Multivariable Calculus
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Lecture –18
Error Approximation

Hello friends. So, welcome to lecture series on multivariable calculus. So, in the last lecture I have to discuss about Taylors theorem, that how can we write Taylors theorem for 2 or more than 2 variable functions.

So, how can you write that? That if you have say f of x naught plus h y naught plus k.

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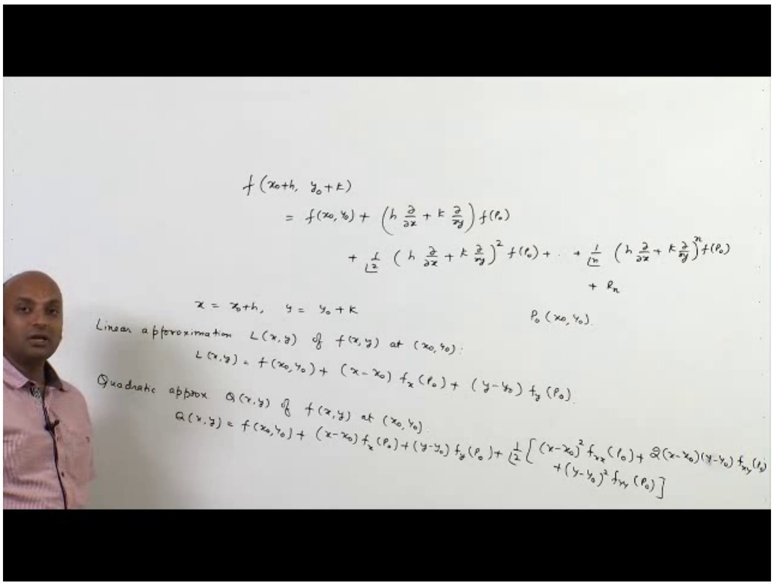

So, that will be equal to f of x naught y naught plus h del by del x plus k del by del y of f at p naught, plus 1 by factorial 2 h del by del x plus k del by del y whole square f at P naught and so on 1 by factorial n h del by del x plus k del by del y whole raise to power n at P naught plus reminder term ok. Where P naught is x naught y naught.

Now, if you want to find out linear approximation of this f at x naught y naught. So, how can I do that? We simply we simply put x equals to x minus we simply replace x is equals to x naught plus h and y is equals to y naught plus k. When you replace x by x naught plus k and y and y by y naught plus k and takes the term up to power 1 then we get the linear approximation of this f.

So, what a linear approximation? L xy of f xy at x naught y naught is simply fx naught y naught plus this h will be x minus x naught fx at P naught plus y minus y naught fy at p naught. So, this will be the linear approximation of this f at x naught y naught.

And what is the quadratic approximation? Quadratic approximation Q xy of f xy at x naught y naught will be simply 2 xy is equals to f x naught y naught plus now you will take term up to here, you first expand this and then replace h by x minus x naught and k by y minus y naught.

So, what we will obtain? They simply x minus x naught fx at P naught plus y minus y naught fy at P naught plus this will be 1 by factorial 2. So, this term is x minus x naught whole square f xx at P naught plus this is 2 xy x minus x naught y minus y naught f xy at P naught plus y minus y naught whole square f yy at P naught. So, this is the quadratic approximation of this f at x naught y naught.

Now if you find linear approximation or quadratic approximation, how accurate our approximation is or how can we find out the bound of the error. So, this lecture is deal with error approximation. So, let us go with this lecture now.

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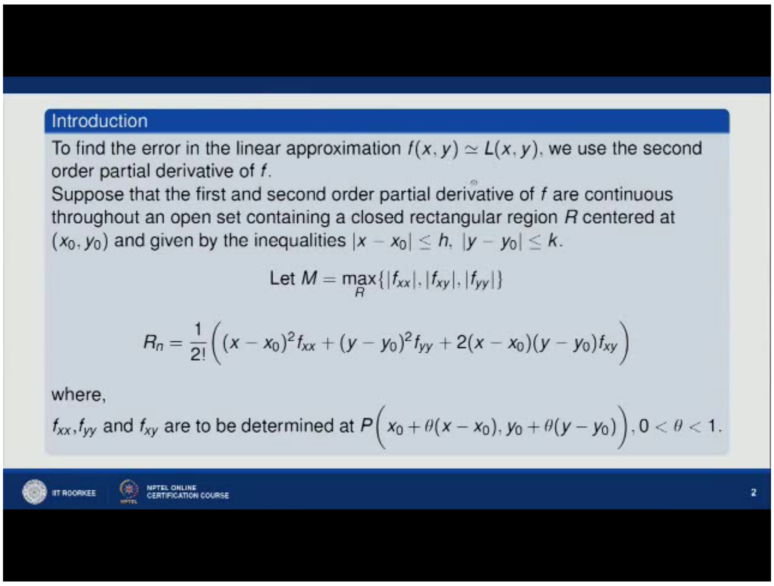

Now, suppose you find linear approximation L xy of f xy. Now to find to find the error in the linear approximation first let us suppose that first and second a partial derivative of f are continuous throughout an open set containing a closed rectangular region r centered at x naught y naught and given by mod x minus x naught less than equals to h and mod y minus y naught less than equals to k. So, let us discuss this.

