Numerical Methods

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Lecture No 6

Introduction to Nonlinear Equations and Bisection Method

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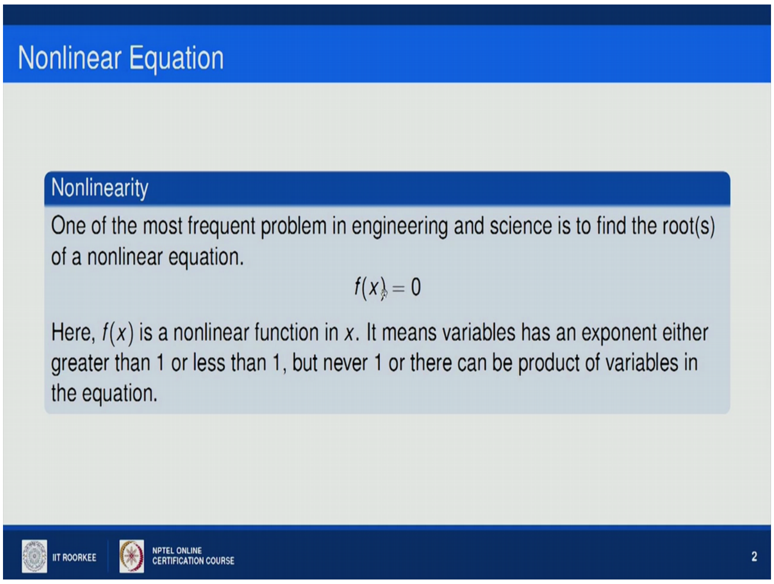

Hello everyone so today we are going to start a new unit in which we will learn few Numerical methods to solve nonlinear equations, so 1st of all what is a nonlinear equation? One of the most frequent problems in real life in engineering is to find the roots of unknown linear equation fx equal is to 0. The equation is nonlinear due to the unknown non-linearity of the function f. Here a function is said to be nonlinear if the variables or variable has an exponent either greater than 1 or less than 1, but never one or if you are having an equation having more than one variable you are having the cross product terms of the variable in the equation.

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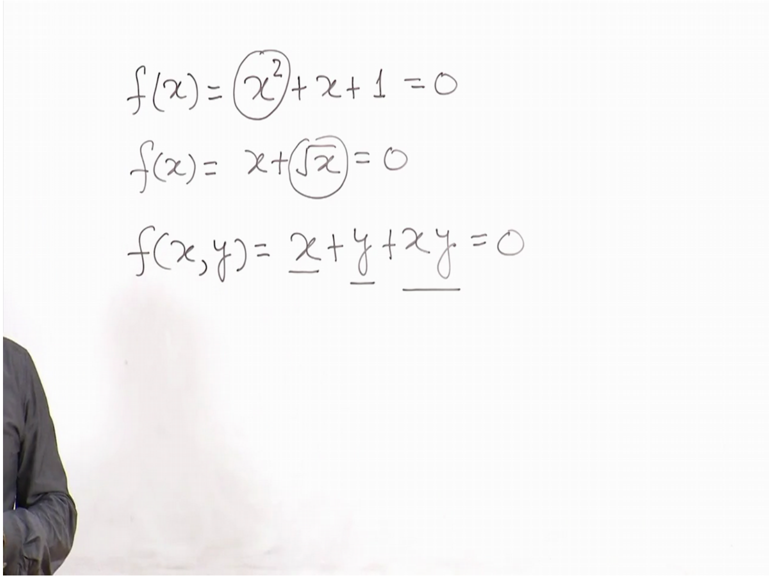

If we take some example of nonlinear equation or nonlinear function here x square plus x plus 1 equals to 0, so here this equation is an nonlinear equation and it is because this particular term is having exponent as 2. Similarly if I am having some other functions let us say x plus root x and again it is a nonlinear equation because here the exponent is 1 by 2 of this particular x in this particular term, so in a nonlinear equation if any of the term is having exponent less than 1 or greater than 1 but not exactly 1 then the equation is called nonlinear equation.

If you are having an equation is in more than one variables let us say f is a function of x and y, where x and y are independent variables, so if I am having this function as xy equals to 0, so here you can see this term is linear, this term is linear but here we are having separately x and y having exponent one but with a dear product this particular function becomes nonlinear, so in next few lectures we will learn how to solve these nonlinear equations. Now what we mean by the solution of nonlinear equations?

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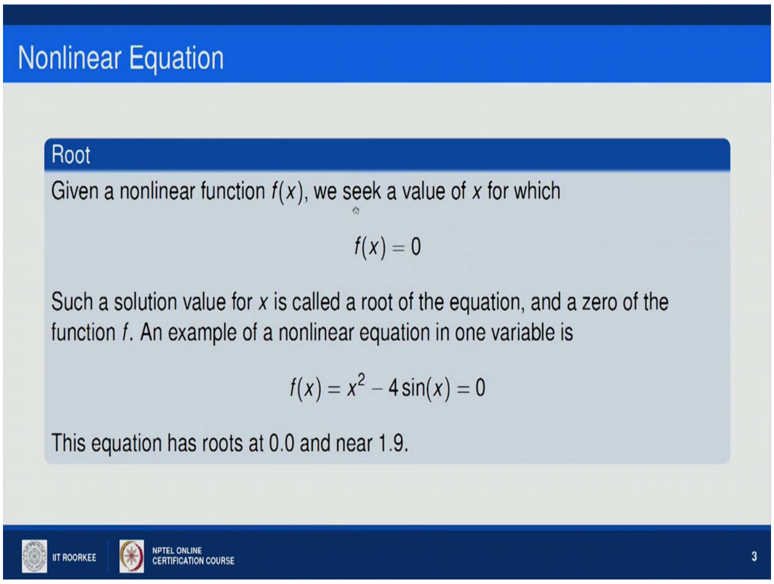

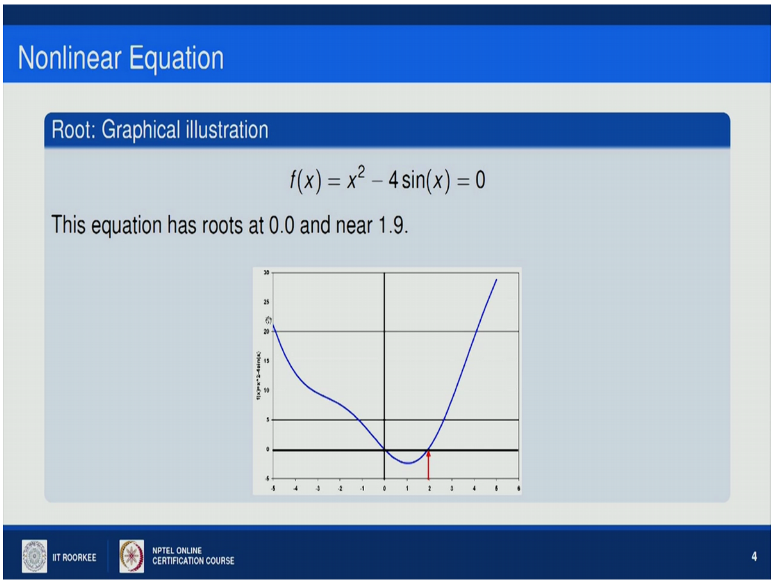

So given a nonlinear function f we seek of value of x for which f is 0. Such a solution value for x is called a root of the equation and a 0 of the function f. An example of a nonlinear equation in one variable if you take f equal to x square minus 4 sin x equals to 0 and if I see the graft of this function, it can be clearly seen that the function is 0 when x is 0 and another point when x is near to 1.9, so hence this particular equation is having 2 roots 1 is 0 another one is just near to 1.9. So graphically we can say the solution of a nonlinear equation means we have 2 find the curve of nonlinear function and we need to see where this particular curve is touching or intersecting x axis.

