power minus u. So, with this minus this will become plus into e raise to power minus u d u. Now, just see this term when u is 0, this term will become 0, because this will be 1 and this will be 0. So, this is 0, when u is infinity.

(Refer Slide Time: 08:05)



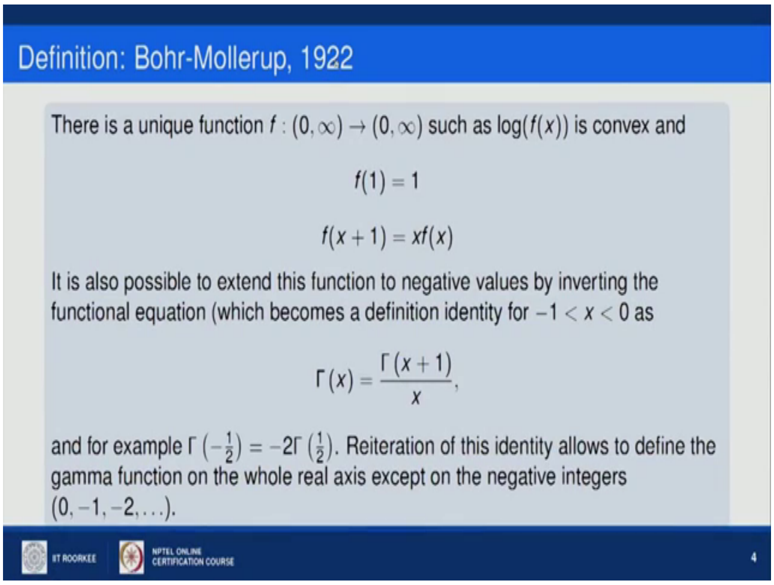
So, it is something limit u tending to infinity, I can write it u raise to power x upon e raise to power minus u sorry minus u is here. So, I can take it down. So, e raise to power u. Now, it is the form of infinity upon infinity. So, using L-hospital, it will become x u raise to power x minus 1 upon e raise to power u doing it finally, I will be having limit u tending to infinity here I will be having x into x minus 1 x minus 2 up to 2 into one like this. And in the denominator I will be having e raise to power u. Here this will be some finite number; however, this will be infinity. So, I can write it as zero because growth of exponential function will be more than the function given in numerator.

(Refer Slide Time: 09:24)



So, hence this term is 0. So, I can write it I can take this x out. So, x 0 to infinity u raise to power x minus 1 e raise to power minus u d u. Now, this is quite familiar to us, this is gamma x, so it will become x gamma x. So, basically we ended up as the second property that gamma x plus 1 equals to x gamma x. So, from this definition, we have obtained two important properties of gamma function one is gamma 1 equals to 1, another one is gamma x plus 1 equals to x gamma x. So, together with this definition over the time several other definitions were proposed for this particular function.

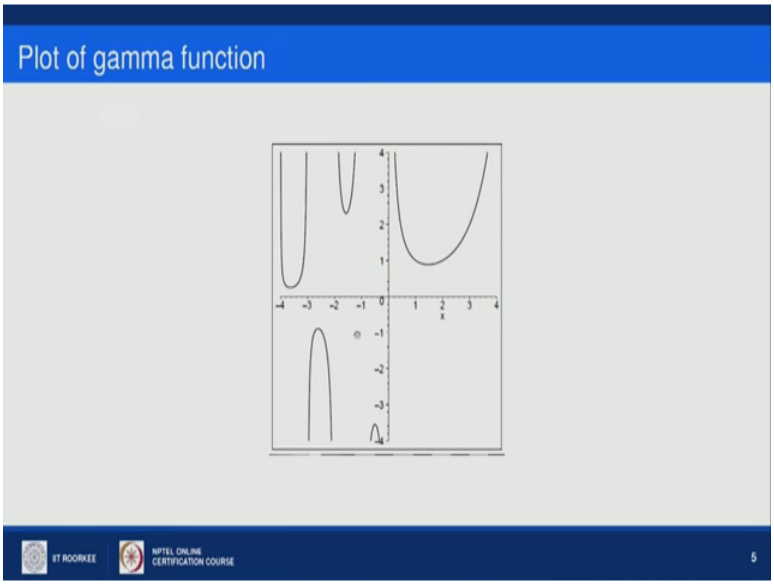
(Refer Slide Time: 10:33)



