Numerical Linear Algebra
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Lecture – 55
Basic theorems on eigenvalues and QR method

Hello friends, welcome to this lecture. In this lecture we will discuss some Eigen pairs, Eigenvalue problem and some elementary properties of Eigen pairs. Also we will discuss basic QR method to find out Eigenvalues of a given matrix. So, let us start, say discussing the lecture here.

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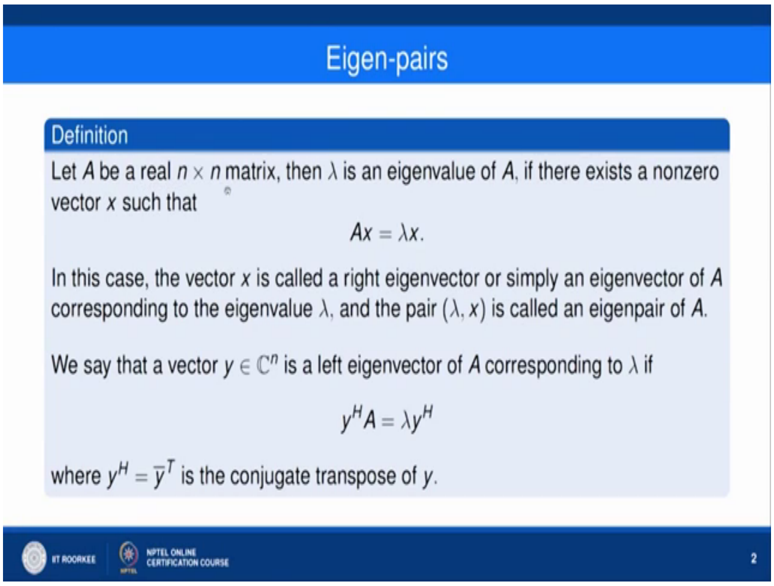

So, first we define what is Eigen pairs. So, let A be a real n cross n matrix, then lambda is an Eigenvalue of A, if they exist a non 0 vector x such that Ax equal to lambda x. So, in this case the vector x is called a right eigenvector or simply an eigenvector of A, corresponding to the Eigenvalue lambda and the pairs this lambda and x, is known as Eigen pair of matrix a.

This Eigen, finding the Eigenvalue of A given matrix and eigenvector of A given matrix, is very very important say problem in linear algebra. In fact, many a times when you discuss differential equation or say difference equation or buckling problem, there are many more problem, where your system a your say finding say solution of the given system is reduced to finding the say Eigenvalue of a given matrix.

For example if we want to consider the stability of a, solution of a linear system, then we convert our system into x dash equal to ax, where x is a vector state vector. And when we write x dash is equal to Ax then any, stability of any solution is governed by the behavior of the Eigenvalue of the matrix a so, that is why the problem Eigenvalue problem, finding the Eigenvalue of a given matrix is very very important problem. So, in this lecture we will discuss what is Eigen pairs and how to find out what are the properties of Eigen pairs and how to find out eigen value of a given matrix.

So, we have defined lambda x as an Eigen pair of A, if Ax equal to lambda x is here and we say that a vector y is a left eigenvector of A of a corresponding to the Eigenvalue lambda, if y H a is equal to lambda y H. Here y H is a conjugate transpose of y, is the conjugate transpose of y. So, based on the definition of Eigen pair now we observe certain things.

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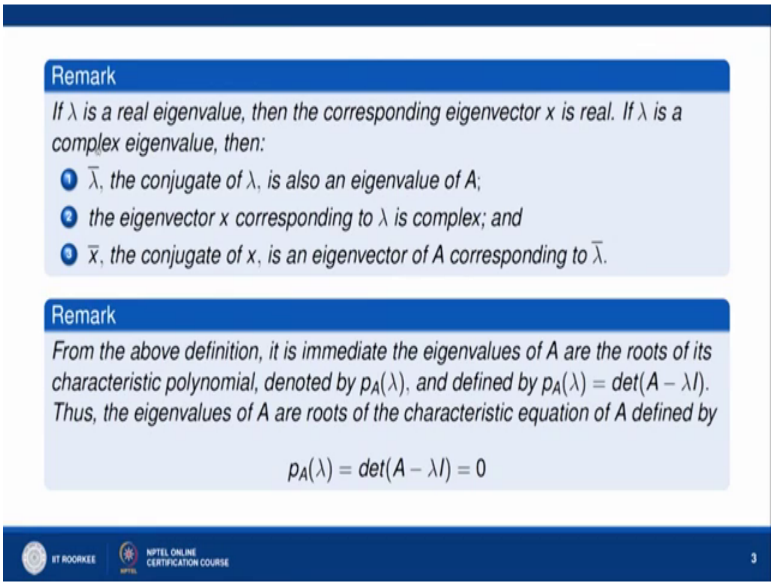

First that if lambda is a real Eigenvalue then the corresponding eigenvector is also real, and if lambda is a complex Eigenvalue, then lambda bar, which is the conjugate conjugate of lambda is also an Eigenvalue of a and the eigenvector x corresponding to lambda is complex, and x bar the conjugate of x is an eigenvector of A corresponding to lambda bar; that is quite trivial.

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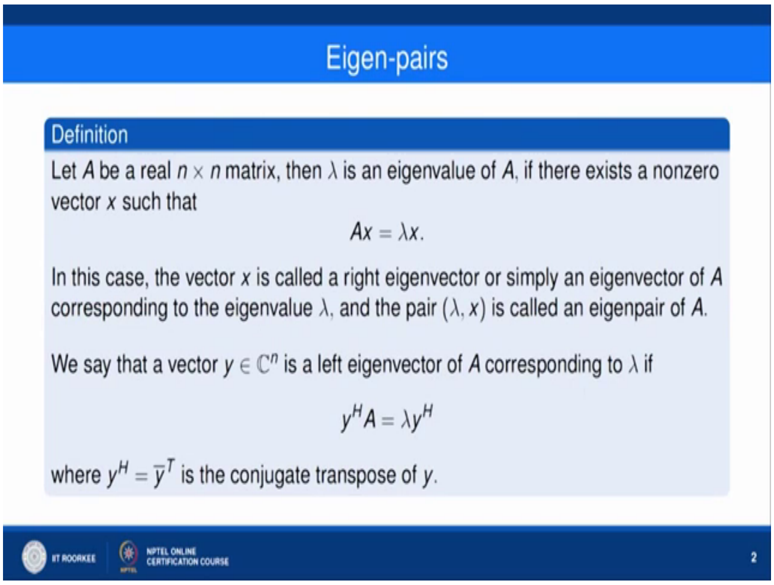

When you consider this problem Ax equal to lambda x and if we take the conjugate transpose you can conjugate you can see that it is true. Now, second remark is that from the above definition it is immediate, that the Eigenvalues of A are the roots of its characteristic polynomial and characteristic polynomial it is denoted by p A lambda and it is defined as a determinant of a minus A lambda I, this is the Eigenvalues of A are the roots of the characteristic equation of a defined by p A lambda equal to 0, where p A lambda is equal to determinant of A minus lambda I

In some book you find that characteristic polynomials are defined as determinant of lambda I minus A. So, here we may have, in some book it is determinant of A minus lambda I and in some books it is determinant of lambda I minus A, but in finding the problem of Eigenvalues of A given matrix whatever definition you will take, it is just a root of characteristic equation and characteristic equation is determinant of A minus lambda I is equal to 0 here.

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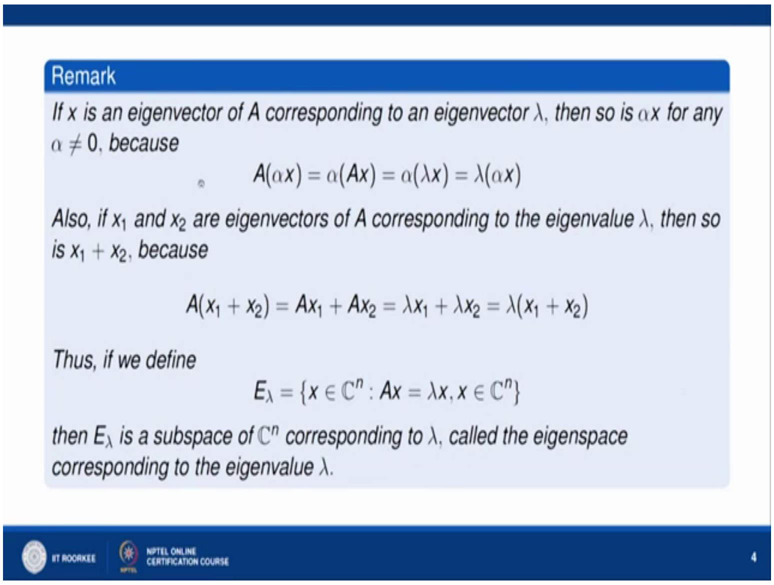

And we also saw that that the Eigen, we try to define what is Eigen space. So, if x is an eigenvector of A corresponding to an eigenvector lambda. Then alpha x is also an Eigen vector corresponding to the same Eigen Eigenvalue lambda. So, A alpha x equal to alpha A x; that is due to linearity and Ax is nothing, but lambda x. So, we can write A alpha x equal to lambda alpha x

So, it means that if x is an eigenvector corresponding to lambda then any non0 constant multiple of x is also an eigenvector corresponding to this lambda. And similarly if x 1 and x 2 are two eigenvectors of A corresponding to the Eigenvalue lambda, then we can also show that x 1 plus x 2 is also an Eigen eigenvector corresponding to the same Eigenvalue lambda.