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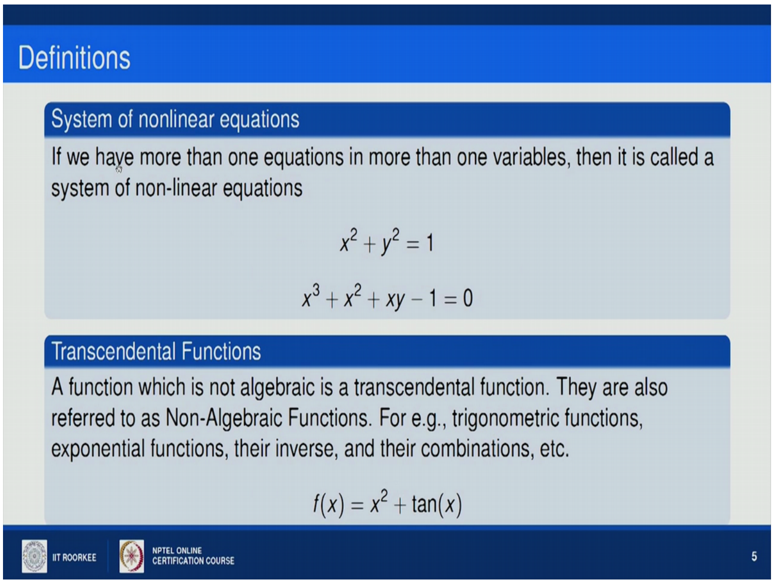

Like we had the system of linear equations in the previous unit in this unit also we can have system of non-linear equations. If we have more than one equation in more than one variables then it is called a system of non-linear equations. For example if you take these 2 equations that is 1st equation is x square plus y square is equal to 1 and the 2nd equation is x cube plus x square plus xy minus 1 equals to 0. So here we are having to equations in 2 unknown and both the equations are non-linear in x as well as in y.

So we will learnt in the last lecture this unit that how to solve such a system using some numerical technique. As you see in all these examples which I have given to you are basically algebraic functions. Algebraic function means they are the polynomials in x or in y. Apart from that you are having another type of nonlinear equation or nonlinear function those are called transcendental function, so a function which is not algebraic is a transcendental function. They are also referred to as non-algebraic function or example if you are having trigonometric term in your function, exponential term, logarithmic term or their combinations. For example if you see this equation fx equal to x square plus tan x. Here it is a trigonometric term in x and hence it is a transcendental function. So we will learn how to solve or how to find out zeros of such transcendental functions using numerical techniques.

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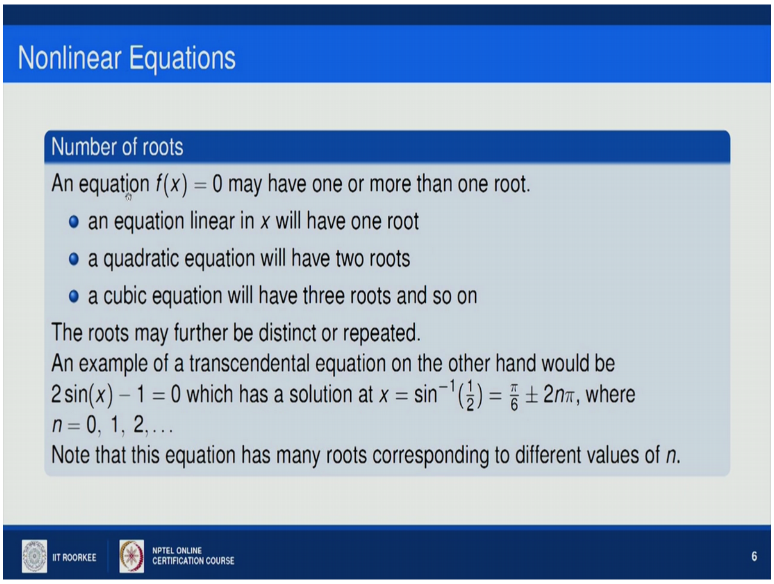

Now given a nonlinear equation we can have multiple roots. For example if you take a nonlinear equation fx equal is to 0. If it is a linear equation then it will be having one root but if it is a quadratic equation means it is a second degree polynomial in x, it will have 2 roots. If it is a cubic polynomial in x then the equation will be having 3 roots and so on. If the roots may further be distinct or repeated for example if you take a simple function x square minus 4 equals to 0.

So basically this quadratic polynomial is having 2 roots x equals 2 plus 2 and minus 2 and both the roots are distinct but or on the same time if you take another function let us say x square plus 4 minus 4x basically x minus 2 whole square then the roots are 2 and 2 and hence the 2 roots but both are same, so roots are repeated twice. In case of transcendental equation if we consider an example like twice of sin x minus 1 equals to 0 then if we find a solution of this it will be something like x equals to sin inverse 1 by 2 and it is coming out pi upon 6 plus minus twice n pi where n equals to 0, 1, 2. So you can see that this equation as many roots corresponding to different values of n.

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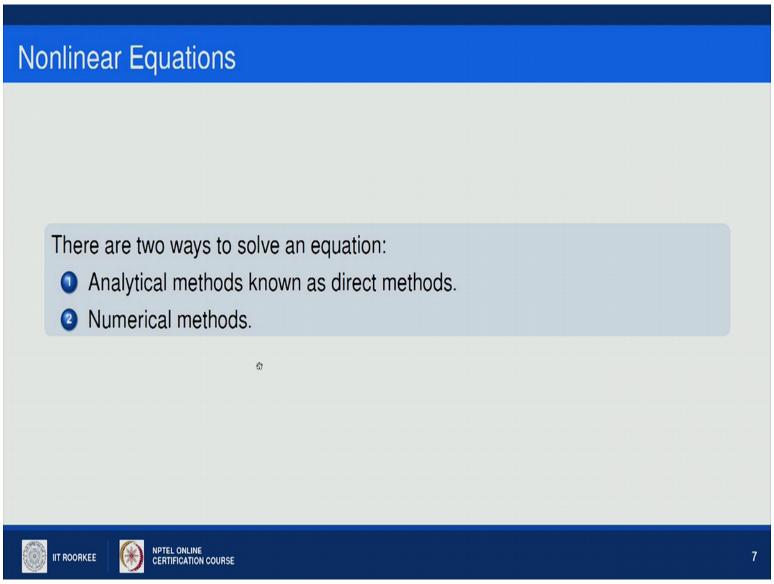

