Numerical Methods
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Lecture No 4
Jacobi and Gauss Seidel Methods

Hello everyone in this lecture I am going to introduce a new way of solving linear systems that is called iterative methods, so in the past couple of lectures I have introduced the direct methods those includes Gaussian elimination, Gaussian elimination with partial pivoting and then LU decomposition. Today iterative methods are quite beneficial in terms of numerical computation when you are going to solve the linear system of equations.

In direct method example in Gaussian elimination we need to do order of n cube operations for solving a n by n linear system, so it will be something order of n cube. However the method which am going to introduce today Jacobi and Gauss Seidel we need to do only order of any square operations in each iteration, so if number of iterations are less that is less than n then these methods are quite beneficial when compared to the direct method.

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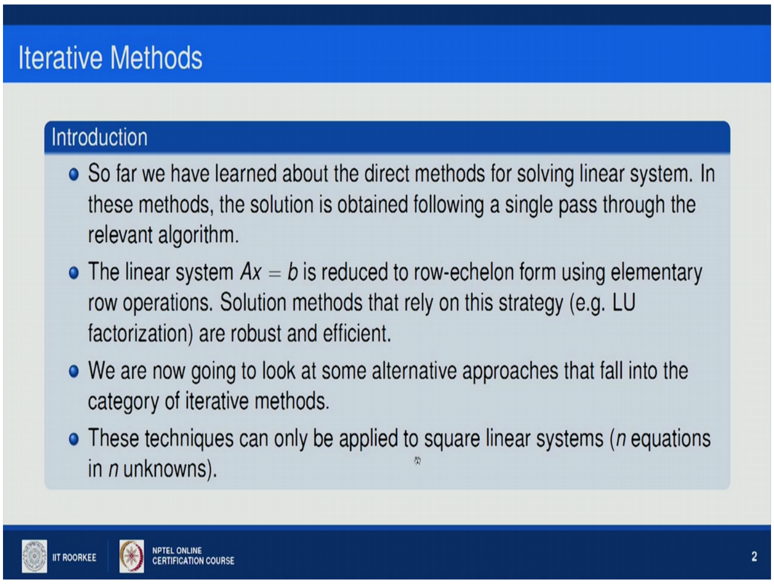

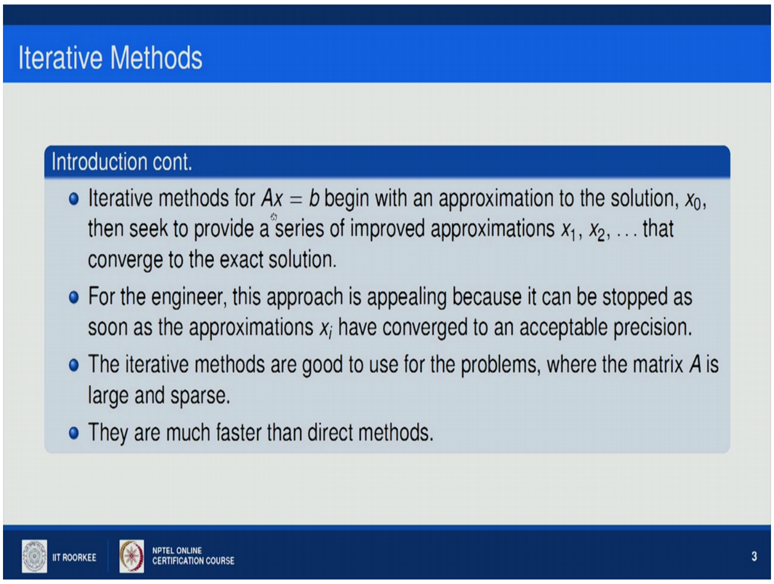

So however these techniques can only be applied or square linear system, Square linear system means you are having n number of equation together with n number of unknown variables, so basically iterative methods for Ax equals to b being with an approximation to the solution x naught that we use to say initial solution then we seek to provide a series of improved approximation x 1, x 2 et cetera that converge to the exact solution.