So, that we can verify easily that A x 1 plus x 2 is equal to Ax 1 plus x 2 and x 1 is lambda x 1, because x 1 is an eigenvector corresponding to lambda. Similarly x 2 is lambda x 2 and if we take lambda common then it is lambda x 1 plus x 2. So, it means that if x 1, x 2 are eigenvector of A corresponding to lambda, then x 1 plus x 2 is also an eigenvector corresponding to lambda

So, we have seen that if x is an eigenvector of A then alpha x is also an eigenvector of A and x 1 plus x 2 x 1 and x 2 are eigenvectors of A corresponding to the Eigenvalue lambda, then x 1 plus x 2 is also an eigenvector corresponding to the same Eigenvalue lambda

So, here we can say that x 1 plus x 2 is eigenvector and constant multiple of x is also an eigenvector. So, we can define a set E lambda which consists all those x in Cn; such that Ax equal to lambda x, then this set E lambda is a subspace of Cn corresponding to the lambda, corresponding to lambda and it is called as Eigen space corresponding to the Eigenvalue lambda

So, basically E lambda is what? E lambda is the these set of all the eigenvectors including 0. So, it means that, it contains all the Eigenvalue eigenvectors corresponding to lambda and the 0 vector. 0 vector is not a eigenvector corresponding to lambda, is it ok. So, it is a basically the set of all eigenvectors corresponding to lambda union 0

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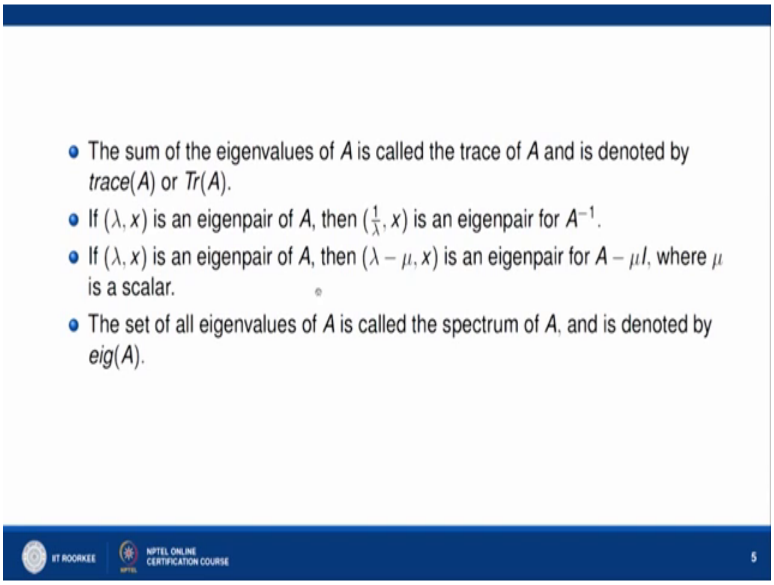

Now, based on this we have certain more remarks that the sum of the Eigenvalues of A is called the trace of a and it is denoted by trace of A or trace of A. And similarly we can also define that the product of all the Eigenvalues of A is known as determinant of A. So, determinant of A you can calculate by taking the product of all the Eigenvalues of the matrix same. And if lambda x is an Eigen pair of A then 1 by lambda comma x is an Eigen pair of a inverse

So, here when we define 1 by lambda or we define A inverse then we assume that none of the Eigenvalue is a 0 Eigenvalue. So, that is why we are able to define 1 by lambda. So, it means that that if lambda x is an Eigen pair of A then 1 by lambda comma x is an Eigen pair of A inverse

Now, next remark is that if lambda x is an Eigen pair of A then lambda minus mu comma x is an Eigen pair from a minus mu y, where mu is any scalar, and the set of all Eigenvalues of A is called the spectrum of A and it is denoted by e i g of A. So, that gives you the spectrum of all the Eigenvalues of matrix same.

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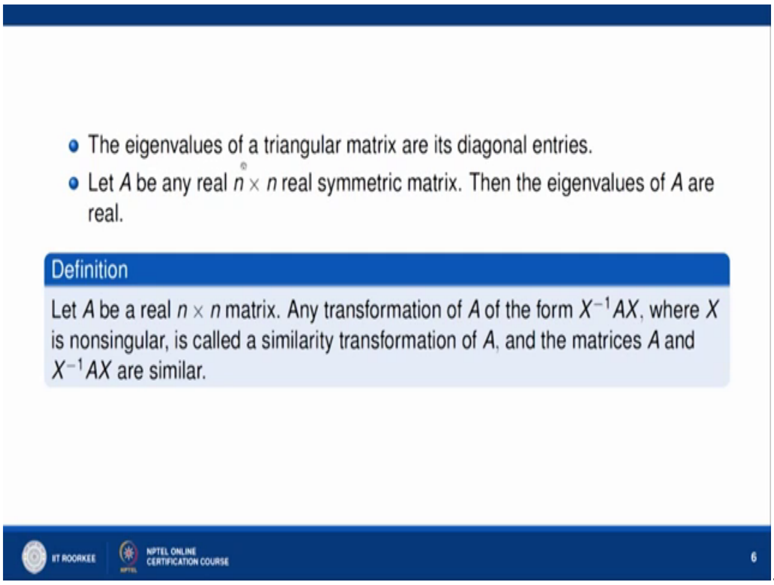

And the Eigenvalues of a triangular matrix are its diagonal entry. So, if we have a triangular matrix or diagonal matrix, then the diagonal entries are nothing, but the Eigenvalues of a given matrix. And let A be any real n cross n real symmetric matrix then the Eigenvalues of a are real.

Now, we also define a similarity transformation what is that, let A be a real n cross n matrix, then any transformation of A of the form X inverse AX, where X is nonsingular is called a similarity transformation of A and the matrix A and X inverse AX are similar right and.

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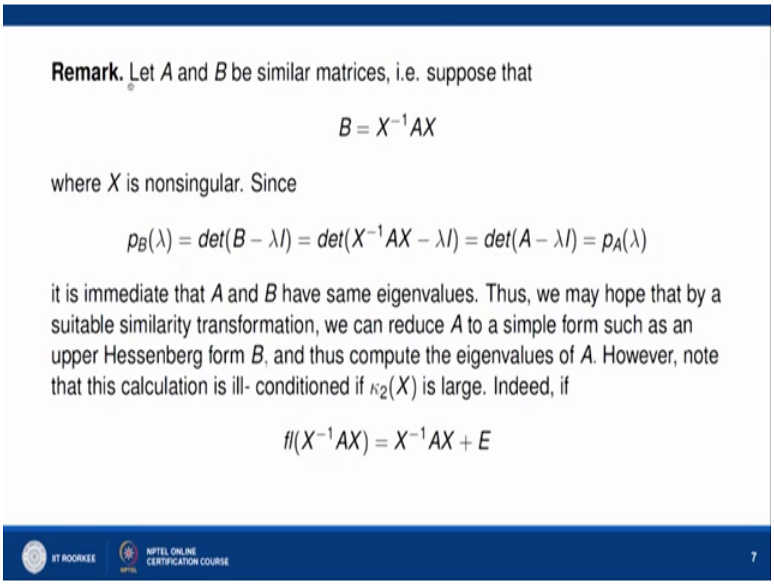

Here we say that let A and B be similar matrices, it means that B and A are related by this relation that B can be written as X inverse AX. Then our claim is that Eigenvalues of B A A and B will share the same Eigenvalues

So, here we can say that p B lambda, which is nothing, but determinant of B minus lambda I. Now here we can replace B by X inverse Ax minus lambda I can be written as X inverse X. So, determinant of X inverse Ax minus lambda X inverse X, and if we take the x inverse and X common here then it is nothing, but determinant of A minus lambda I and which is nothing, but the characteristic polynomial of matrix A

So, it means that characteristic polynomial corresponding to B and characteristic polynomial of A are same, when B and A are similar matrices. So, it means that it is immediate that A and B have same Eigenvalues. Thus we may hope that by a suitable similarity transformation we can reduce A to a simple form; such as an upper Hessenberg form A B and thus comp compute the Eigenvalues of A.

So, this we will give you an idea how to find out the Eigenvalue of A given matrix. So, so by suitable transformation we try to convert our matrix into a triangular matrix or A diagonal matrix. And once it is converted in to a triangular matrix or diagonal matrix, then the a diagonal entries of the reduced form it will give you the Eigenvalue of the matrix A.

But here we, we may note one thing that that in this calculation we may create some kind of error, because if we look at the computer representation of the matrix X inverse AX then it is nothing, but X inverse AX plus some error and this error is basically written as.