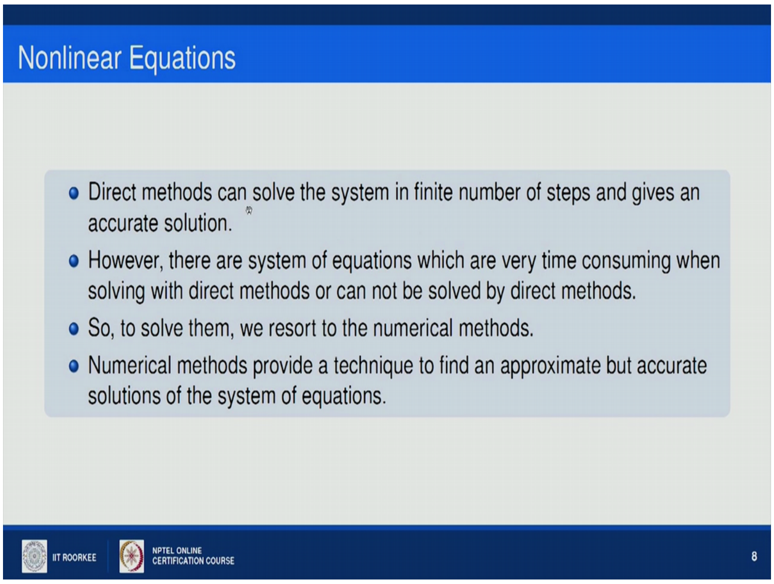

So using numerical techniques we can find single root given in a particular interval or more than one roots. For this we are having 2 types of techniques as I told you one is analytical methods or direct methods like in case of linear equations and the other one is numerical methods. So direct method can solve system in finite number of steps and gives an accurate solution. However, there are systems of equations which are very time consuming when solving with direct method or there are number of equations which you cannot solve using the direct method hence we need to rely on the numerical methods to find out a solution of such equation those are not solvable using direct method, so numerical methods provides a technique to find an approximate but accurate solution to the system of equations.

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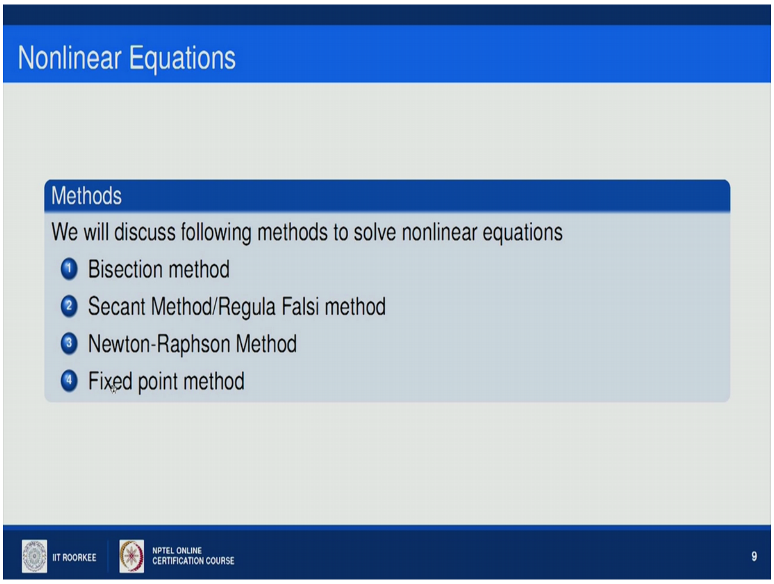

So in the next few lectures we are going to discuss following numerical techniques like bisection method which we will cover in this lecture itself then in the next lecture we will take Secant method and regular Falsi method, then in the 3rd lecture of this unit we will go for Newton-Raphson method, in the 4th lecture of this unit we will discuss about fixed point method and in the last lecture of this unit will learn how can be solved system of non-linear equation using these techniques. So what we mean by numerical solution of a nonlinear equation.

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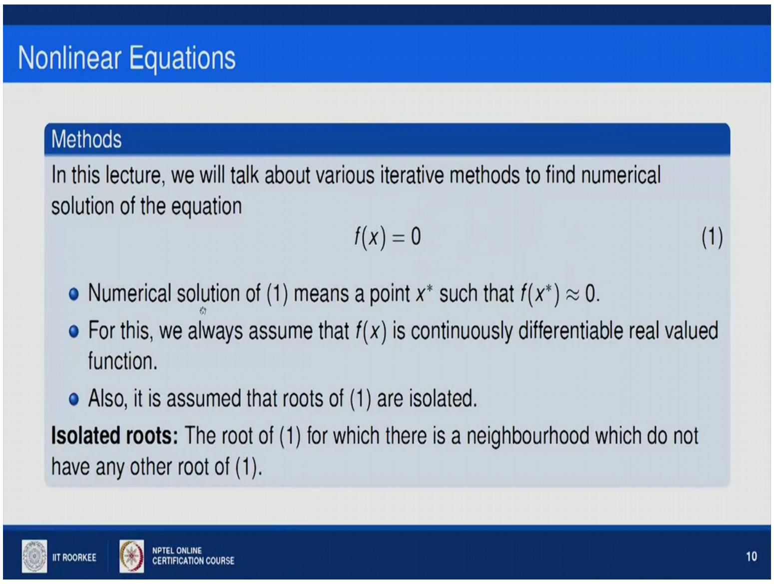

So given a nonlinear equation fx equal is to 0 numerical solution means a point we need to find out an approximate x star such that f x star is approximately 0. For this we always assume that fx continuously differentiable and real valued function also it is assumed that root of this particular equation fx equal is to 0 are isolated. What we mean by isolated root? The root of fx equal to 0 for which there is a neighborhood which do not have any other root of 1 is said to be isolated root for example if you take fx equals to x square minus 4 equals to 0 again, so here we are having 2 roots one is x equals to minus 2 another one is x equals to 2, so both the roots are isolated because in the neighborhood of 2 as well as neighborhood of minus 2 we do not have any other root.