In engineering problems with this approach is appealing because it can be stopped as soon as the approximation x i has converge into an acceptable precision means whenever the difference between the exact solution and the solution in any iteration is less than a given threshold. The iterative methods are good to use for the problems where the matrix A is large and sparse, okay and in these cases if the matrix A is large and sparse they are much faster than the direct methods.

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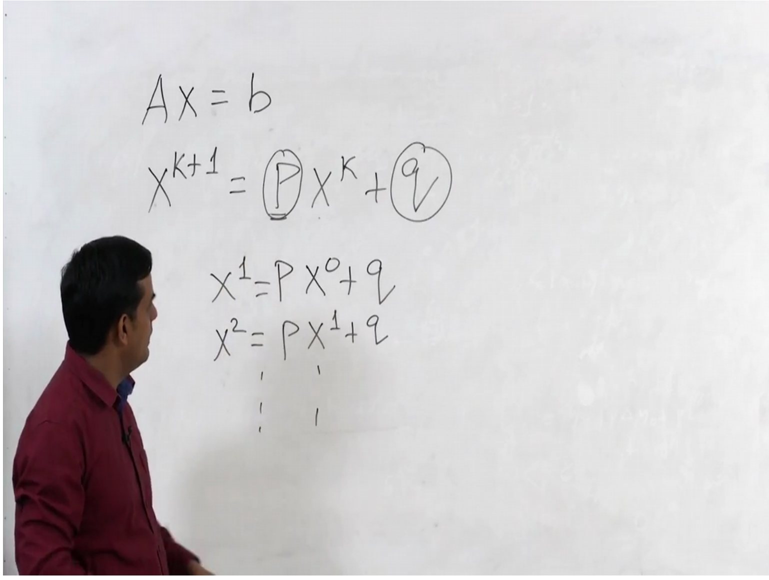

So let me introduce these matters in a general setting, so if we are having a system Ax equals to b where A is the coefficient matrix x is the vector of unknown variables and b is the right-hand side vector then an iterative method for solving this linear system can be written as in this form where P is called the iteration matrix, q is a column vector and for any x obtaining K iteration we can get the update on x in K plus 1 iteration is x K plus 1.

Now how to write this P and q that is the difference in various methods those coming under the category of iterative methods. As I told you we start with an initial solution x naught, so P x naught plus q will give us x 1 then in the 2nd iteration x 2 will be P x 1 plus q and so on, so in each iteration we update our solution and we with an assumption that solution is going to converge to the exact solution however is not true always, solution may diverge also for a given iterative scheme. Now how to write this particular matrix P that is the iteration matrix and the column vector q.

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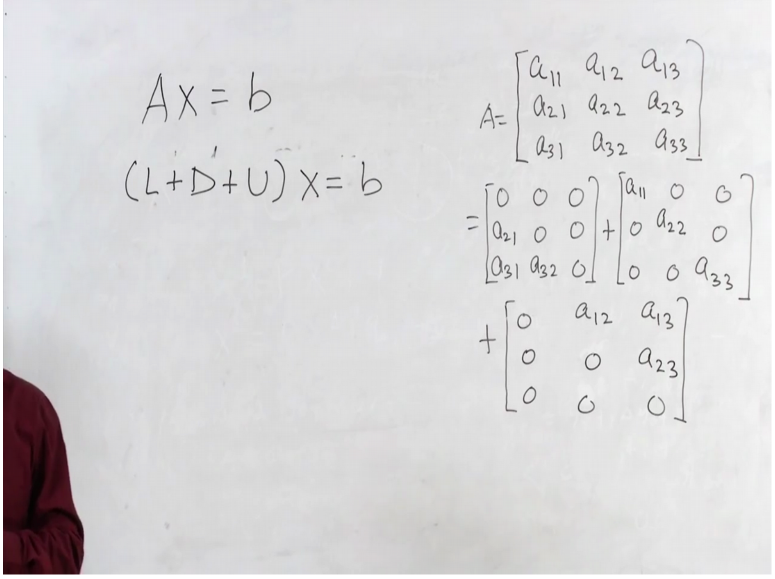