So, if you if we write linear approximation of f at x naught y naught.

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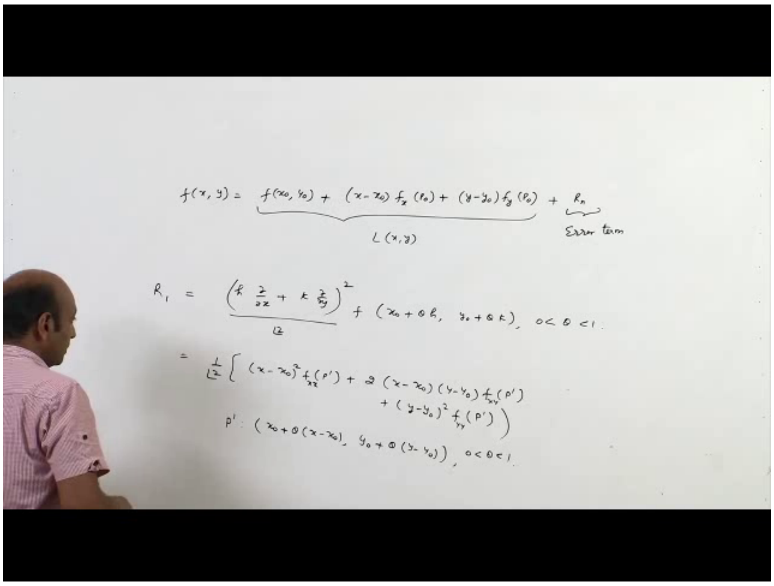

So, that will be f xy is equals to f x naught y naught plus x minus x naught f fx at P naught plus y minus y naught fy at P naught and plus we write remainder term. So, this is L xy which is the linear approximation of this f at x naught y naught and this is a remainder term or the error term.

How you find Rnx now? Rn now Rn will be given by here we are taking here we are finding linear approximation I mean n equals to 1. So, we will put n equal to 1 here. So, this will be this is h del by del x plus k del by del y whole raise to the power n plus 1 that is 2 upon factorial 2 f second derivative at x naught plus theta h and y naught plus theta k where theta between 0 to 1 ok.

Now, when you replace when you replace h by you simply expand this and replace h by x minus x naught and y by and k by y minus y naught. So, what you will obtain. So, this will be sorry this derivative will not be here because this is included here ok. Now this will be when you open this it is 1 the factorial 2 h is x minus x naught it is f at say it is P dash plus 2 x minus x naught y minus y naught f at P dash plus y minus y naught whole square f at p dash, it is f xx it is f xy it is f yy.

Now you simply expand this take the power 2 simply expand this and replace h by x minus x naught and y by and k by y minus y naught. So, you will get this expression at P dash where P dash is simply where P dash is this point x naught plus theta into x minus x naught and y naught plus theta into y minus y naught where theta is between 0 and 1.

Now how we can find the bound of this error? So, this will be given by the error term for linear approximation how can we find out an upper bound of this error. So, so that we can say that the approximation that the error concerned with this approximation is, the maximum amount of error will be this which this much.

How can you find that? So, to find this so, this is over R 1.

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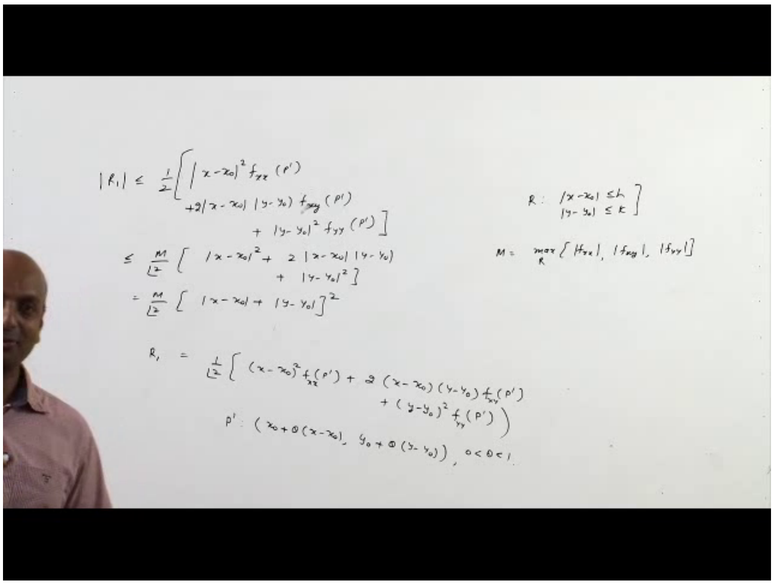

So, mod of R 1 will be equal to 1 by 2 mod of. So, it will be less than equals to you can say mod of x minus x naught whole square f xx at P dash plus mod of x minus x naught mod of y minus y naught f yy f xy at P dash plus mod y minus y naught whole square f yy at P dash by this expression where P dash is this point.

Now let us suppose in this region R which is x minus x naught less than equals to h and mod y minus y naught less than equals to k in this rectangular region the maximum value of mod f xx mod f xy and mod yy is R is M the maximum value of mod f xx mod f xy and mod yy over this region R, suppose this is M.