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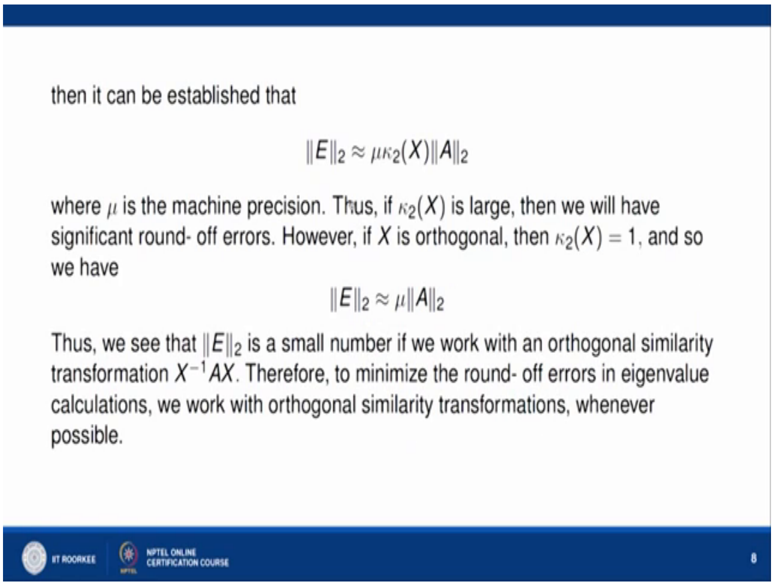

This condition number of X and two norm of A and mu, where mu is the machine precision and here X is the matrix, which we are considering here for this similarity transformation.

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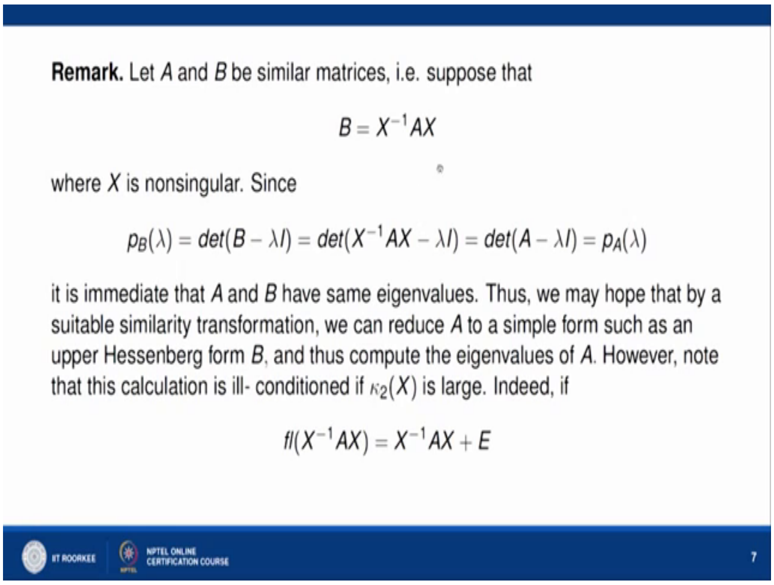

Now, if this condition number of this a matrix X is large, then we can say that this error is quite large. So, this procedure is done in efficient manner, if we choose our matrix X anyway such that this condition number X is very small or you can say that we have already seen that if X is orthogonal, then condition number of this orthogonal matrix is 1.

So, we can say that in this case the two norm of E is nothing, but mu times two norm of A and this is a. So, if we can say that this two norm of error matrix is a small. And if we work with an orthogonal similarity transformation X inverse X AX, then the error is not going to increase. So, therefore, to minimize the round off errors in Eigenvalue calculation, we work with orthogonal similarity transformation whenever it is possible. Of course, it may not be possible for all the time, but whenever it is possible we always try to hunt for orthogonal similarity transformation.

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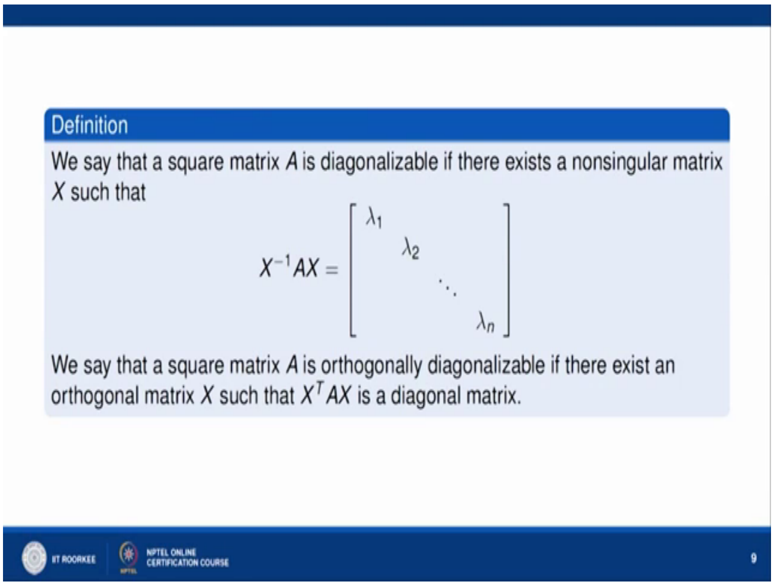

Now, we define the diagnosable matrices. So, we say that a square matrix A is diagnosable if they exist a nonsingular matrix X; such that X inverse AX is a diagonal matrix and we say that a square matrix A is orthogonally diagonalize diagonalizable, if they exist an orthogonal matrix X ; such that X transpose AX is diagonal matrix. So, it means that it is diagonalizable, if we have a nonsingular matrix X as A X X inverse AX is a D matrix, diagonal matrix and orthogonally diagonalizable, if this matrix X is an orthogonal matrix.

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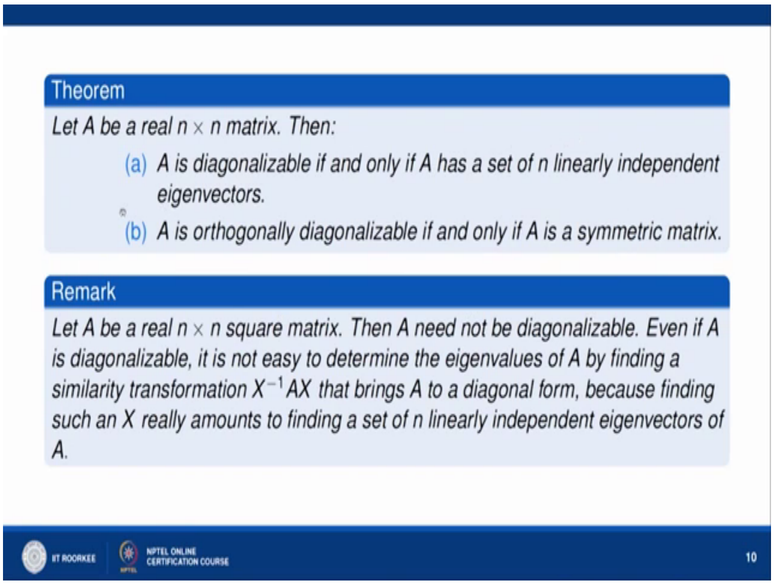

Now, we have some result on diagonalizability. Here I am just giving without any proof that you can see in any good book of linear algebra. So, here the result is says that let A be a real n cross n matrix, then A is diagonalizable if and only if A has set of n linearly independent eigenvectors. So, diagonalizable is possible if and only if we have n linearly independent eigenvector, where n is the size of the matrix M and A is orthogonally diagonalizable if and only if A is a symmetric matrix.

So, if A is a symmetric matrix then it is orthogonally diagonalizable and if A has a n linearly independent eigenvectors, then also A is diagonalizable. And next remark is that let A be real n cross n matrix, then A need not be diagnosable, it may not possible, because one such possible case is that we do not have n linearly independent eigenvectors. So, in that case A not may not be diagonalizable.

Then even if A is not diagonalizable, even if A is diagonalizable, it is not easy to find out the Eigenvalues of a by finding this similarity transformation X inverse AX, because here we what we have seen is that similarity matrices have same number of, same Eigenvalue, but to convert this X inverse AX you know in a diagonal form.

It is not easy, because we have to calculate the nonsingular matrix as X and that we can calculate only when we have n linearly independent eigenvectors of A. Basically this X is nothing, but the column matrix of consisting eigenvectors of the matrix A. So, when we have n linearly independent eigenvectors then we can concert Xs um, say v 1 to vn, where all these v 1 to v n are eigenvectors of the matrix A ok.

Next is a real Schur Form Theorem. So, if A is real n cross n matrix then they exist.

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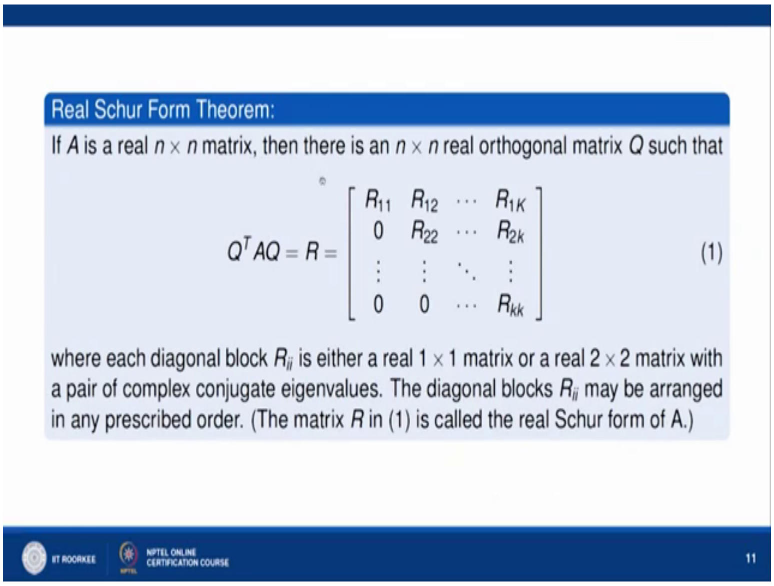

They exist an n cross n real orthogonal matrix Q; such that Q transpose A Q is R, where R is this block triangular matrix here. So, here this block matrices R I I is either a real one cross one matrix or real 2 cross 2 matrix with a pair of complex conjugate Eigenvalues.