So as we know you can always write given n by n matrix say as the sum of 3 matrices L plus D plus U into x equals to b where L is a lower triangular matrix, D is a diagonal matrix and U is an upper triangular matrix. For example if you are having a 3 by 3 system given by the coefficient matrix A as this one then I can write this coefficient matrix A equal to a lower diagonal matrix plus a diagonal matrix plus an upper triangular matrix which is given as a 11, a 12, a 13, 0 here a 11 will not come as I have taken a 11 already in diagonal matrix that is capital D, so this is the matrix capital L this is the matrix capital D and this is the matrix capital U.

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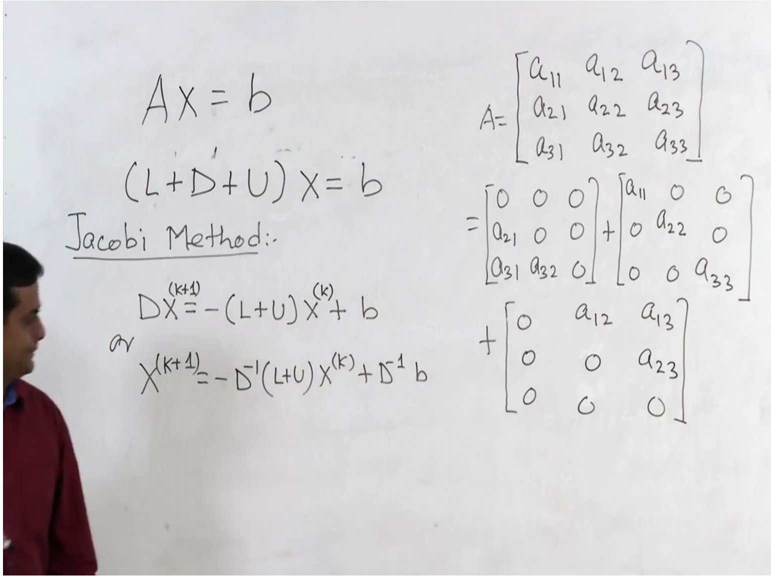