So, we can say that this is less than equals to this is less than equals to M this is less than equals to M and this is less than equals to M and M we can take out. So, this will be M by factorial 2 this will be mod x minus x naught whole square plus 2 2 is here. So, it is 2 mod x minus x naught 2 mod y minus y naught plus mod y minus y naught whole square. So, this is equals to M by factorial 2 and this is mod x minus x naught plus mod y minus y naught whole square. So, this is an upper bound of the error term. So, basically this will give an error term concerned with a linear approximation of f xy.

So, let us solve some problems based on this find the linearization L xy of the following function at P naught and also find the maximum error in the specified region R.

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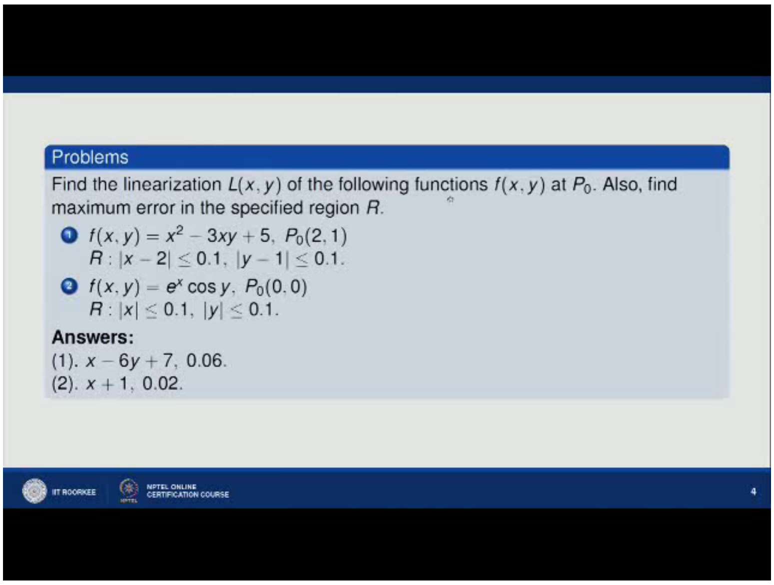

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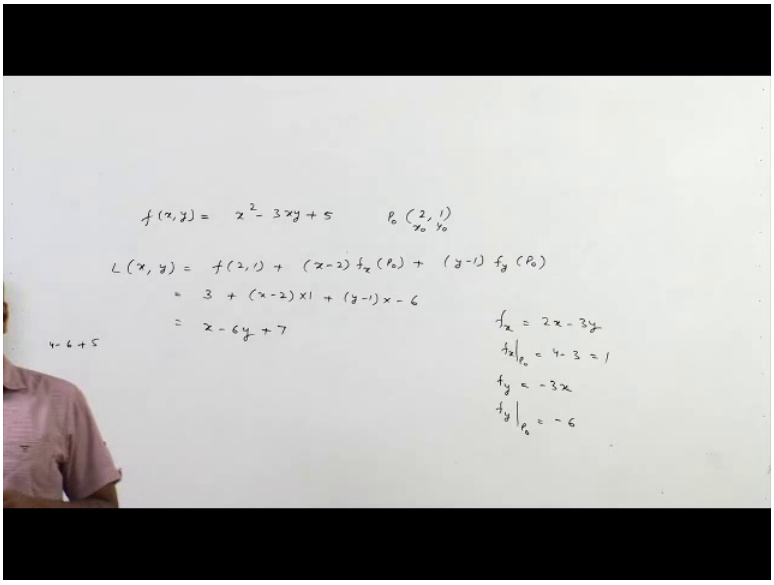

Suppose these are the first problem; the first problem is f xy is equal to this is x square minus 3 xy plus 5 and P naught is 1 comma it is 2 2 and sorry it is 2 comma 1.

So, first we have to find out the linear approximation of this f. So, linear approximation is given by f at 2 1 plus x minus x naught this is x naught and this is y naught. So, x minus x naught fx at P naught plus y minus y naught fy at P naught. So, what is fx at P naught? For this f fx is given by 2 x minus 3 y and fx at P naught will be 4 minus 3 that is 1.

Now, what is fy del f by del y? This will be minus 3 x and fy at P naught will be from here it is minus 6. So, this will be equal to when you substitute x equal to 2 and y equal to 1. So, it is 4 minus 6 plus 5 that is 9 minus 6 is 3 plus x minus 2, fx as P naught is one. So, it is into 1 plus y minus 1 into minus 6. So, it is x minus 6 y it is plus 6 plus 3 plus 9 9 minus 2 is 7. So, it is 7. So, this will be approximation of this f.

Now, how can you find the maximum error? So, for error term for error we first find f xx.