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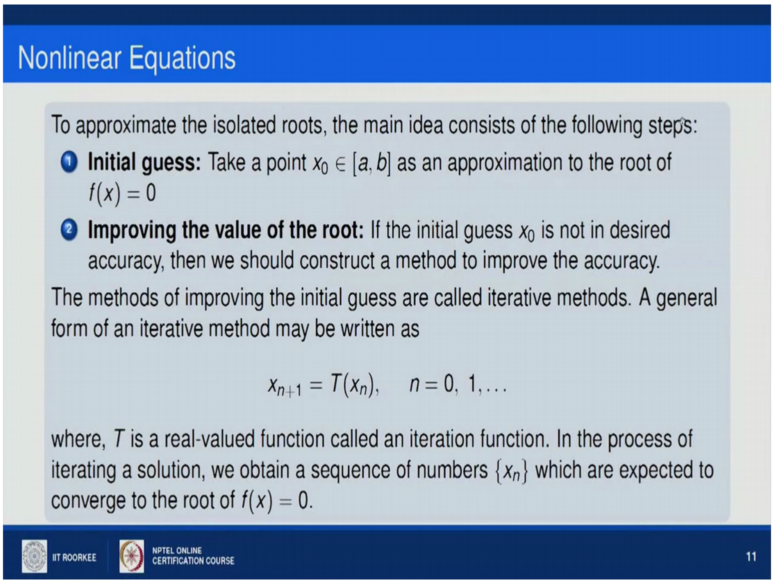

To find out such isolated roots the main idea consist of the following steps in a numerical technique first of all we need an initial guess that is take a point x naught belong to a close interval a and b as an approximation to the root of fx equals to 0 and then we need to improve this initial root or initial solution by using an iterative equation that is x of n plus 1 equals to T of x n, and equals to 0, 1, 2 and so on. Here T is a real valued function called an iteration function. In the process of iterating a solution we octane a sequence of numbers x n which are expected to converge to the root of fx equals to 0. Like here we will start with x naught then using this particular equation, iterative equation we will find out x1 then using x1 here we will find x2 and so on.

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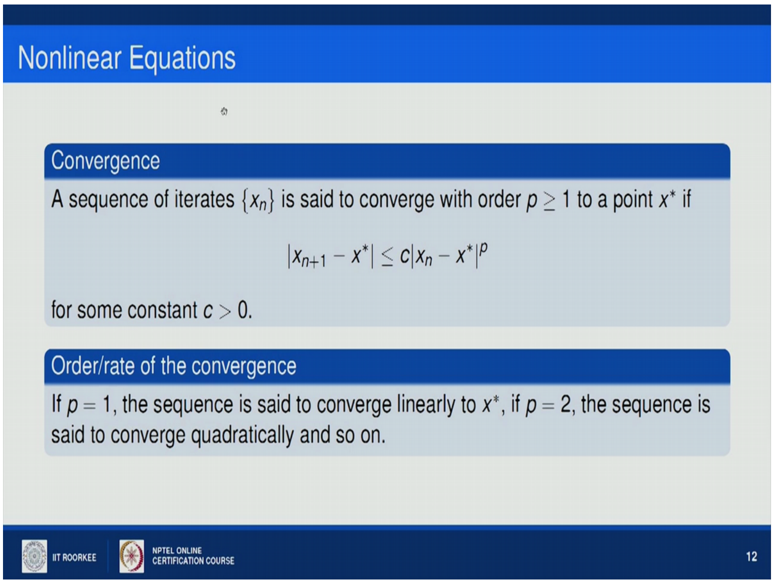

Now in the previous I told you that we obtain a sequence of numbers x n which are expected to converge to the root of fx equals to 0. Now what we mean by convergence of such an iterative scheme, a sequence of iterates x n is set to converge with order p and p will be always greater than equals to 1 to a point x star if x n plus 1 that is the approximate in n plus 1 iteration minus x start that is the approximate solution is less than equals to c, x n minus x star raise to power p for some constant c greater than 0. So if p equals to 1, the sequences said to converge linearly, if p equals to 2 we will say that the sequences converging quadratically and so on.

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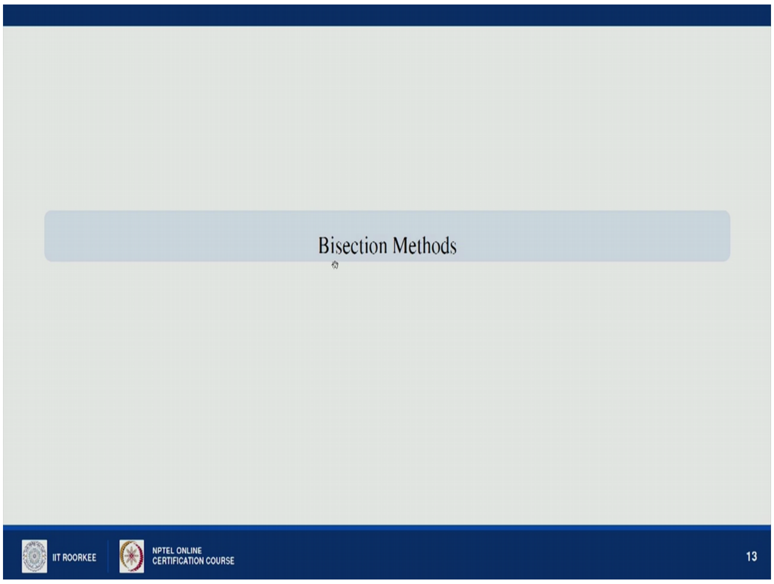

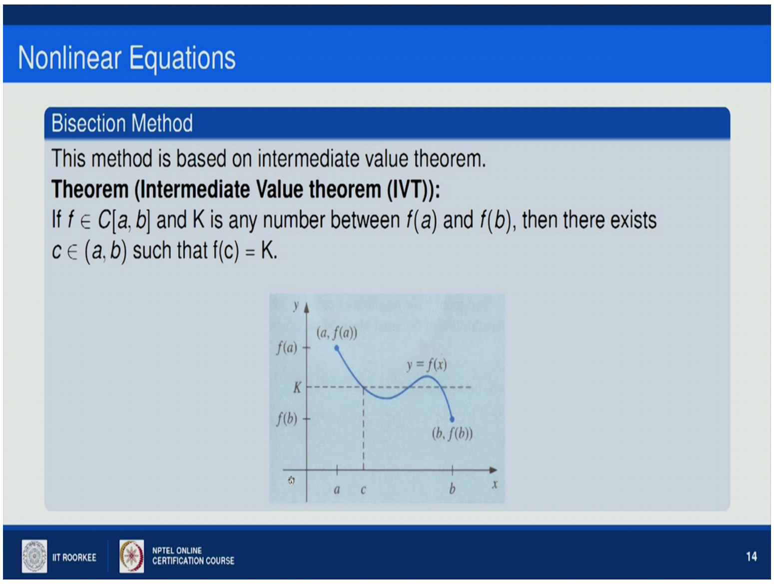

So the 1st numerical technique which we are going to discuss is Bisection method. The method is based on intermediate Value Theorem, so what this theorem tells us that if f is continuous in a close interval a to b and K is any number between fa and fb then there exists c belongs to open interval ab such that fc is equal to K. So if you see this graphics here we are having x on the horizontal axis fx on the vertical axis and this blue curve is showing the graph of fx between a and b. Now if I take a number K between fa and fb, so there will be some number c such that image of c under this function f equals to K.