In 1922, Bohr-Mollerup proposed this particular definition. So, definition is like this. There is a unique function f from the 0 to infinity to 0 to infinity such as log of f x is convex and f 1 equals to 1 and f of x plus 1 equals to x times f of x. So, if we look in these two particular statements these are quite similar which we have just proved that gamma of 1 equal to 1, and gamma of x plus 1 equals to x gamma x. And please note that it is defined for positive x. It is also possible to extend this definition to negative values, because whatever we have taken so far we have taken only for positive x.

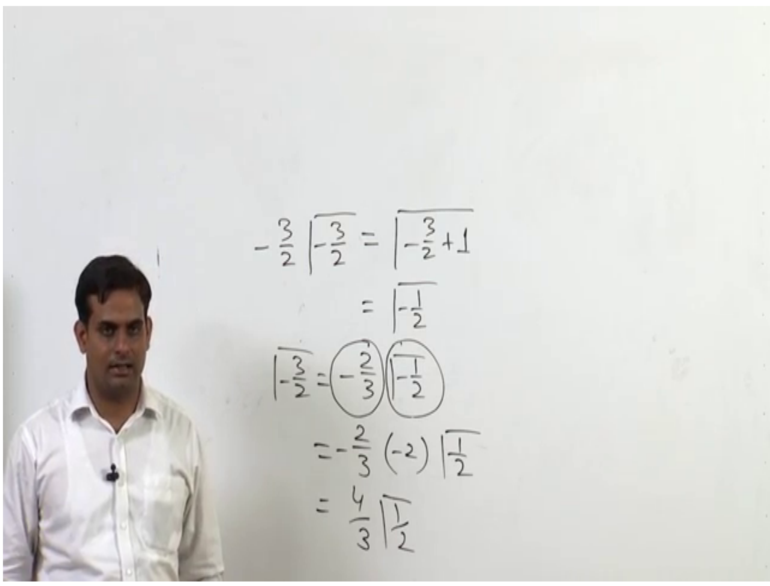
So, we know that gamma of x plus 1 equals to x times gamma x. So, I can write that particular property in this expression. So, gamma x equals to gamma x plus 1 upon x and this is definition this gives the definition of gamma function for negative numbers except the negative integers like minus 1 minus 2 etcetera. For example, if I want to find out gamma of minus half, so here if I put minus half, it will become minus half plus 1. So, gamma half in the numerator and the denominator it will become minus half. So, it will become gamma minus half equals to minus 2 times gamma half. Reiteration of this identities allow to define the gamma function on the whole real axis as I told you except on the negative integers and 0.

(Refer Slide Time: 12:36)



So, this particular plot gives the graph of gamma function. So, here you can see we have just plotted it from minus 4 to 4. So, if we talk about positive x, the function will be like this. So, it is decreasing up to 1, a bit more than 1, and then it is increasing ok. So, it is something convex type of thing. For the negative x what we are having between 0 to minus 1 as I told you if we look at the value gamma minus half, it will be minus 2 times gamma half. Gamma half will be something positive quantity so minus two times some positive quantity. So, it will become negative quantity. So, between 0 to minus 1, the value of gamma function will be always negative.