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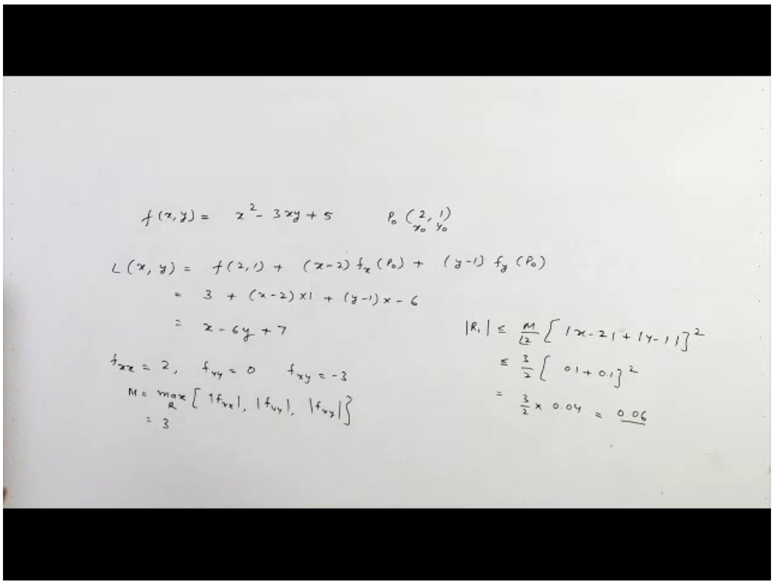

What is f xx? f xx from here is 2 f yy is minus 3 and fx f yy is 0 sorry and fx y is minus 3. So, M which is the maximum of over r of mod of f xx mod of f yy and mod of f xy what the region r which is given by mod x minus 2 less than equals to 0.1 and mod y minus 1 less than equals to 0.1 ok. Here these values are constant they are not dependent on x or y. So, their maximum value will remain the same on any region R.

So, this is 2 this is 0 and this mod is 3. So, maximum will be 3 3 and the bound of the error will be given by M by factorial 2 as we have discussed x minus x naught mod whole x mod x 1, x naught plus mod y minus y naught whole square this M is 3. So, it is 3 by 2.

Now, mod of x minus x naught is less than equals to 0.1 as given in the problem and mod of y minus 1 is less than equals to 0.1 both are 0.1 x naught is 2 and y naught is 1. So, this would be less than equals to 0.1 plus 0.1 whole square. So, this is 3 by 2 into 0.2 square that is 0.04. So, this will be a simply 0.06. So, the bound of the error is 0.06.

So, similarly we can solve second problem also now for a second problem how can you find out m. So, let us discuss this.

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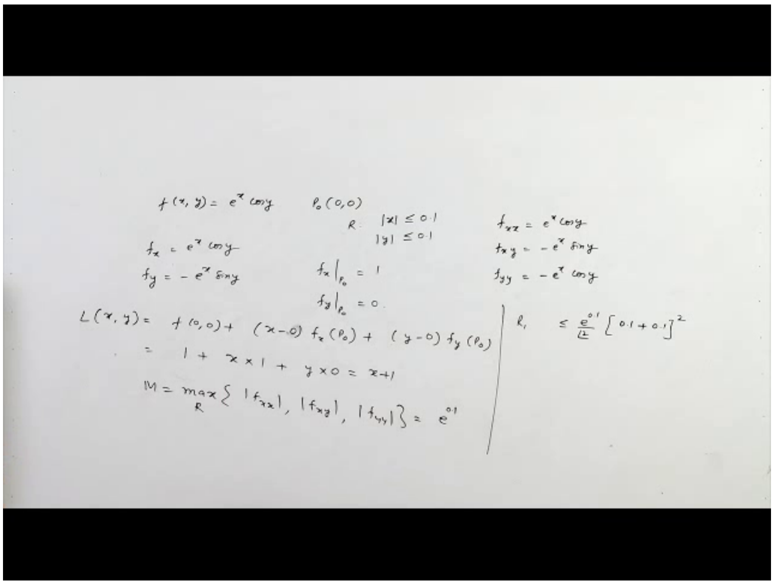

Here, f xy is e raised to power x cos y point is 00 and the region R is mod x less than equals to 0.1 and mod y less than equals to 0.1.

Now, what is fx? Fx is e raised to power x cos y what is fy? Minus e raised to power x sine y fx as P naught is simply 1 and fy at P naught is simply 0. So, linearization will be given by I mean linear approximation of this f will be given by f 0 0 plus x minus 0 x minus 0 fx at P naught plus y 0 fy at P naught. This is the linear approximation of this f at P naught.

So, this is f 0 0 is 1 x and fx is 1 at P naught and this is y into 0. So, this is x plus 1. So, the linear approximation of this surface is x plus 1 now what is the maximum error how can you find that. So, first we find f xx f xx is e raised to power x cos y, f xy is minus e raised to power x sine y and f yy is minus e raise to the power x cos y.

Now in this region, in this region you see the massive value of mod of cos or mod of sine is always 1 and in this region in this region it will attain maximum when x is 0.1.

So, we can say that M which is maximum of over r of mod xx mod yy mod it is mod xx, mod xy, and mod yy this will be e raise to power 0.1 because because the maximum value of sine or cos is always 1 and maximum value of e raise to power x in this region is the e raise to power 0.2 1. So, a maximum will be value will be value will be raised to power 0.1.

So, what will be the bound? Bound of the error will be less than equals to M by factorial 2 mod x minus x naught plus mod y minus y naught whole square. So, we can solve this and find out the maximum error.

Now, similarly we can do linearization of 3 variable functions, for 3 variable functions we will be having.