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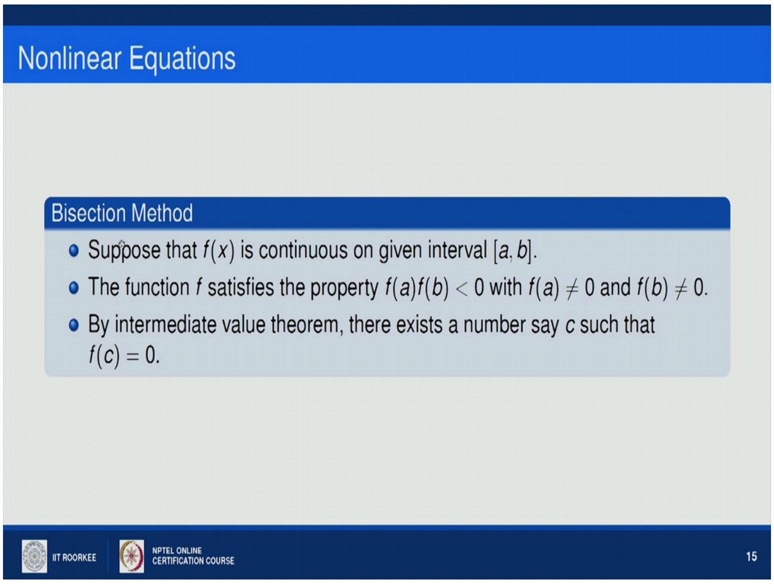

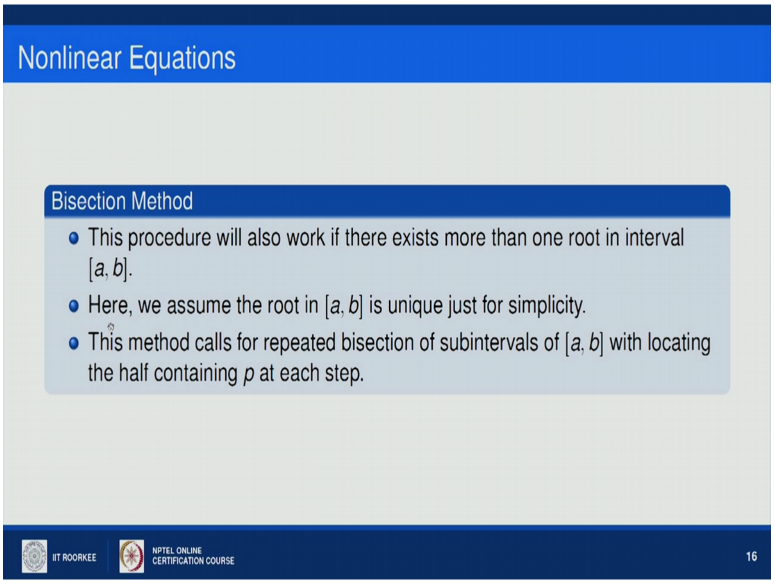

It means if we are having a function fx which is continuous on giving interval ab the function f satisfies the property fa into fb is than 0 with fa not equals to 0 and fb not equals to 0 that a and b are not root of a then by intermediate value theorem, we can see there exists as c or a root between a and b such that fc equals to 0. This procedure will also work if there exist more than one root in the interval ab. Here we assume that root in ab is unique just for simplicity. This method calls for repeated bisection of subintervals of ab with locating the half containing p at each step.

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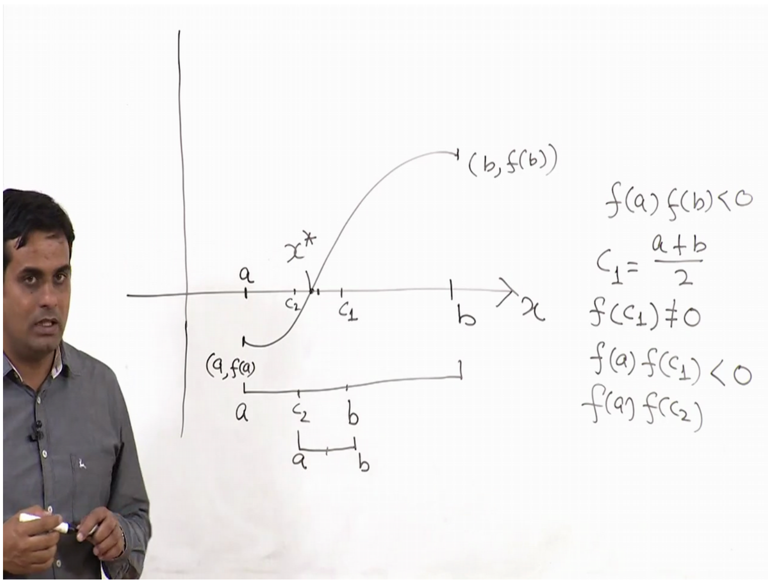

So geometrically how this method will work? Let me explain here. Suppose we are having a function like this, so this is the point a, this is point b. Now as you can see this function is having a root at this particular point, so what I will see that this point is a fa and this point is b fb. So what you can see fa is a negative value while fb is a positive value, so it means fa into fb means there is a root between a and b. Now what I will do? I will calculate c 1 which will be midpoint of a and b, so if I take the midpoint this interval a and b midpoint will be somewhere here, so let us say this is my c 1.

Now if I check f of c 1 and if it is equals to 0 I will say that c 1 is a root of the equation, if it is not 0 or I will check? I will check fa into fc 1. So product of these 2 if this product is negative the root will lie between a and c 1. If it is positive the root will lie in the right half interval that is c 1 to b, like in this example root is here, so what I will do? I will update this as a, and again this will become b, so what will happen in the next iteration where the length of the interval was b minus a in the 1st iteration it reduced to b minus a upon 2. Now again I will find out the midpoint of these 2, so let us say this one c 2, so c 2 will be somewhere here and again have the check fa into fc 2 which is coming positive.

Here it means root is in the right half interval that is in this interval, so what I will do? I will name it as a and this as b, so it means and then I will find out midpoint of this which will be somewhere here and continuing this process my method will converge to a root which is given here as b let us say or x star. So this is the geometric explanation of the bisection method in each iteration we reduce the half of our search intervals and then we will find out root by repeating this half and half and half like this.