So, our aim is to when, whenever we try to find out say I Eigenvalue of a given matrix, we try to use this real Schur Form Theorem and try to convert our matrix A into this. We need to find out a orthogonal matrix Q; such that Q transpose AQ can be reduced to in this form and the diagonal blocks RII may be arranged in any prescribed order, and the matrix R here is known as real Schur Form of A.

So, that is the basic point to consider QR, QR say QR method to find out the Eigenvalue of the matrix a. So, we will discuss it later.

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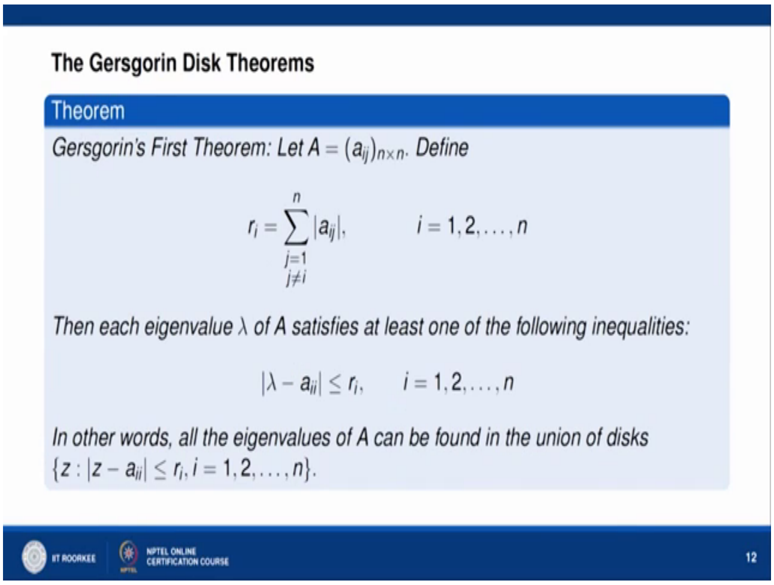

Now, here certain more result based on say localization of the Eigenvalues. So, here we discuss two theorem given by Gersgorin Disk Theorem. So, first theorem of Gersgorin is this, that let a is a ij is n cross n matrix. We can define ri as summation j equal to 1 to n ai j, where j is not equal to i. So, other than the diagonal entry, we find out the sum of the absolute term of the ith row here.

So, that we denote as ri. Then each Eigenvalue lambda of a satisfied at least one of the following inequalities; that is modulus of lambda minus a I i is less than or equal to ri or we can say that all the Eigenvalues of matrix a can be found in the union of this, this z; such that modulus of z minus a I i is less than or equal to ri, where I is from 1 to n.

So, at least this will give you an a bound with that your Eigenvalue will lie in which kind of disk.

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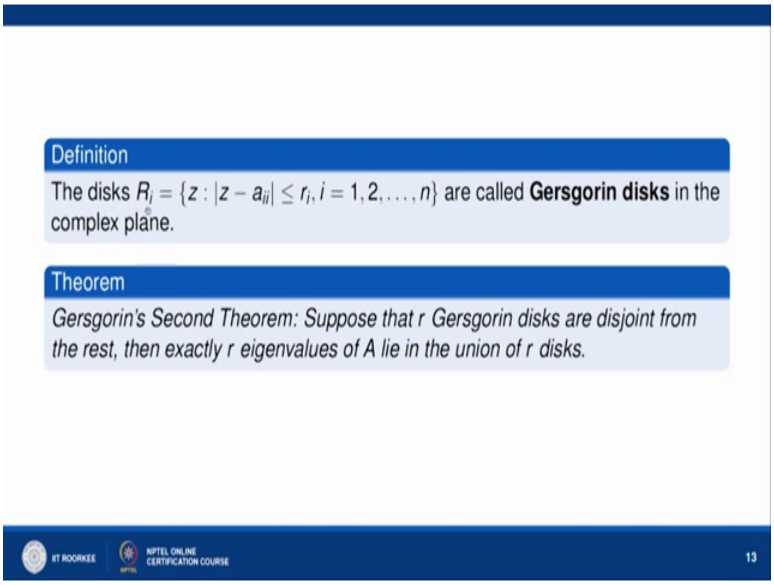

So, that will just give you an idea and the disk Ri, which is define as z such that modulus of z minus aii is less than or equal to Ri, where I is running from 1 to n are known as Gersgorin disk in the complex plane.

And the second theorem says that if suppose that R Gersgorin disks are disjoint from the rest, then exactly R Eigenvalues of A lie in the union of R disks. So, it means that if we have R disjoint Gersgorin disk then every Gersgorin disk will contain one Eigenvalue of matrix A.

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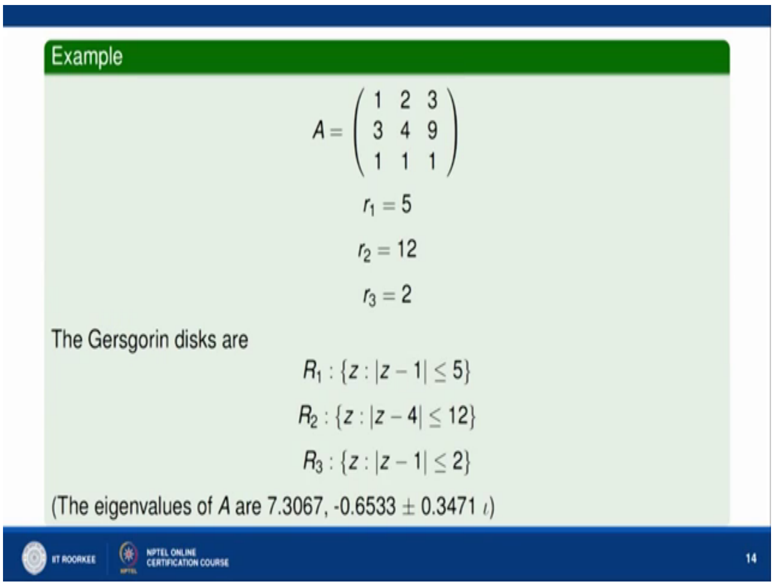

So, let us consider the example based on this Gersgorin disk theorems. So, first let us consider this a as 1 2 3 3 4 9 1 1 1 then if you look at the first Gersgorin disk; that is, if this is a diagonal entry one. So, based on removing this one we have 2 plus 3. So, it means that R 1 is 5 and if you look at the R 2 remove this 4. So, 3 plus 9 R 2 is 12 and here diagonal entry is 1.

So, R 3 is going to be 1 plus 1. So, that is 2. So, it means that R 1 is what, z minus the diagonal entry that is 1 less than or equal to R 1; that is 5 here. R 2 is z minus 4 4 is the diagonal entry here less than or equal to R 2 that is 12 and R 3 is z set of all z; such that modulus of z minus 1 is less than or equal to 2.

So, here we have three Gersgorin disks and it is defined like this. And if we calculate the Eigen values of the matrix using MATLAB or some other means, we can say that the Eigenvalues of a is coming out to be 7.3067 minus 0.5, 0.6533 plus minus 0.347 I and you can see that it is all lying inside your the Gersgorin disk R 2. So, that will give an idea that where your Eigenvalues will lie.

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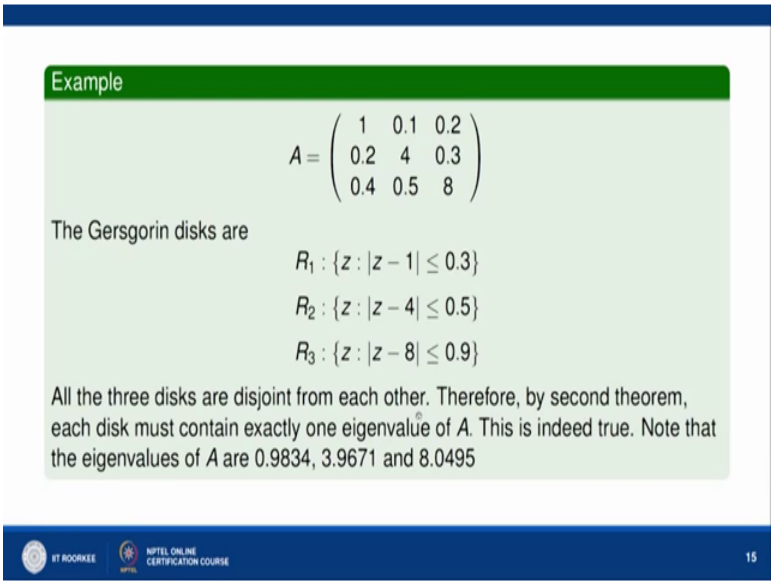

Similarly, look at the example based on the second Gersgorin theorem that if A is given as 1 0.1 0.2 and we have a matrix like this, then Gersgorin disks are defined as z minus 1 less than or equal to 0.3 and from second Gersgorin z minus 4 less than or equal to 0.5 and z minus 8 less than or equal to 0.9.

And if you look at, if we draw these three Gersgorin disk and we can see that these disks are disjoint to each other. So, all the three disks are disjoint from each other and we know that this matrix A must have three, say complex Eigenvalues. So, we can say that by second theorem of Gersgorin we can say that each disk must contain exactly one Eigenvalue of A and that we can check through MATLAB or some other means.