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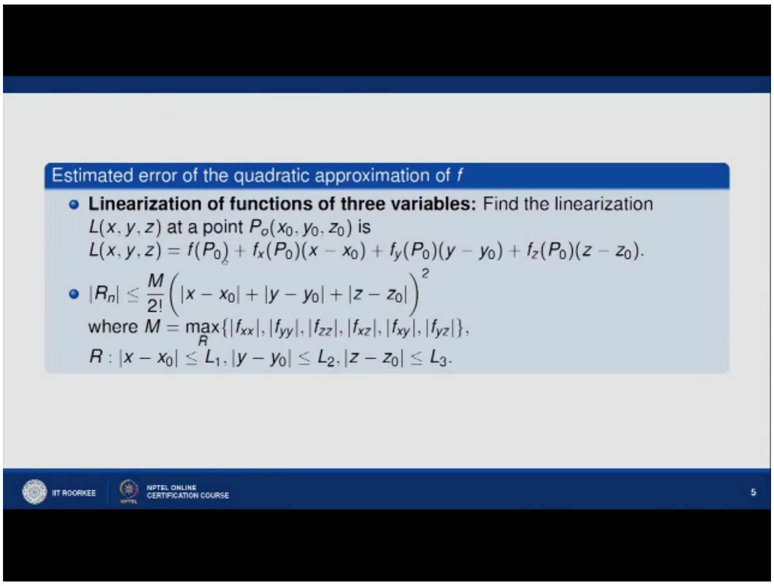

F P naught plus f x P naught x minus x naught plus fy P naught y minus y naught and this term will be there which is fz P naught z minus z naught and similarly the linear approximation I mean the error bound of this linear approximation is given by M by factorial 2 mod x minus x naught plus mod y minus y naught plus mod z minus at naught whole square. Where M is a maximum of all second order partial derivatives all combinations and R is given by this region.

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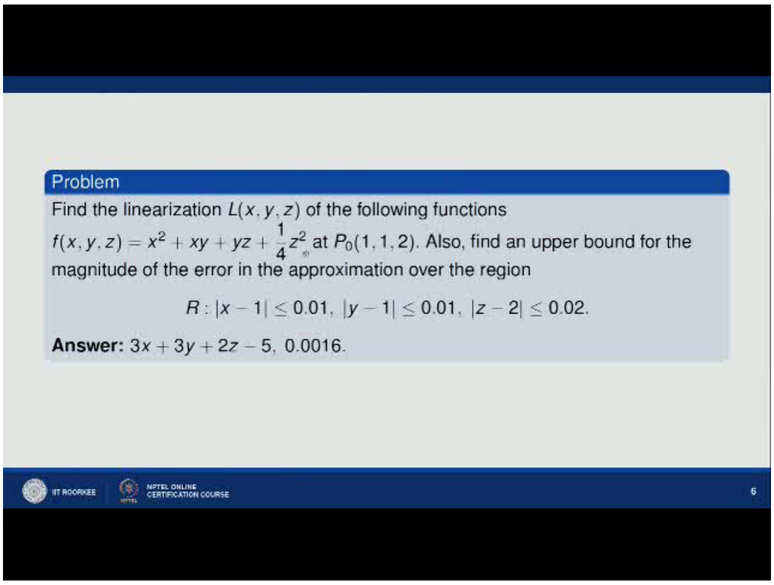

Say, we want to solve this problem of 3 variable functions.

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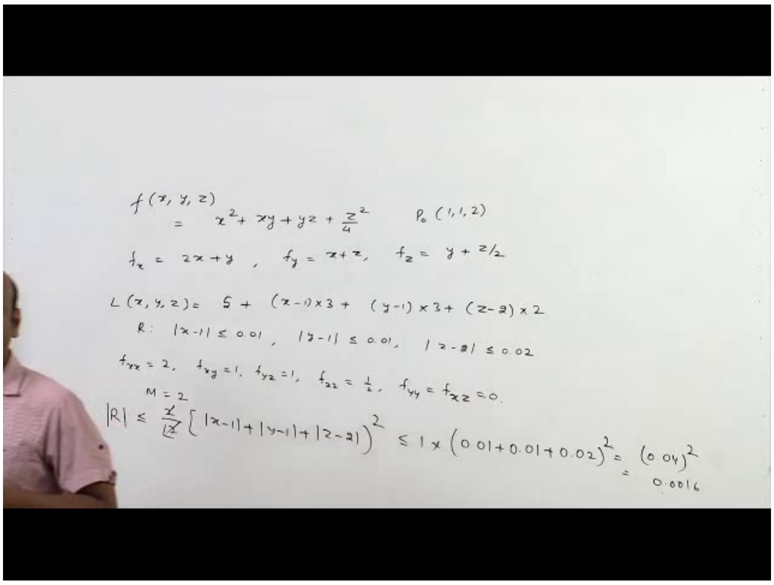

So, what is f xyz it is x square plus xy plus yz plus z square by 4 and P naught is 1 1 2. First you find the linear approximation of this f. So, what is fx is 2 x plus y what is fy is x plus z and what is fz is y plus zy 2.

Now, linear approximation of this f will be f at P naught f at P naught this is 1, I mean this is x naught y naught z naught this is 1 plus 1 is 2, 2 plus 2 is 4 4 plus 1 is 5 plus then it is x minus x naught that is x minus 1 into fx at P naught. So, fx at P naught is 2 plus 1 that is 3 plus y minus y naught and fy at P naught this is 1 plus 2 that is 3 plus z minus z naught into fz as this which is 2. So, you can solve this which will be the linear approximation of this f at 1 1 2.