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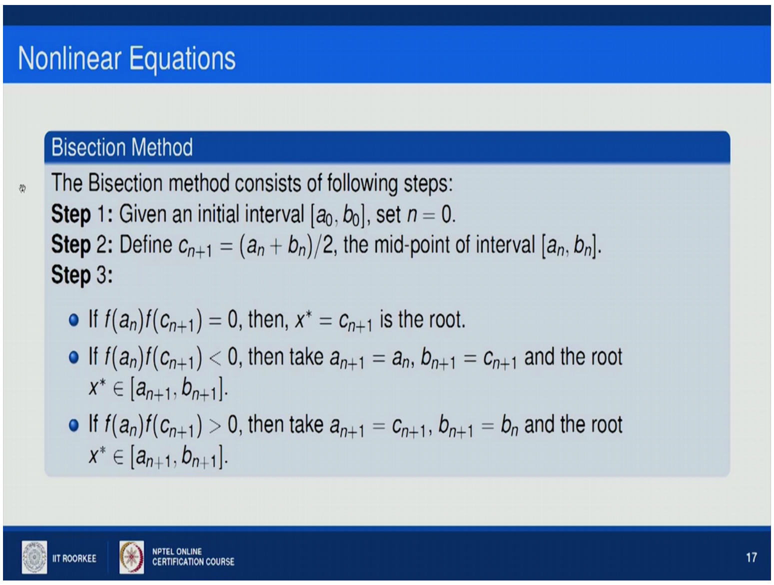

So the algorithm for bisection method can be described in the following steps, the 1st step is given an initial interval a naught b naught set n equals to 0. In step 2 define c n plus 1 equals to a n plus b n upon 2 like you we are finding the midpoint of the original interval if f of a n into f of c n plus 1 equals to 0 then the root is c n plus 1 if it is the product of f of a n with f of c n plus 1 is negative then we take root as means root will lie in the interval a n plus 1 and b n plus 1 where n plus 1 is a n and b n plus 1 is c n plus 1. Similarly if this product is positive then take a n plus 1 equals to c n plus 1 and b n plus 1 equals to b n means the root will lie in the 1st half of the interval and so on.

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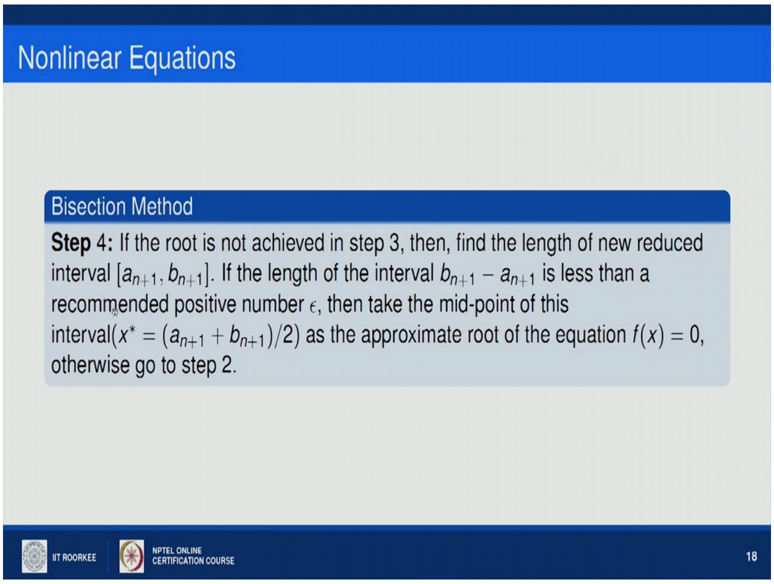

Step 4 if the root is not achieve in step 3 then find the length of new reduce interval and plus 1 b n plus 1. If the length of the interval b a plus 1 minus n plus 1 less than a threshold epsilon then take the midpoint of this interval as the root otherwise go to step 2 for the next iterations.

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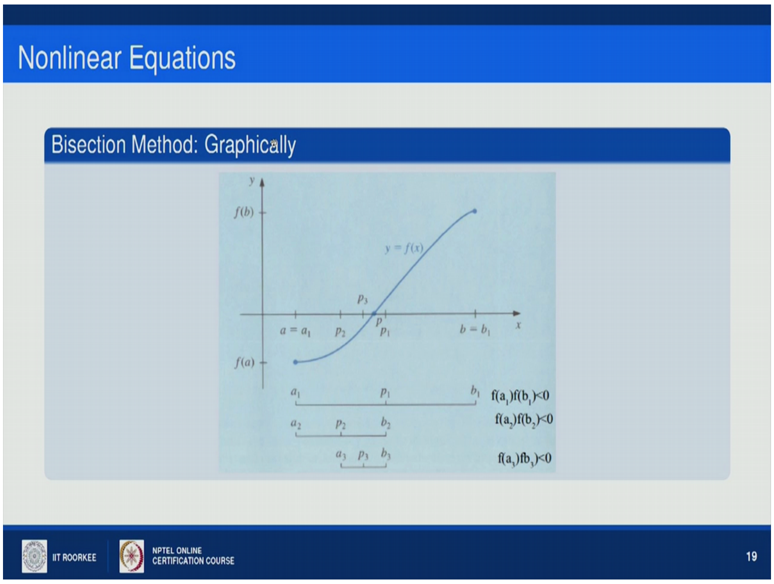

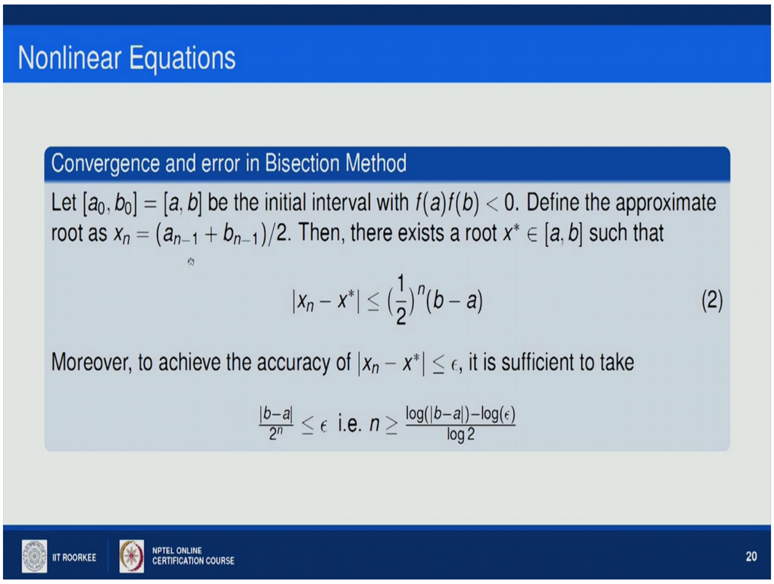

So this is the same which I have described on the board graphic really how bisection method works. Now if you talk about the convergence of this particular method let a naught b naught equals to a b be the initial interval with fa into fb less than 0 that is negative. Define the approximate root as x n which is the midpoint of a n minus 1 plus b n minus 1 upon 2 then there exist a root x star a b such that x and minus x star less than equals to 1 by 2 raise to the power n into b minus a where b a to b is the original interval.

Moreover to achieve the accuracy it is given that the accuracy should be of order of epsilon where epsilon is a small positive number. It is sufficient to take b minus a raise to power b minus a upon 2 raise to power n less than equal to epsilon because we need this number less than equals to epsilon for getting the desired accuracy in our numerical solution, so it is less than equal to epsilon. It means if we take the logarithm I can write it n should be greater than equal to log b minus a minus log epsilon upon log 2. So this particular equation tells us about the number of iterations required to get a given accuracy epsilon.