We can say that the Eigenvalues of A in this case is it is coming out 0.9834 which lies in the disk are one, 3.9671 which lies inside your disk R 2 and 8.0495 which lie in disk R 3. So, here we can say that since R 1 R 2 R 3 are all disjoint. So, we can say that each and each disk will contain at least one Eigenvalue of the matrix A..

So, in this way, these localization theorems are important.

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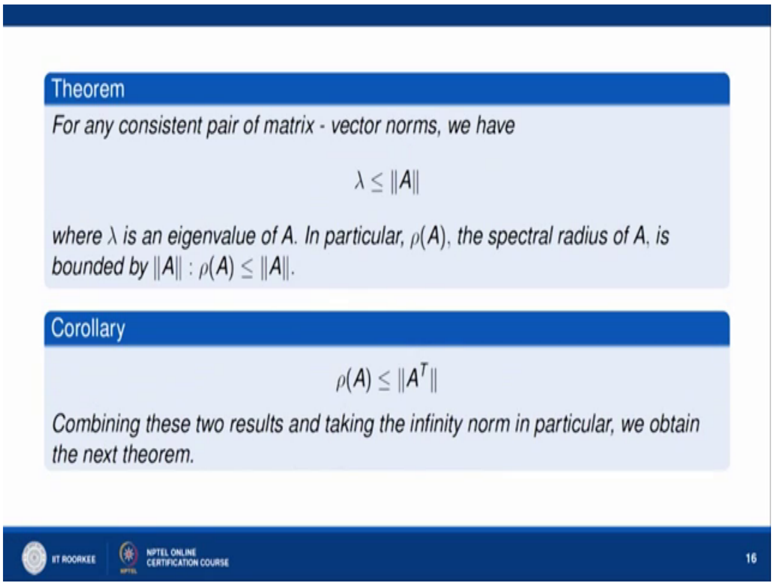

Now, next result which we have already discussed that, for any consistent pair of matrix vector vectors norm we have lambda less than or equal to norm of A, that we have discussed in the lecture of vector in matrix norm. So, there we have discussed it norm. In fact, we have proved something more; that is spectral radius of matrix A is less than or equal to A. By spectral radius of A is denoted as maximum of all the Eigenvalues of the matrix A.

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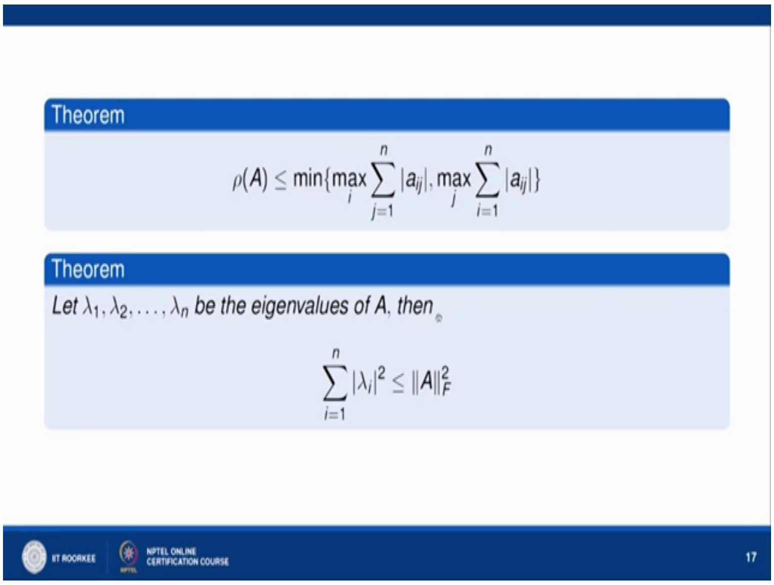

So, and there is one more corollary which says that the spectral radius of A, it is less than or equal to a transpose. And if we combine these two A result then we can have this theorem that spectral radius of A is less than or equal to minimum of, maximum of j equal to 1 to n modulus of ai j and maximum of j summation i equal to 1 to n ai j.

So, this is some result which we generally use to look at the information about the Eigenvalue of a given matrix. And there is one more theorem, which says that let lambda 1 to lambda 1 be the Eigenvalues of a then summation I equal to 1 to n modulus of lambda I square less than or equal to Frobenius Norm of a A square.

Ah why we are considering all these, is that many a times finding the Eigenvalue is quite difficult and in many application we really need not to find out the exact value of the Eigenvalue, but rather than finding the say bound of the Eigenvalue is sufficient. So, these theorems, these results which we have just illustrated will help you, say giving qualitative information about the matrix A, but without calculating the Eigenvalue of the matrix A.

But now let us consider certain, basic results, basic method to find out all the Eigenvalues of the matrix A. So, this QR method is a very very basic method and it will help you to find out all the Eigenvalue of a given matrix A. And of course, this is a basic method, so it is not applicable for very large matrix or a very same large dimension sparse matrices, because when we calculate very large matrices and it is quite slow and sometimes your a sparsity may be lost. So, this QR basic method is applicable very well in small size matrices.

So, first let us consider what is QR method and how we can apply this. So, basically QR method applies the real Schur Form matrix form of a given matrix A. So, here we will use the theorem of a real Schur Form of a given matrix A and we consider QR sequence from a given matrix A.

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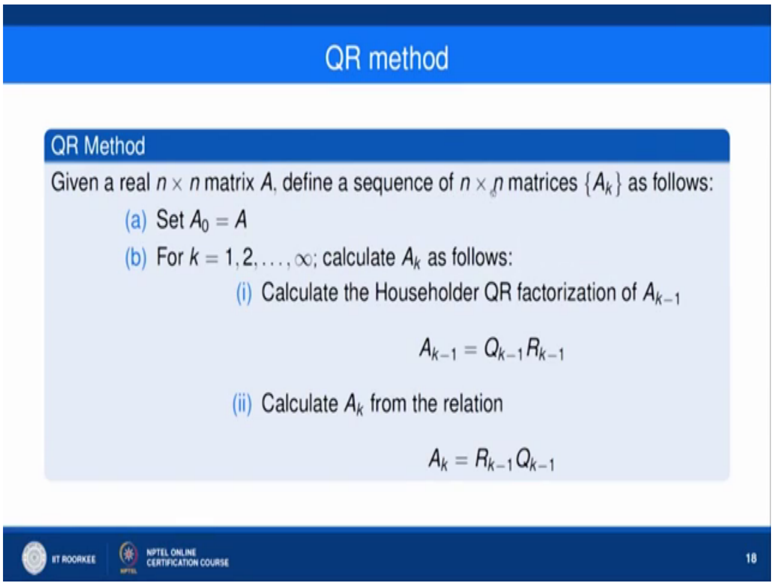

So, let us consider what is QR method. So, given a real n cross n matrix A define a sequence of n cross n matrices Ak as follows. So, first thing is that you define A naught as the given matrix A and then for k equal to 1 2 to infinity, we can calculate the A 1 A 2 in a form. So, once we have A naught we find out the QR decompo Q QR factorization of A naught.

And once we have Q naught R naught then we can define A 1 as R R naught Q naught or we can say that if for k equal to 4 any arbitrary k. We first calculate the householder QR factorization of Ak minus 1 and let us say that Ak minus 1 is Qk minus 1 into R k minus 1. Then we just reverse the order and we define Ak as R k minus 1 and Qk minus 1 and in this way we can find out the sequence of the matrices Ak.

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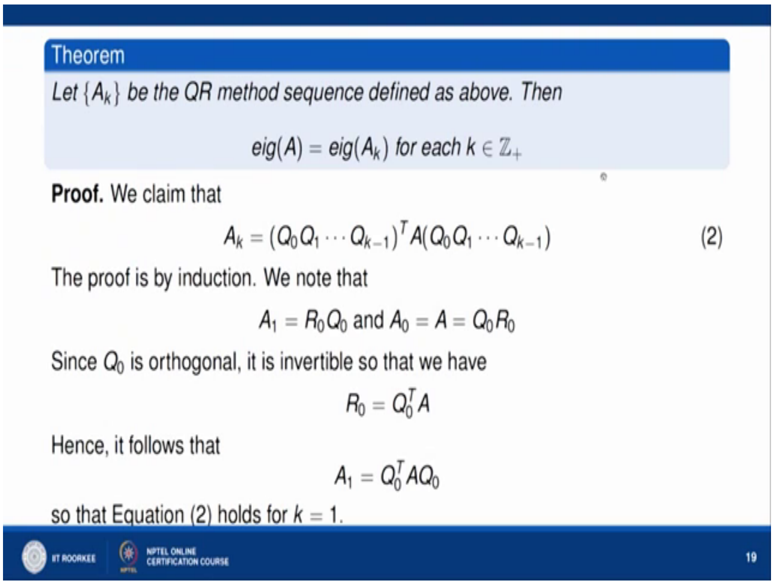

Now, once we have this sequence, then how it will help you to find out the Eigenvalue of a given matrices. So, here in for this, this theorem is quite important that let Ak be the eig Ak be the QR method sequence defined as above, then Eigenvalue of the matrix A is nothing, but Eigenvalue of Ak for any k in z plus. So, it means that if you have A 1 you find out the Eigenvalue and it is same as the Eigenvalue of the matrix a. So, spectrum of matrix a is same as the spectrum of Ak.