Now, suppose you want to find out an upper bound for the magnitude of error in the approximation over this region. So, here region is mod x minus 1 less than equals to 0.01 mod y minus 1 less than equals to 0.01 and mod z minus 2 less than equals to 0.02.

So, now you find f xx, f xx is clearly 2 f xy is 1 f yz is 1 f zz is 1 by 2 and all others second order partial derivatives are 0 like f yy is 0 like f xz is 0 f xz is 0. So, what is the maximum of all these is 2. So, M will be 2 because it is free from x y and z. So, it is constant. So, over this region or over any other region the maximum value will remain the same maximum of mod of this mod of this mod of this and mod of all these that is 2.

So, what is the bound of the error? Bound to the error will be less than equals to M by a factorial 2 mod x minus x naught mod x minus 1 plus mod y minus 1 plus mod z minus 2 whole square and this is less than equals to 1 into 2 two cancels out this is mod x minus 1 is less than equals to 0.01, this is less than equal to 0.01 and then as it goes to a 0.02. So, it is 0.01 plus 0.02, 0.01 plus 0.02 whole square. So, it is 1 plus 1 2 plus 1 plus 1 that is 4 and 4 square is 0.04 whole square that is 16 and. So, that will be the error approximation.

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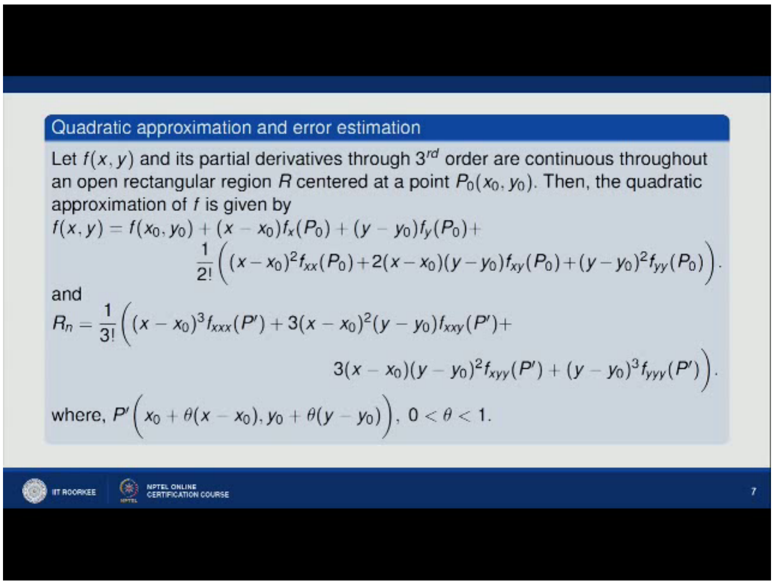

Now, for quadratic approximation as we already did. So, let us discuss this.

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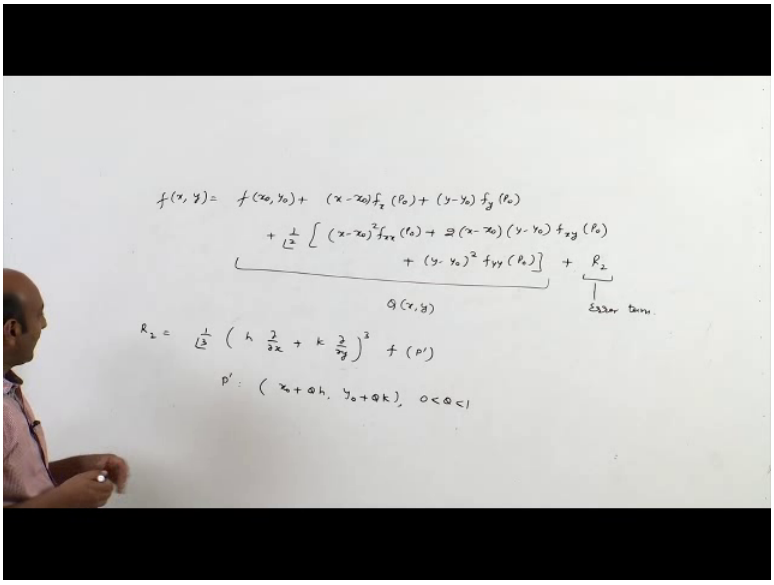

So, f xy is given as fx naught y naught plus x minus x naught fx at P naught plus y minus y naught fy at P naught plus 1 by factorial 2, x minus x naught whole square f xx at P naught plus 2 x minus x naught y minus y naught f xy at P naught plus y minus y naught whole square f yy at P naught plus error term. So, this you can replace by 2 R 2. So, this entire term is the quality approximation of this f and this will be the error term or the remainder term.

So, how can we find error term here? We can find the quality approximation of any f using this using this polynomial you find the values of fx, fy f xx, f xy, f yy at P naught substitute it over here. So, you will get quadratic approximation of a function f.

Now, for the bound of the error now here are 2 will be given by 1 by factorial 3 h del by del x plus k del by del y whole cube and f at P dash, where P dash is simply x naught plus theta h and y naught plus theta k where theta between 0 and 1.