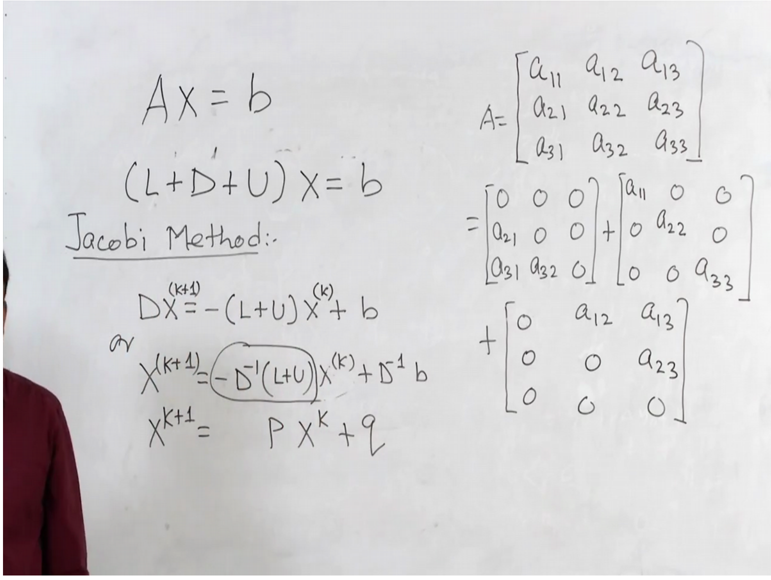

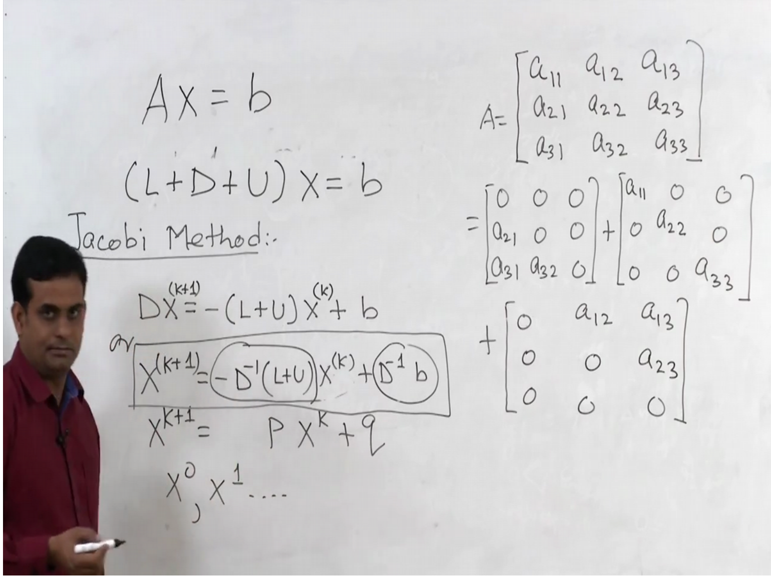

Now we are having different scheme under the category of iterative methods 1st of all I want to drive or I want to introduce the most simple scheme that is called Jacobi Method, in Jacobi method what we use to do, we use to write this system as Dx equals to minus L plus U x plus b, so what we have done? We have taken these 2 terms into right-hand side. Now what I will do? My iterative scheme will work like this that at K plus 1 iteration x is given as from the estimate of x in K iteration or I can write x K plus 1 equals to just multiplying both side pre multiplying by D inverse.

So it will become minus D inverse L plus U x K plus D inverse into b. Here if I compare this particular scheme with a general formula of iterative scheme that is given by x K plus 1 equals to P times x K plus q, so the iteration matrix P is given by this particular matrix. The column vector q is given by this term it will be n by n matrix and this will be a n by 1 column vector, so this particular scheme is called Jacobi iterative scheme. Here we will start with an initial solution x naught then I will find x 1 using this formula and so on.

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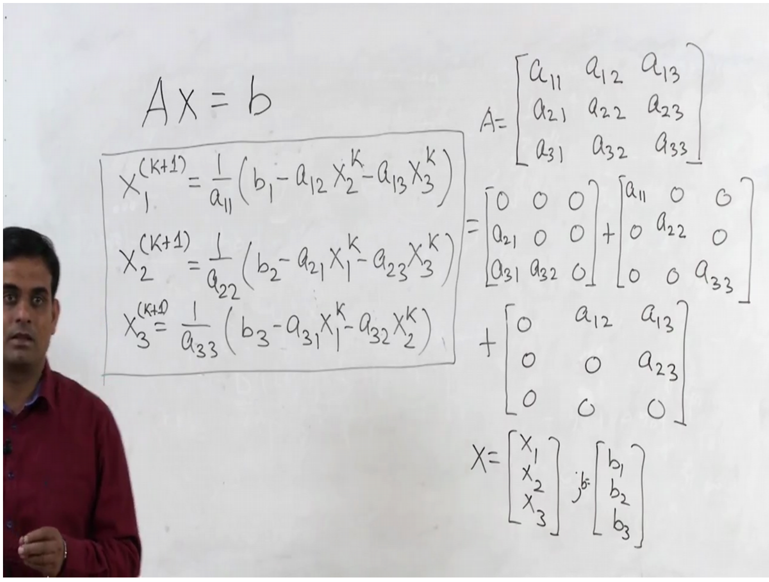

In a more simple setting if I want to write the iterative equation for this particular scheme, I need to consider a system of 3 equations with 3 unknown let us say the coefficient matrix as given by this one and vector x is x 1, x 2, x 3 and right hand side vector b is b 1, b 2, b 3 okay, so now this is the linear system Ax equals to b, the 1st equation can be written like x 1 K plus 1 equals to 1 upon a 11 into b1 minus a 12 x 2 at K iteration minus a 13 x 3 at K iteration.