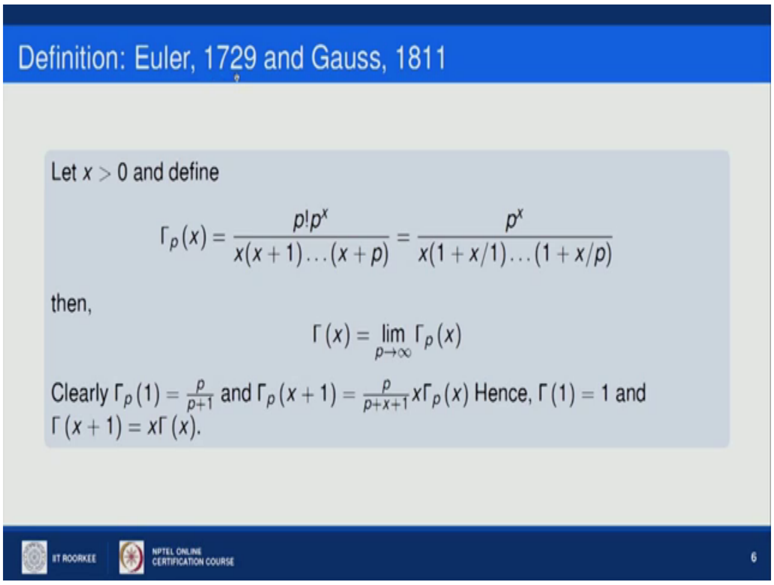
(Refer Slide Time: 13:43)



If we talk between minus 1 to minus 2, so suppose we need to find out gamma minus 3 by 2, so need to find out gamma minus 3 by 2. So, how can I write this gamma minus 3 by 2. So, this can be written as I told you gamma x plus 1 equal to x gamma x. So, this minus 3 by 2 times minus 3 by 2 will be gamma minus 3 by 2 plus 1. So, this will be this value will be gamma minus half. So, hence gamma minus 3 by 2 will be minus 2 by 3 times gamma minus half.

This is a negative quantity as I told you gamma minus half will be a negative number. So, negative into negative will become a positive number or basically it will become minus 2 by 3 minus 2 into gamma half so 4 by 3 into gamma half, since gamma half is a positive quantity, so it will be positive. So, all the that the value of gamma function for all the value between minus 1 to minus 2 will be a positive quantity. So, again from minus 2 to minus 3, it will be negative minus 3 to minus 4, it will be positive. So, it will alternating in this way. And please again note that the gamma function is not define for negative integers.

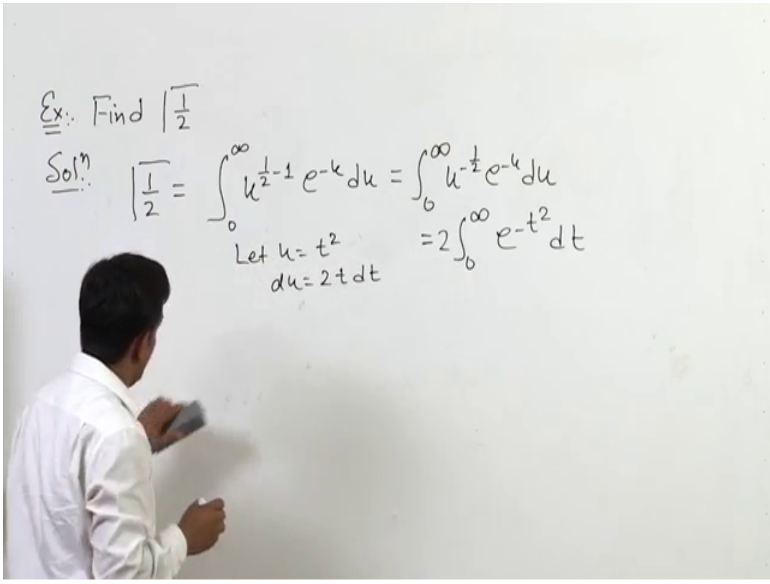
(Refer Slide Time: 15:29)