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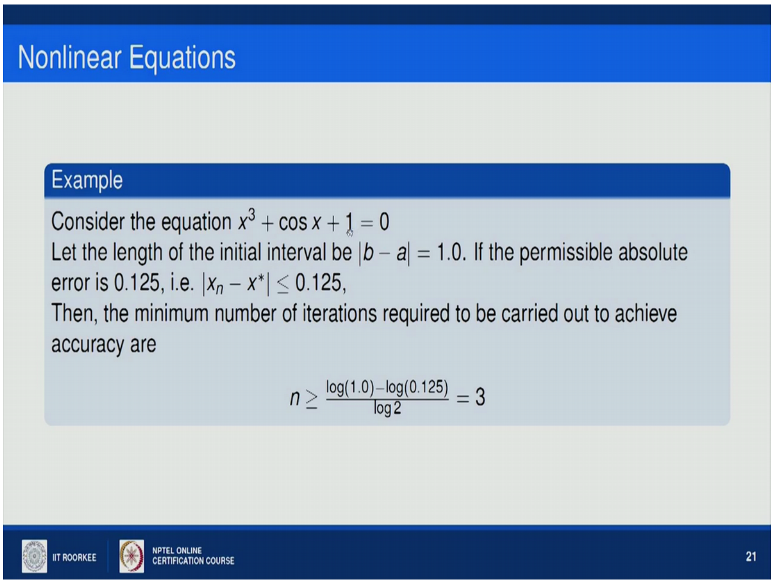

For example consider this function x cube plus cos x plus 1 and this equation equals to 0. Let the length of the initial interval is 1 that is b minus a is 1. If the permissible absolute error is 0.125 that is x n minus x star less than equals to 0.125. Then minimum number of iterations required to be carried out using the earlier formula so it will become n greater than equals to log of be minus a minus log of epsilon upon log 2, so log 1 minus log 0.125 upon log 2 and it is 3. So you need at least 3 iterations to get the accuracy of the order 0.125.

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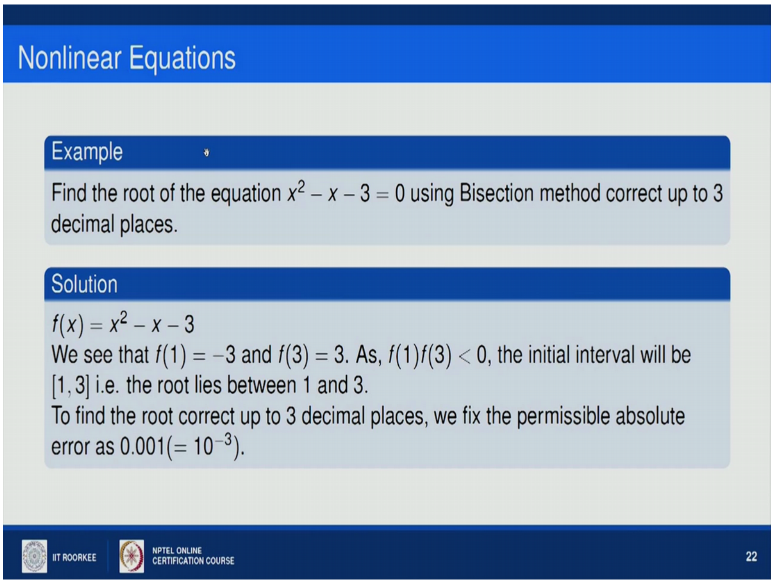

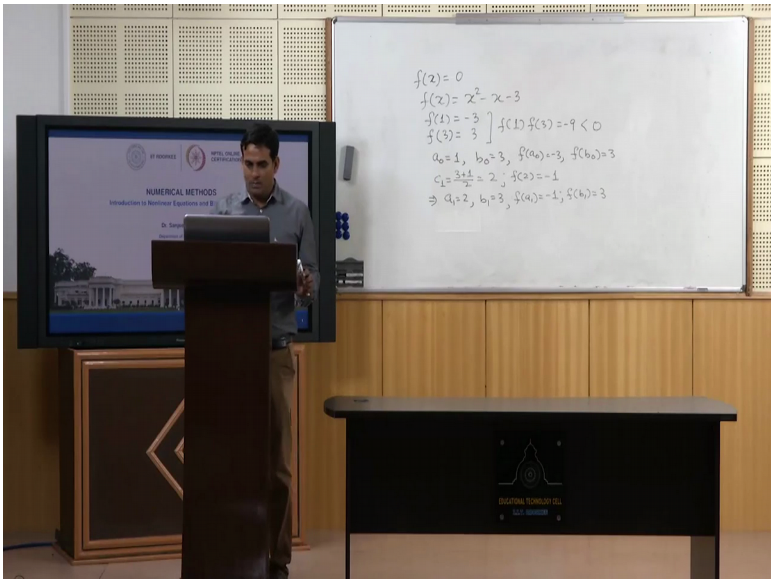

Now we will take an example and we will see how we can solve this particular example using the bisection method, so example is an nonlinear equation and we will follow the steps of bisection method to get the solution, so let I need an nonlinear equation f x equal to 0 where f x is given by x square minus x minus 3. Now I need to find out root of this, so if I check f1, f1 comes out to be 1 minus 1 minus 3 as minus 3 and if I check f2, f2 will be negative if I check f3, f3 will become positive that is 3 into 3 9 minus 3 minus 3 so it is positive.

As we can see that product of f1 into f3 is minus 9 which is a negative number it means the root of this particular equation fx equal is to 0 where fx is given by this function is lie between one and 3, so what I will do in the 1st iteration I will set a 0 equals to 1, b 0 as 3, f of a 0 is minus 3, f of b 0 as 3. Now I will find c 1, c 1 comes out to be the middle point of a naught and b naught so it will 3 plus 1 upon 2 so which is 2, now I will check f 2, f 2 is 4 minus 2 minus 3, so it is minus 1, so check here if I take the product of f 1 into f 2 it is coming out a positive number hence the root lies between 2 to 3 not from 1 to 2. So it means my a 1 becomes 2, b 1 becomes 3, fa 1 is minus 1, f of b 1is 3.