So, how can you find the error term. So, let us discuss this. So, first you expand this term and then you replace h by x minus x naught and y and k by y minus y naught. So, how we will do that is let us let us see.

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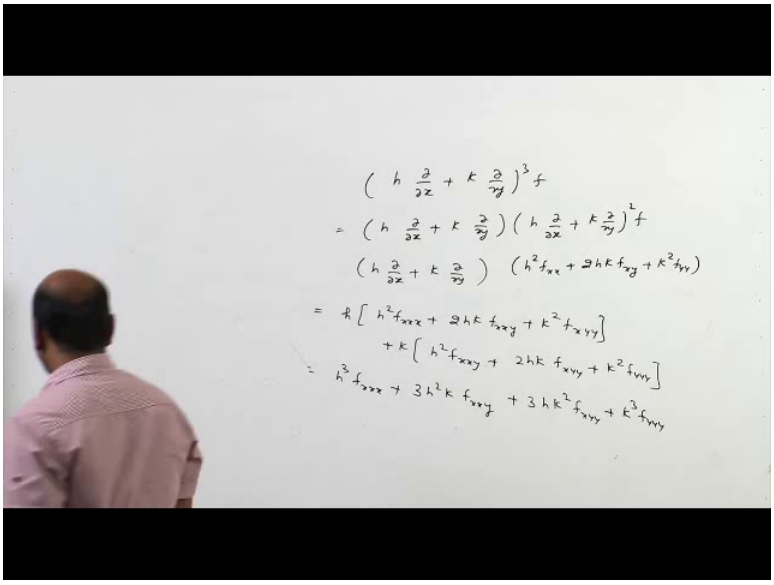

First you will find h del by del x plus k, del by del y whole cube what it is at f of f. So, it is h del by del x plus k del by del y whole bracket into h del by del x plus k del by del y whole square f.

So, we have already seen that this term is h square f xx plus 2 hk f xy plus k square f yy. So, this we have already seen when you operate this operator on this function. So, what you will obtain now? This is h h square f xx plus 2 hk f xxy plus k square f xyy plus. So, this term is over here plus k times h square f xxy plus 2 hk f xyy plus k square f yyy and this will be h cube f xx x plus it is 2 h square k and it is also h square k we are taking both are same. So, 2 h square k f xxy plus. So, it is 3 yeah 2 it is 3 2 times here and 1 times here. So, it is 3 and similarly here 2 hk square f xyy plus k cube f yyy. So, this is the term this is this term.

And now you replace h by x minus x naught and y by and k by y minus y naught. So, what you will obtain.

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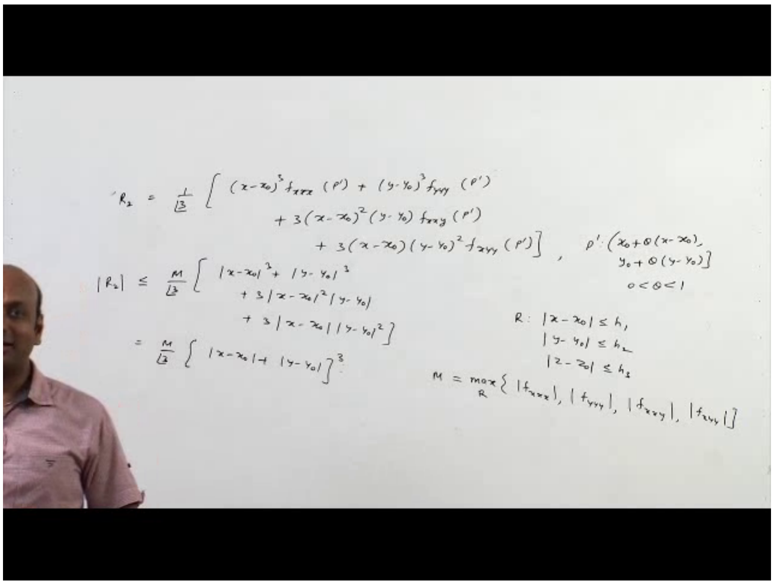

So, what is r R 2 mod of R 2 first find R 2 and then we will find mod it is 1 by factorial 3, it will be it will be x minus x naught whole cube f xxx at P dash plus y minus y naught whole cube f yyy at P dash plus 3 x minus x naught whole square, y minus y naught f xxy at P dash plus 3 x minus x naught y minus y naught whole square f xyy at P dash. Now where P dash is simply x naught plus theta x minus x naught and y naught plus theta y minus y naught and theta between 0 to 1. So, simply replace h by x minus x naught and k by y minus y naught you will get this expression.

Now, suppose you have a region R which is x minus x naught less than equals to say h 1 y minus y naught less than equals to say h 2 and in this region in this region let us suppose let us suppose a maximum of mod f xxx mod f yyy and mod f xxy and mod f xyy this maximum is say M.

So, we can say that R 2 this modulus will be less than equals to now this is maximum is M. So, this will be less than equals to M mod of this less than equals to M, less than equals to M, less than equals to M. So, M will be common M can we take out and inside the bracket you will be having you will be having mod of this cube; I mean this will be mod x minus x naught whole cube plus mod y minus y naught whole cube plus 3 mod x minus x naught whole square y minus y naught plus 3 mod x minus x naught into mod y minus y naught whole square and this is nothing, but mod x minus x naught plus mod y minus y naught whole cube.