The another definition for gamma function given by Euler in 1729 and Gauss in 1811 is like that. So, this definition is based on the product of infinite terms so, let x is a positive number then define gamma p x in this fashion factorial p into p raise to power x upon x into x plus 1 up to x plus p. So, this can be written as in this way then gamma x will be limit p tending to infinity gamma p x. For example, if we want to find out the value of gamma one from this definition. So, gamma p one will be just put x equals to 1 here. So, it will become p upon p plus 1 and gamma 1 will be limit p tending to infinity p upon p plus 1 which is obviously 1. Similarly, gamma p x plus 1 equals to p upon p plus x plus 1 into x times gamma p x. In limiting case, this value will become one gamma one is one. So, gamma x plus 1 will become x gamma x.

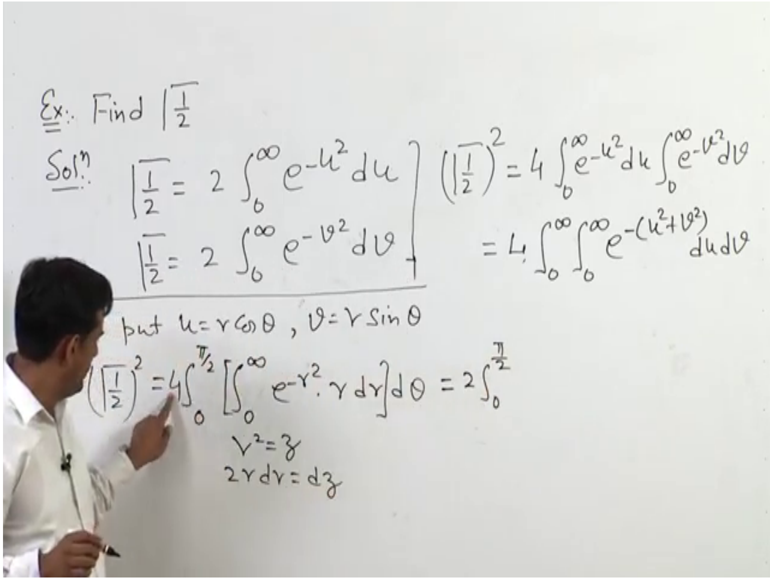
So, again this definition drives the two important properties which we have obtained using the definition given on the first slide of this lecture. Now, we will evaluate some important values. First of all we will find out the value of gamma half.

(Refer Slide Time: 17:09)



So, I will take it as an example. So, find gamma half. So, it is an interesting problem that finding the value of gamma half. So, we know that gamma half will be by the definitions something u raise to power half minus 1 e raise to power minus u d u, this becomes 0 to infinity u raise to power minus half e raise to power minus u d u. If I substitute u equals to t square, so let u equals to t square, then d u will become 2 t d t. So, after substituting this, this integral will be 0 to infinity and then two times so t is basically square root u. So, square root u will cancel this u raise to power minus half. So, it will become e raise to power minus t square d t.

(Refer Slide Time: 18:47)