So, that let us see how it is true. We claim that Ak is nothing, but Q naught to Qk minus 1 transpose A Q naught Q 1 to Qk minus 1 we can. Once we show this thing then Q naught to Qk minus 1 or R or all orthogonal, then this is nothing, but some orthogonal matrix transpose 8 into orthogonal matrix, and it means that Ak and a are similar matrices and hence a spectrum of Ak and a spectrum of a is same.

So, first let us try to prove this the equation number 2. So, that we are going to prove by induction. So, for k equal to 1 it is quite obvious, because A 1 is basically R naught Q naught and A naught is Q naught R naught. So, using here you can find out the value R naught, R naught is basically Q naught transpose A and when you use the value of R naught in A 1 then A 1 can be written as Q naught transpose A Q naught. So, this two equation is true for k equal to 1 and let us assume for k equal to m.

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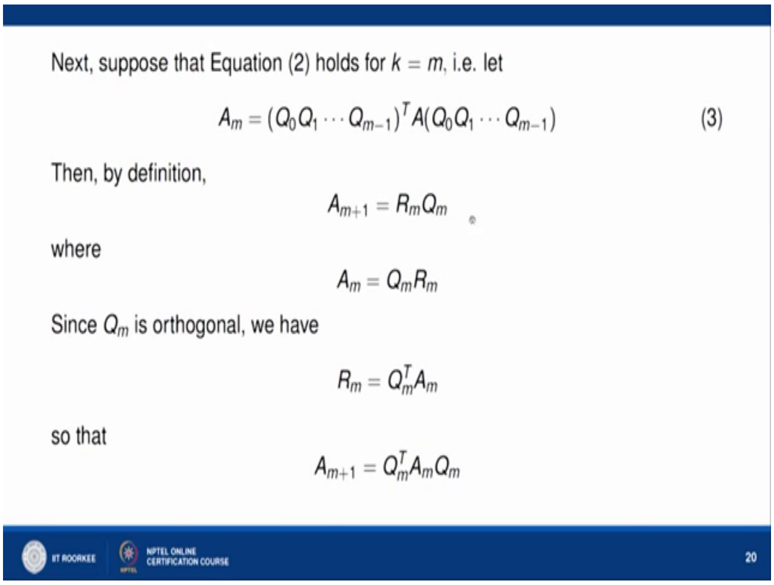

So, it means that a m can be written as Q naught to Q m minus 1 transpose a into Q naught to Q m minus 1. Then we want to show that it is also true for k equal to m plus 1. So, we already know that the Am plus 1 is defined as R m Q m and A m is given as Qm R m.

So, with the help of, this Am equal to Q m R m you can find out the value of R m, Rm is given as Qm transpose Am. So, using this Rm we can put it here and we can have A m plus 1 as Qm Qm transpose A m qm. Now we can find out the expression for this am which is given in 3.

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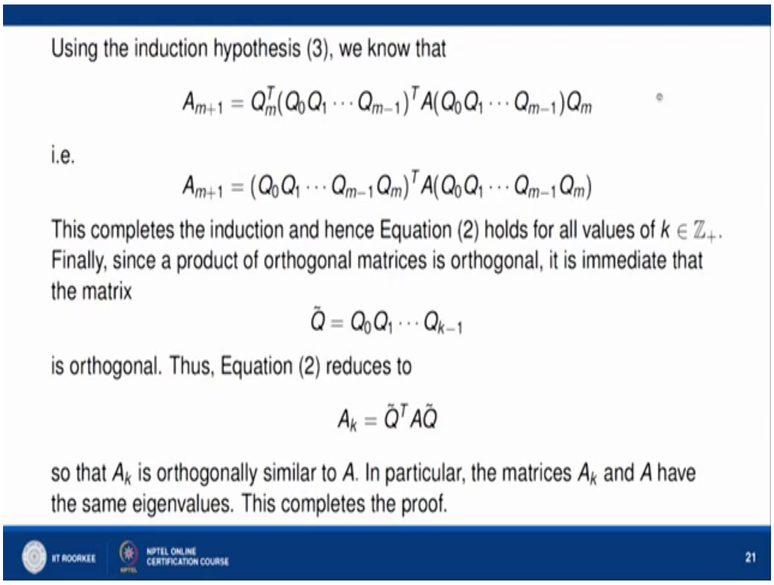

And we can write it here A m plus 1 as Qm transpose Q naught to Q m minus 1 transpose A into Q naught to Q m minus 1 into Qm. And if we combine this we have A m plus 1 as product of these orthogonal matricix matrices in transpose A into Q naught to Q m. So, it means that our relation A 2 is holds for all values of k.

Now, as we pointed out that all these Q naught to Qk minus 1 are all orthogonal matrices and product of orthogonal matrices is orthogonal. So, let us denote this Q naught to Qk minus 1 as Q tilde and we can write that Ak as Q tilde transpose A Q tilde. So, we can say that Ak is orthogonally similar to matrix a and we know that orthogonal matricy, matrices have the same Eigenvalues.

In fact, they have the same characteristic polynomial. So, they will have the same Eigenvalues and this complete the proof. So, once we have this QR sequence, then I am finding, the Eigenvalue of any particular element will give you the Eigenvalue of the original matrix A.

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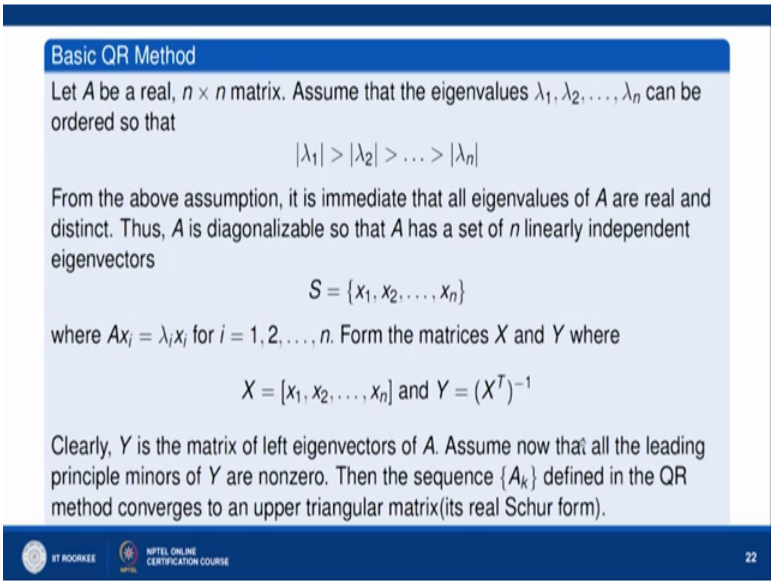

So, we need some kind of convergence criteria to work with this basic QR method. So, here is the result which give you the guarantee that, that this process will converge somewhere. So, let A be a real n cross n matrices, assume that the Eigen is lambda 1 to lambda n can be ordered in a way that modulus of lambda 1 is greater than modulus of lambda 2.

So, we have arranged in a decreasing order. And it is also clear from this arrangement that Eigenvalues of a are real and distinct and once we have Eigenvalue is real and distinct, we can show that the corresponding eigenvectors are also orthogonal to each other and we have n number of linearly independent eigenvectors or we can say that A is diagonalizable and we can write S as x 1 to x n, where each x is are linearly independent eigenvectors of the matrix A.

And from this set of eigenvectors we can form the capital X which is the matrix of eigenvectors and, we write Y as X transpose inverse and this Y is the matrix of left eigenvectors of A. And assume now that all the leading principle minus of Y are non0, this is one assumption which we are assuming. Then this sequence Ak defined in the QR method converges to an upper triangular matrix which we say that it is its real Schur Form.

So, it means that if we assume these condition, first condition is that it can be arranged in this way, another assumption is that all the leading principle minus of Y are non zero.

Then this sequence Ak will converge to the real Schur Form of the given matrix A. So, basically this is the starting point and we will use this point to find out the Eigenvalues of a given matrix A.

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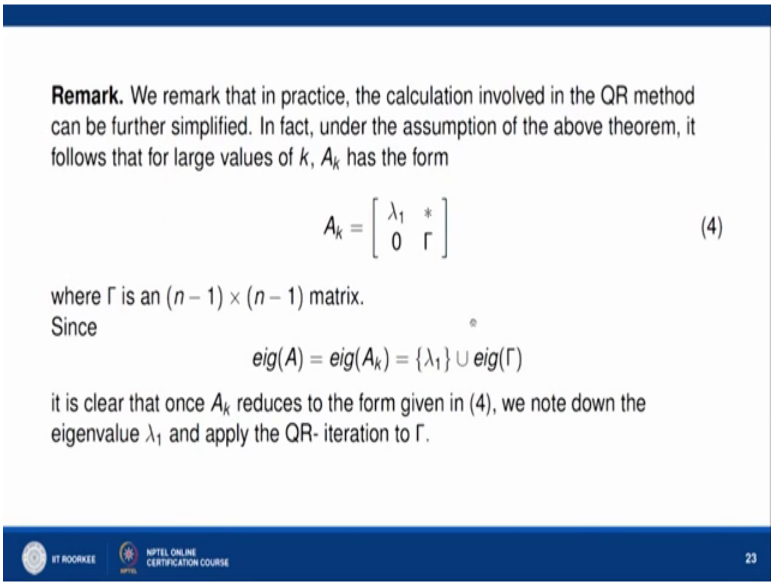

So, here we remarked that in practice the calculation involved in the QR method can be further simplified, and we can say that we when we write Ak as this lambda 1 block matrix, basically lambda 1 0 and something in gamma, where gamma is an n minus 1 cross n minus 1 matrix and we can say that Eigenvalues of Ak is nothing, but this lambda 1 union Eigenvalues of the gamma.