So, in this way we can find out the error bound of the error for quadratic approximation. So, we can discuss this example, this sample is very simple let us discuss this. So, similarly we can think for higher approximations cubic approximation or approximation of degree 4 using the same concept ok. Here I have discussed linear approximation or quadratic approximation and the corresponding error approximation or the error bound.

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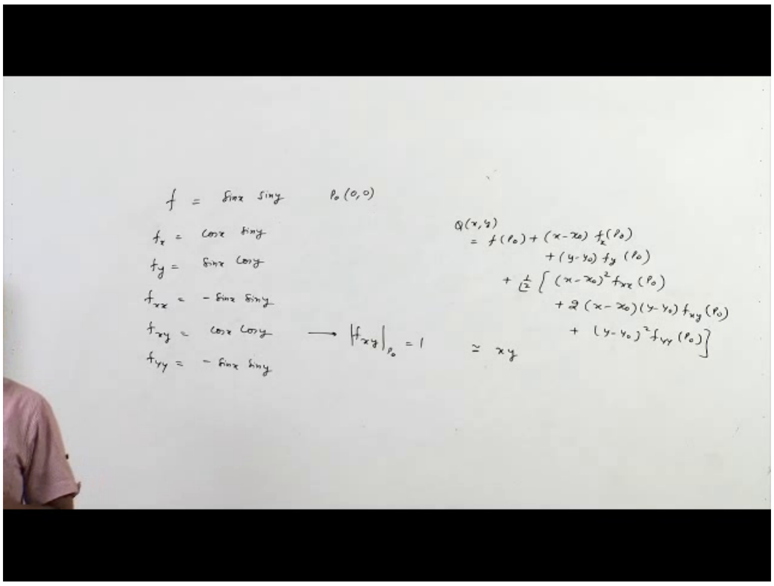

Now function is sine x sine y point is 0 0. Now when you find fx it is cos x sine y when you find fy it is sine x cos y, when you find f xx it is minus sine x sine y, f xy it is cos x cos y, f yy it is minus sine x sine y because you want quadratic approximation.

Now when these values at 0 comma 0 this is 0 because of sine 0 this is 0, this is 0, this is 1, and this is 0. So, we are leaving with only 1 1 value at P naught which is f xy now what is the quadratic approximation of any f quadratic approximation of f will be given by f at P naught plus x minus x naught f at P naught plus fx at P naught y minus y naught fy at P naught plus 1 by factorial 2 x minus x naught whole square f xx at P naught plus 2 x minus x naught y minus y naught f xy at P naught plus y minus y naught whole square f yy at P naught.

Now this is 0, this is 0, this is 0, this is 0, and f at P naught is also 0. So, when f is only this term which is 1 and x naught and y naught are 0. So, this is 2 by 2 which is 1. So, this is given by x into y.

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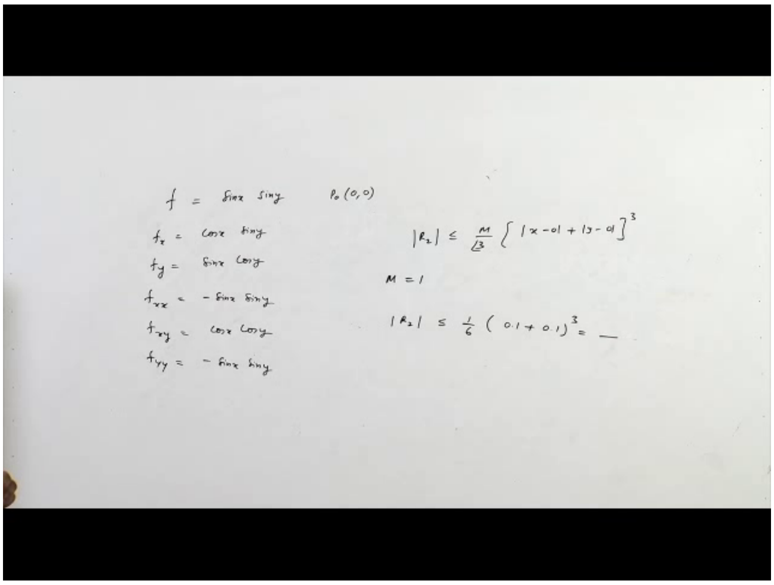

So, x into y is the linear is the quadratic approximation of this f now to find the bound. Now what is the bound? So, error bound is given by M by factorial 3 mod x minus 0 plus mod y minus 0 whole cube.

Now to find M, which is the maximum of mod f xx as all third order partial derivatives all combinations? So, maximum over this region when mod x is less than equals to 0.1 and mod y is than equals to 0.1 how can you find that? Now any combination of like f xxx or f xyy will contain the terms of sine and cos only and the maximum value of sine into cos is always 1 because the maximum value of sine is 1 and cause is 1.

So, we can simply say that that upper bound of M will be 1. So, we can say that R 2 mod of R 2 will be less than equals to 1 by 6, and this is 0.1 plus 0.1 whole cube. So, we can simplify this and this will give error approximation of this quadratic approximation over this region R. So, that is all about this lecture so.

Thank you very much.