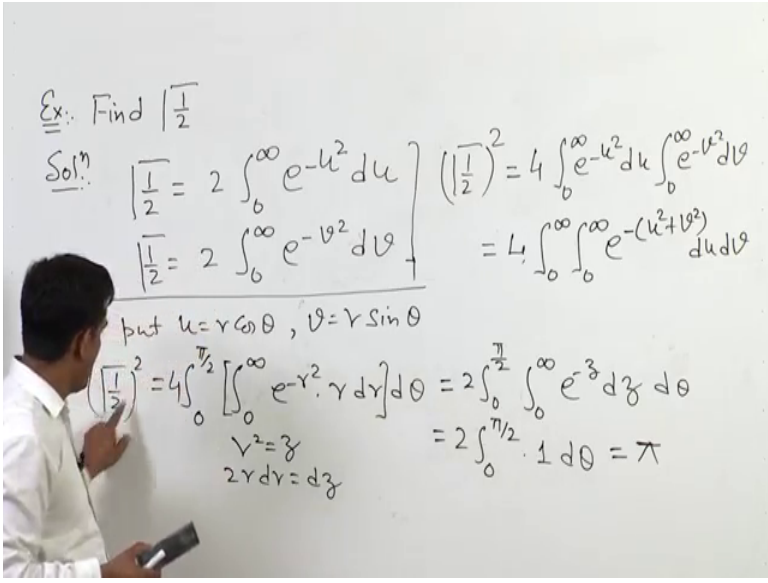
So, basically what I am writing I am writing gamma half equals to 0 to infinity e raise to power minus u square d u just that I have written in terms of u again. Also I can define gamma half as 2 times 0 to infinity, if I take v instead of u e raise to power minus v square d v. If I multiplied this two equations, I will be having gamma half whole square in the left hand side that is gamma half into gamma half. In the right hand side, I will be having 4 times 2 into 2 4 0 to infinity e raise to power minus u square d u 0 to infinity e raise to power minus v square d v.

By the Fubini's theorem I can write since the limits are constant I can write it the product of these two definite integrals as a double integral. So, it will be four times 0 to infinity 0 to infinity e raise to power minus u square into e raise to power minus v square. So, it will become e raise to power minus u square plus v square d u d v. Now, I need to solve this particular integral that is the double integral.

What I will do, I will change the variable in polar coordinates. So, what I am doing put u equals to r cos theta and v equals to r sin theta. So, from here the integral will become 0 to infinity that is gamma half is square 0 to infinity or I am having already here e raise to power minus r square and as you know d u d v will become r d r d theta.

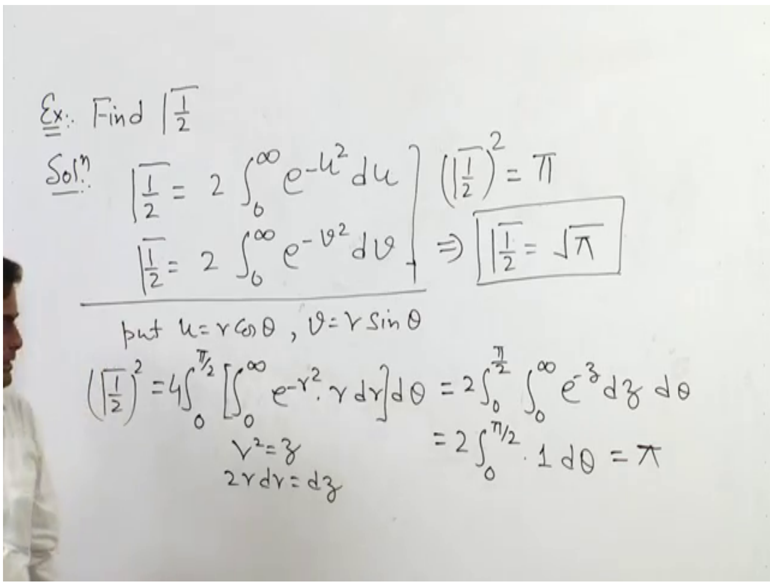
Again for first I will solve this integral. So, for solving this integral I will put r square equals to z. So, what I will be having 2 r d r equals to d z sorry after putting this limit will change, limit will be in now according to polar coordinates. So, as you know it is 0 to infinity and 0 to infinity. So, region is first quadrant. So, first quadrant means r will move from 0 to infinity and theta will be 0 to pi by 2. So, this by this substitution, it will become 0 to pi by 2 because I have taken this two inside 2 r d r. So, 2 r d r will be d z.