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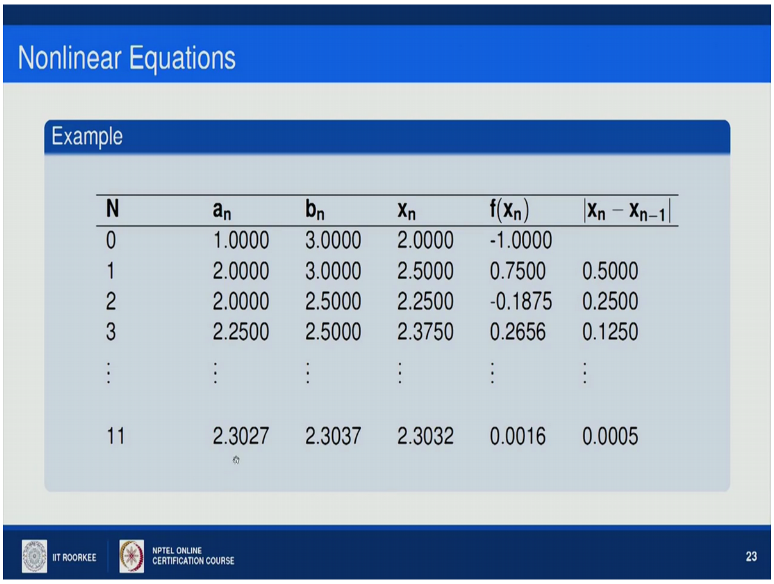
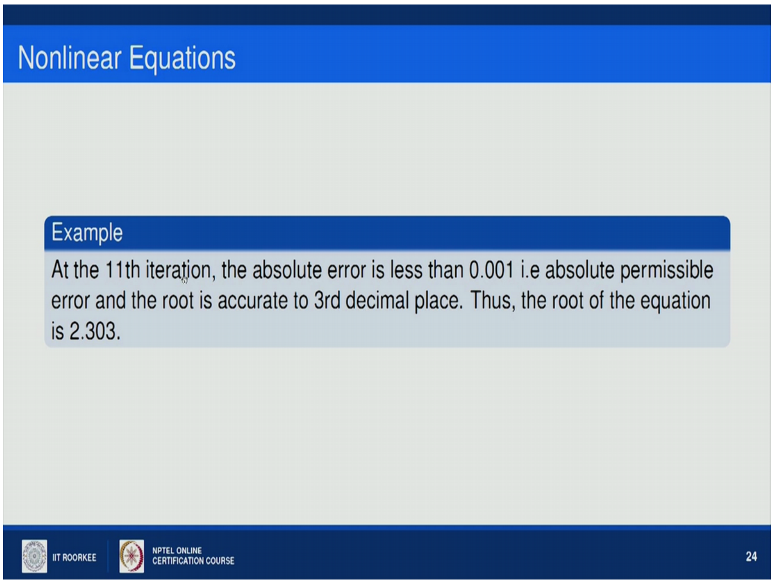


N	a_n	b_n	x_n	$f(x_n)$	$\lvert x_n - x_{n-1} \rvert$
0	1.0000	3.0000	2.0000	-1.0000	
1	2.0000	3.0000	2.5000	0.7500	0.5000
2	2.0000	2.5000	2.2500	-0.1875	0.2500
3	2.2500	2.5000	2.3750	0.2656	0.1250
⋮	⋮	⋮	⋮	⋮	⋮
11	2.3027	2.3037	2.3032	0.0016	0.0005



So if I need to find out n accuracy up to of the order 10 raise to power minus 3 then the iteration will go like this in the 11^{th} iterations I will get a n as 2.3027, b n as 2.3037, x n is 2.3032 and the value of fx at this particular number is 0.0016. If I check this x n minus x n minus 1 when the difference between 2 consecutive iterations it is coming out to be 0.0005 and hence in the 11 iteration the absolute error is less than 0.001 which is permissible error and the root is accurate to 3^{rd} decimal place. Thus the root of the equation is 2.303. We are having some advantage when we are using the bisection method as well as some disadvantage.

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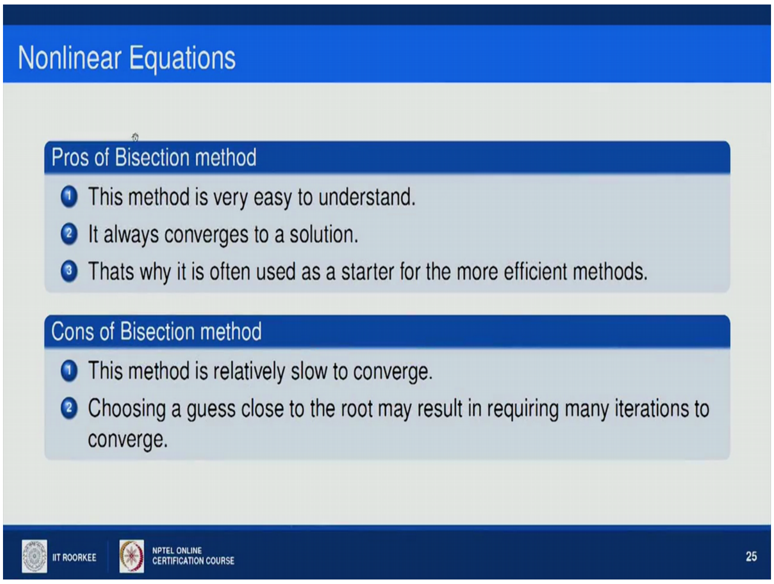

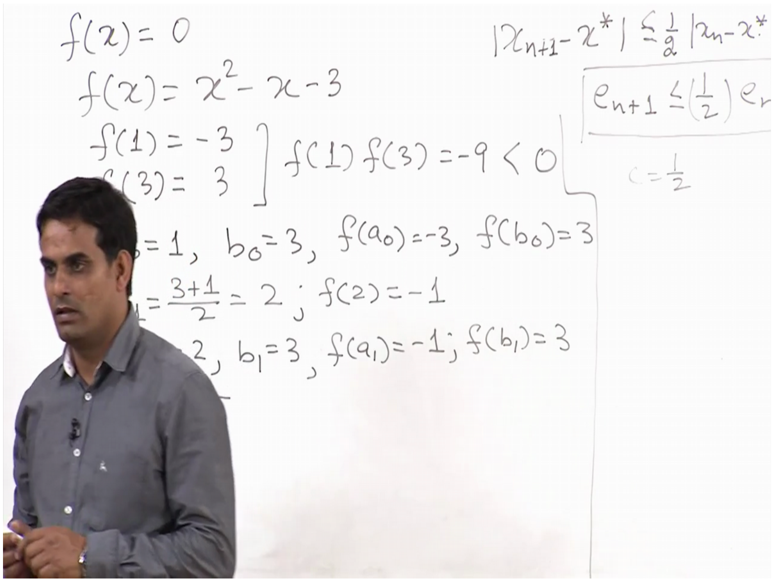

Advantage like this method is very easy to understand. It always converges to a solution. That is why it is often used as a starter for other more efficient numerical techniques. The disadvantages are this method is relatively slow to converge actually you can see if I am having x n plus 1 minus x star, so that is the error in nth iteration. Hence e n plus 1 less than equals to 1 by 2 e n, so if I check with the convergence formula you can see if I take c equals to half, so this method is having exactly linear convergence and hence this method is quite a lot to converge, moreover if you choose a guess close to the root may result in requiring many iterations to converge, so in this lecture we have learn about bisection method. In the next lecture we will learn to other methods those are quite close to bisection method in the same category iteration updates in the same manner however they are having better convergence when compared to the bisection method. Thank you.