Now, so, now, it is reduced to. Now here this gamma is an n minus 1 cross n minus 1 matrix. So, now, we in n place of a now we start with this gamma matrix, and we say that it is clear that once Ak reduced to the form given in 4, we note down that the Eigenvalue lambda 1 and lamb apply the QR iteration for this matrix gamma and in this way we can calculate all the Eigenvalues of the matrix A.

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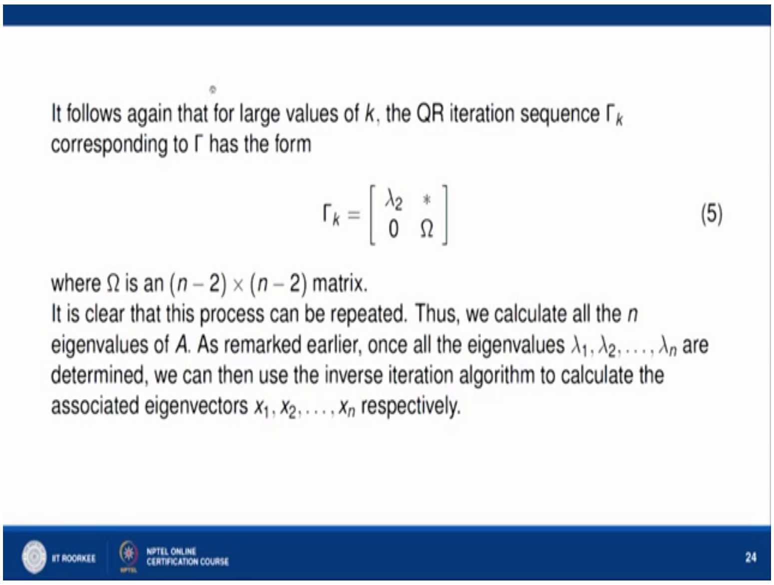

And it follows again that for large values of k, the QR iteration sequence for gamma k corresponding to gamma has this form, gamma k equal to lambda 2 0 and non0 some value and omega, where omega is any n minus 2 cross n minus 2 matrix and this process can be repeated. So, that we can find out all the Eigenvalues of the matrix A and.

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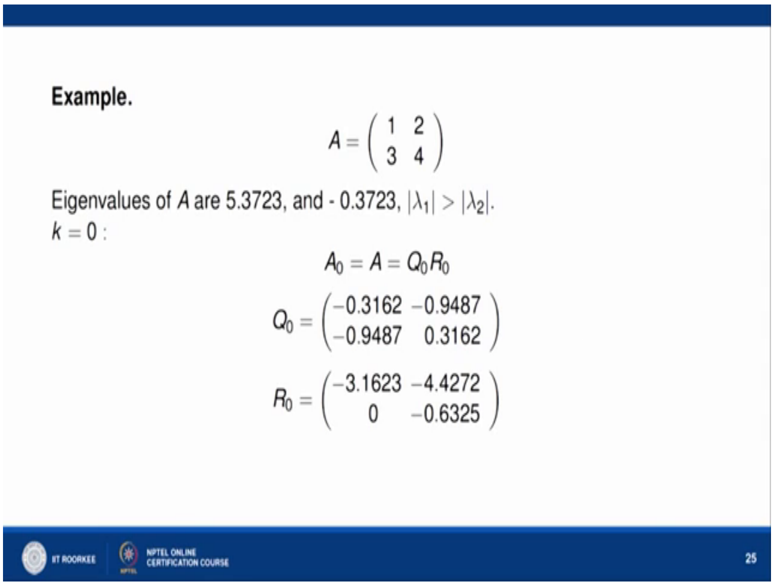

Once Eigenvalues are given to us then we can use reverse iteration method to find out eigenvectors corresponding the matrix A. Now, let us consider two example based on this basic QR method. So, first is 2 cross 2 matrix. So, A is 1 2 3 4 and we can easily calculate the Eigenvalues of this matrix A and it is given as 5.3723 and minus 0.3723. And here we can say that modulus of lambda 1 is greater than modulus of lambda 2. So, it is also satisfying the condition of given result.

So, now let us consider the QR sequence. So, A naught is a and we find out Q naught R naught. I hope that you remember the say command for this that is you can write this as this.

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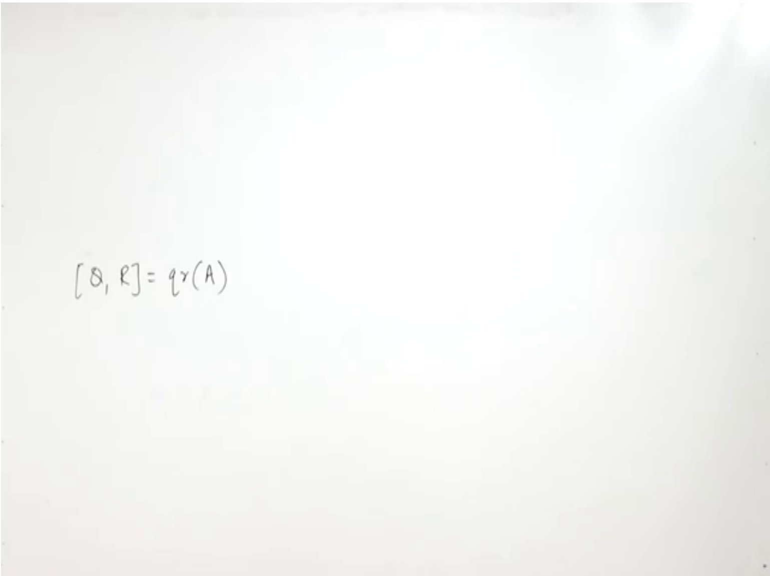

So, here when we write QR is equal to QR of the matrix a. So, that will give you the QR decomposition of a given matrix A. So, it means that given a matrix A you can easily find out Q naught R naught. So, Q naught and R naught is given by this.

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$k = 1:$

$$A_1 = R_0 Q_0 = \begin{pmatrix} 5.2 & 1.6 \\ 0.6 & -0.2 \end{pmatrix} = Q_1 R_1$$

$$Q_1 = \begin{pmatrix} -0.9934 & -0.1146 \\ -0.1146 & -0.9934 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} -5.2345 & -1.5665 \\ 0 & -0.3821 \end{pmatrix}$$

Then you can find out A 1 as R naught Q naught. So, A 1 is given as R naught Q naught. And once we have A 1 then we can find out the QR decomposition of A 1, which is given as Q 1 R 1 and then we can define A 2 as R 1 Q 1, which is given here A 2 as R 1 Q 1. So, once we have A 2, we again find out the QR decomposition and that is Q 2 R 2, where Q 2 is given by this and R 2 given by this.

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$k = 2:$

$$A_2 = R_1 Q_1 = \begin{pmatrix} 5.3796 & -0.9562 \\ 0.0438 & -0.3796 \end{pmatrix} = Q_2 R_2$$

$$Q_2 = \begin{pmatrix} -1 & -0.0082 \\ -0.0081 & 1 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} -5.3797 & 0.9593 \\ 0 & -0.3718 \end{pmatrix}$$

$k = 3:$

$$A_3 = R_2 Q_2 = \begin{pmatrix} 5.3718 & 1.0030 \\ 0.0030 & -0.3718 \end{pmatrix} = Q_3 R_3$$

$$Q_3 = \begin{pmatrix} -1 & -0.0006 \\ -0.0006 & 1 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} -5.3718 & -1.0028 \\ 0 & -0.3723 \end{pmatrix}$$

Then A 3 we define as R 2 Q 2 and it is given by this and once we have this, then again we find out QR decomposition of A 3, it is given as Q 3 R 3, where Q 3 is given by this and R 3 is given by this.

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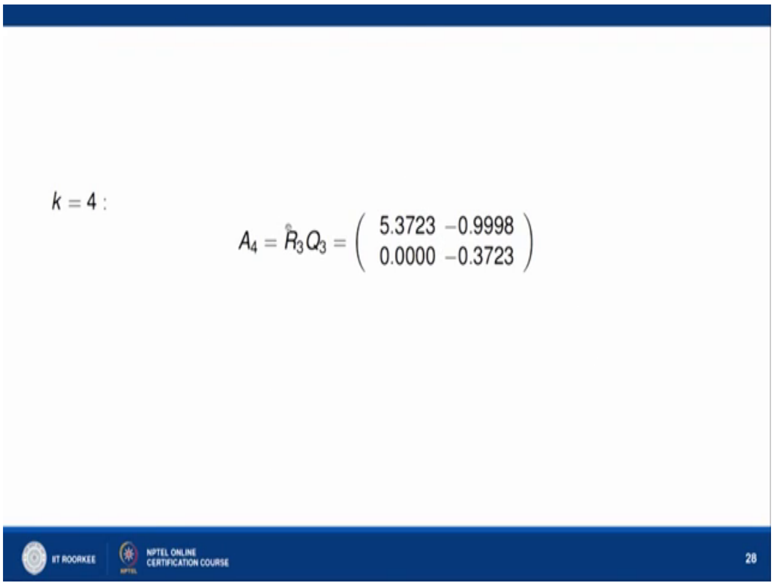

And then we define A for as R 3 Q 3, and if you look at R 3 Q 3 then it is of this form. Here we have a lambda 0 and whatever some value here and here it is your gamma. Now, since we started with the 2 cross 2 matrix and this gamma is nothing, but 1 cross 1 matrix. So, here we can say that this A 4, if you look at this is a upper triangular matrix and Eigenvalue of A 4 is 5.3723 and minus 0.3723.