(Refer Slide Time: 22:30)



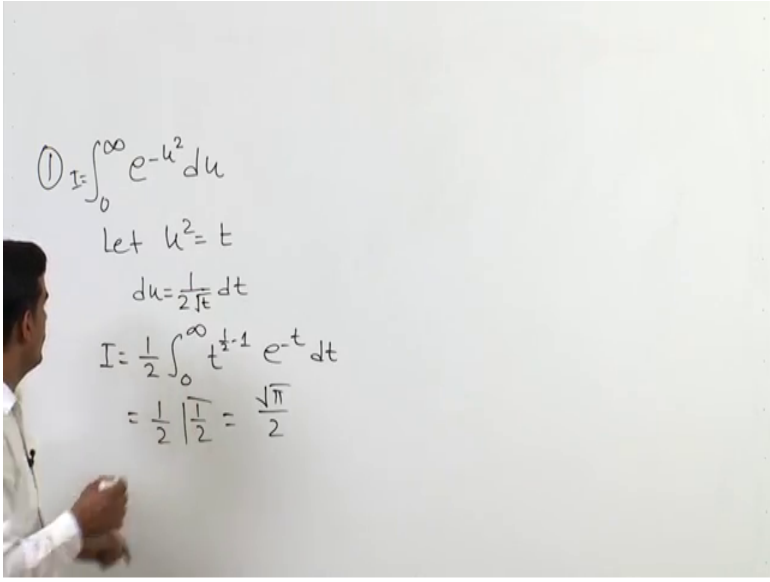
So, 0 to infinity e raise to power minus z d z 2 0 to pi by 2 and this will come out as 1 and sorry d theta is also here, so d theta. Now, 2 into pi by 2 will come out pi. So, what I got here I got gamma half square equals to pi. So, it will give me the value of gamma half, so gamma half square is pi.

(Refer Slide Time: 23:24)



So, from here, I will get gamma half equals to root pi and which is one an important result in gamma function ok. So, just now we have seen that gamma half equals to root pi. Based on this result, we can solve many other integral, which is not possible to solve without the knowledge of gamma function.

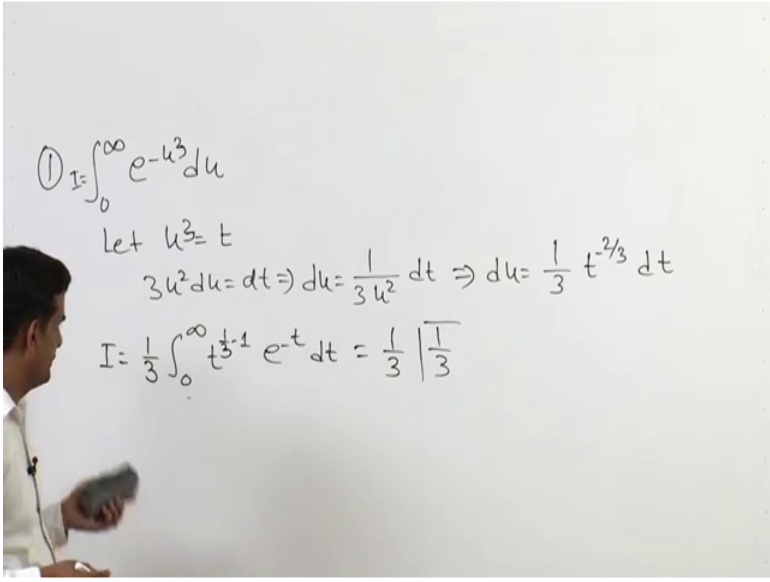
(Refer Slide Time: 23:56)



So, one of them let us take 0 to infinity e raise to power minus u square d u. So, if we do not have the knowledge of gamma function, for solving this integral, we need a u here just before e, but we do not have it here. So, we cannot make any proper substitution here. However, since we have done only job gamma function, so we use the property or I will said the definition of gamma function here. So, if I assume let u square equal to t then I can write d u equals to 1 by 2 root t d t.

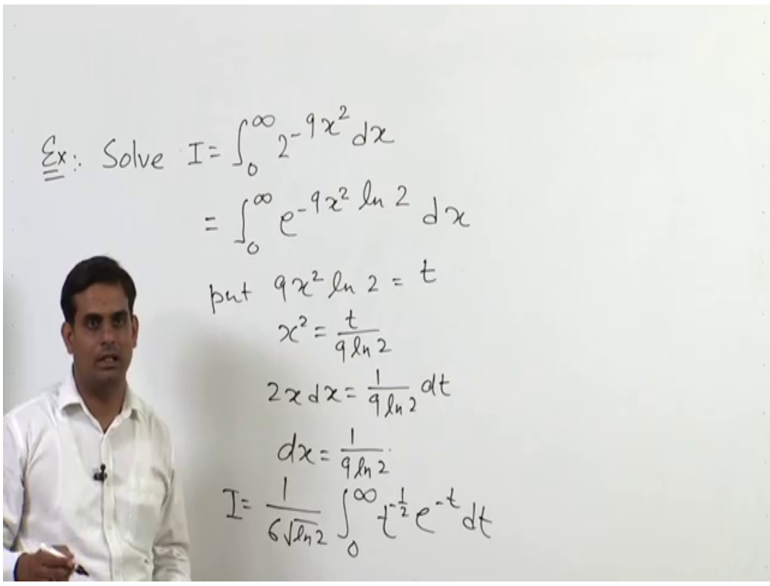
So, after making this substitution, this integral I can be written as 1 by 2 0 to infinity t raise to power minus half, so t raise to power minus half can be written as t raise to power half minus 1 and then e raise to power minus t because earlier it was u square. So, now, just in place of u square I have written t d t. If you see this definition, this is gamma half by the definition of gamma function. So, this will be root pi upon 2. So, this particular integral is root pi upon 2. Similarly, we can solve it for any power of u like u cube.

(Refer Slide Time: 25:59)



So, it is u cube, I will write u cube equals to t. So, then it will become 3 u square d u equals to d t from here d u will be 1 by 2 u square d t, and it will give me d u equals to 1 by 3 u raise to power minus 2. So, u raise to power 3 u equals to t raise to power 1 upon 3. So, u raise to power minus 2 will become t raise to power minus 2 upon 3 d t. So, here I will write this integral as 1 by 3 and minus 2 by 3 can be written as 1 by 3 minus 1 e raise to power minus t d t. So, it will become 1 by 3 gamma 1 by 3. Similarly, e raise to power minus u raise to power 4 will become 1 by 4 into gamma 1 by 4 and so on.

(Refer Slide Time: 27:17)



Let us take one more example solve the integral I equals to 0 to infinity 2 raise to power minus 9 x square d x. So, this is a bit difficult example if we see here because what sort of substitution we can make here, but let us use the property of gamma function here. So, put or before that I can write this as 0 to infinity e raise to power minus nine x square into l n of 2, I can write it in this way into d x. Now, put nine x square l n 2 equals to t.

So, what I will be having x square will be t upon 9 into l n 2, and 2 x d x will become 1 upon 9 l n 2 d t or simply d x will be 1 upon 9 l n 2 into 1 upon 2 x. So, 1 upon 2 x we need to find out from here what it can be and after putting here it will come in the standard form of gamma function. And from there we can write I equals to 1 upon 6 is square root l n 2 0 to infinity t raise to power minus half t raise to power minus t d t for this integral can be written in this form, and you know it is gamma half. So, root pi upon 6 is square root l n 2 will be the value of this integral.

So, with this example I will end this lecture. So, in this lecture, we have seen few definitions of gamma function; we have seen the value of gamma 1, how can we write gamma x plus 1 in terms of gamma x. And then we have seen how can we define gamma function for negative non-integer values. Finally, we have taken few examples, and we have solved them means examples of definite integrals improper integrals, and we have solve them using the definition of gamma function.

Thank you very much.