So, Eigenvalue of A 4 we can easily calculate, and we already know that and that Ak and A will share the same characteristic Eigenvalue. So, the Eigenvalue of A 4 is the same as Eigenvalue of A which we started with here and that we are matching here that Eigenvalue of A is this and that we have achieved here and the fourth iteration. So, A 4 is an upper triangular matrix and we have the Eigenvalues listed here 5.3723 and minus 0.3723.

So, this is a the example on QR method sequence, how to find out, how to utilize this QR method sequence to find out the Eigenvalue of the matrix A. Let us consider one more example based on this.

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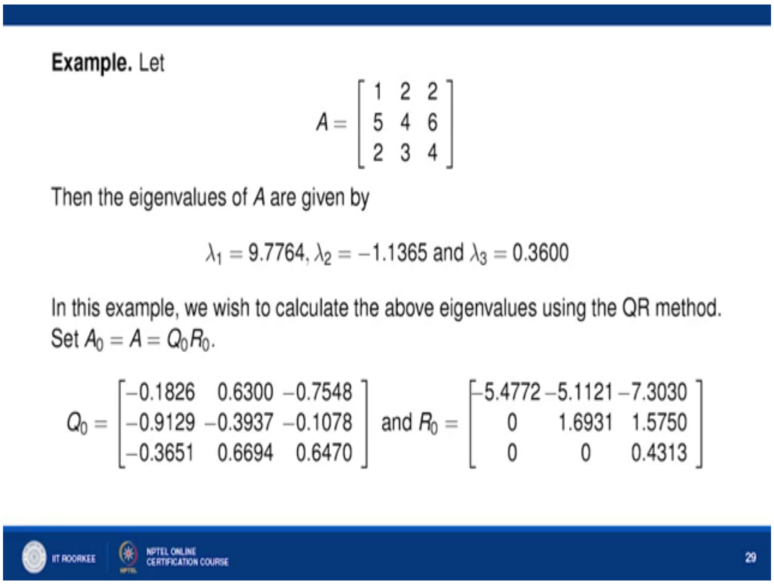

So, now let A is 122546 thru 234. So, this time it is 3 cross 3 matrix and we can find out the Eigenvalue of A as lambda 1 is a 9.7764 lambda 2 equal to minus 1.1365 and lambda 3 as 0.3600.

And now we try to find out eigenvalues using the QR method. So, first thing is A naught as A. Once we have A we can find out say QR decom decomposition using this and we can write Q naught R naught as this, Q naught is given by this matrix and R naught is given by this. Now just reverse the order and we can find out the A 1 matrix. So, A 1 is R naught Q naught.

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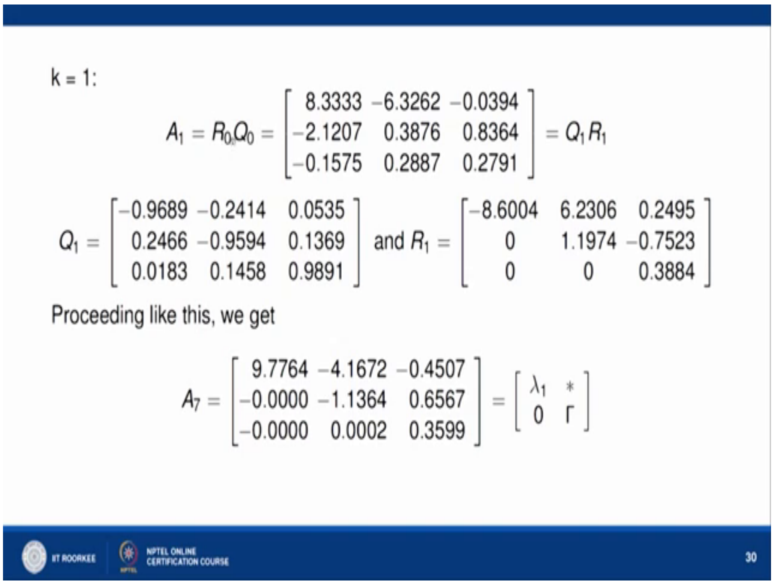

If you look A naught is Q naught R naught and A 1 is R naught Q naught. So, once we define A 1 then now repeat the same procedure, it means that now find out the QR decomposition of this A 1. So, let us call this as Q 1 R 1. So, once we have Q 1 R 1 that is the QR decomposition of the matrix A 1. So, Q 1 is given by this R 1 is given by this right. And again we reverse order and we define A 2 and in this way we can proceed, and we have seen that in a sen A 7 we have this form.

Now, in A 7 if you look at the first column have only non0 diagonal entry and rest are all 0. So, it is basically this form, lambda 1 equal to 0 and lambda 1 0 and some non0 and gamma. Here gamma is this matrix 2 cross 2 matrix. So, here we can see that the one Eigenvalue that is the largest Eigenvalue is given by this lambda 1; that is 9.7764 and once we have one Eigenvalue with us, now we start working with this gamma. So, it means that we have to start working with this gamma.

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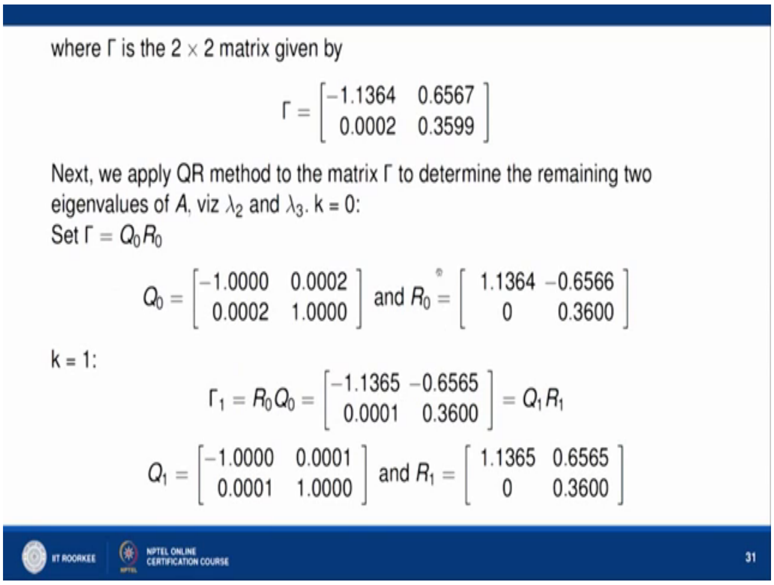

So, now let us gamma is 2 cross 2 matrix given by the. Now again we repeat the process for this gamma and. So, you assume gamma as Q naught R naught, we can find out the QR decomposition of v gamma and Q naught is given by this and R naught is given by this and we can define gamma 1 as just reverse order; that is R naught Q naught and it is given by this [vocalized-noise. Then find out the QR decomposition of gamma 1 that is Q 1 R 1 and Q 1 is given by this and R 1 is given by this.

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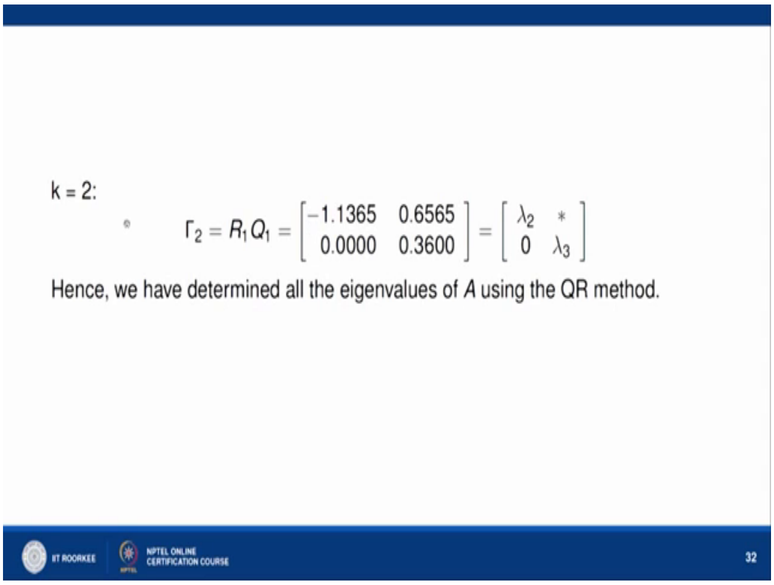

Then for k equal to 2 gamma 2 is equal to R 1 Q 1, and if you look at R 1 Q 1 that is gamma 2, it is basically in upper triangular matrix. So, when we have when we have upper triangular matrix we stop and we can see that the Eigenvalue of gamma 2 is minus 1.1365 and 0.3600 and we can say that we have determined all the Eigenvalues of A using the QR method.

So, with this we stop. In fact, in this lecture, we have discussed some what is the definition of Eigen pairs and some elementary properties of Eigen pairs I have, we have not discussed any proof of that that we can find out in any good book of linear algebra, and we also discussed the QR, basic QR method to find out the, all the Eigenvalue of the given matrix A. And when we do not require to find out all the Eigenvalues of a given matrix A then we may consider a power method and many more method which we can discuss in coming lectures. So, here we stop thank you for listening us.

Thank you.