So what I have done? I have taken the variable which is corresponding to diagonal elements of the coefficient matrix in the left-hand side and I have taken the rest of the 2 terms of the coefficient matrix in the right-hand side. Similarly from the 2nd equation we can get the iterative equation for the variable x 2 and it is given as 1 upon a 22 b 2 minus a 21 x 1 K minus a 23 x 3 K. Finally from the 3rd equation I can write the iterative scheme for x 3 so x 3 K plus 1 equals to 1 upon a 33 b 3 minus a 31 x 1 K minus a 32 x 2 K. So these 3 iterative equations can be upgraded simultaneously to get the iterative values of x in each iteration.

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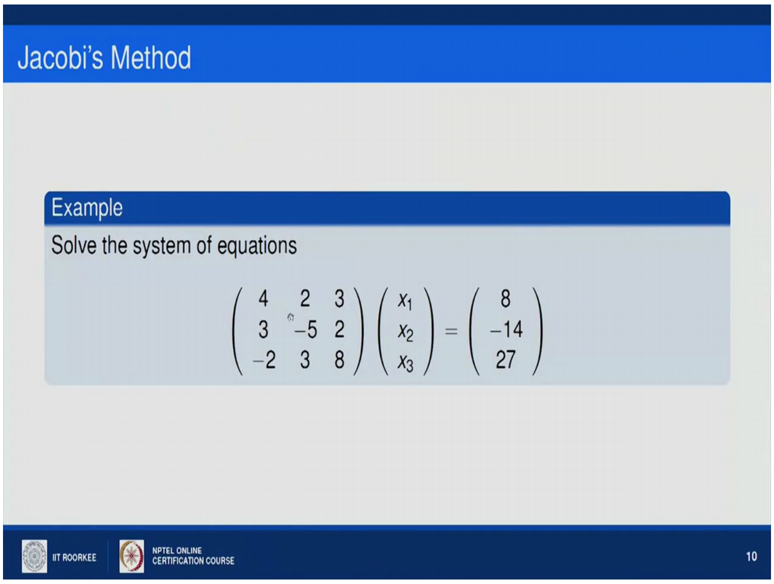

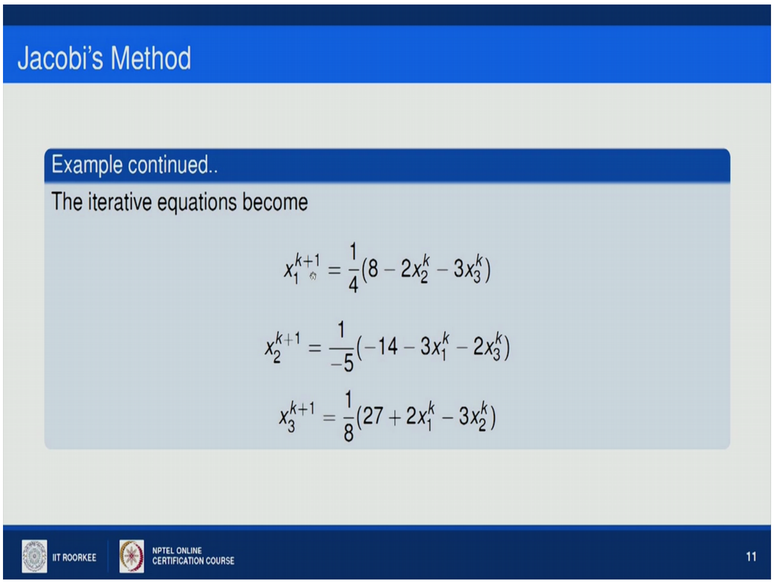

Now I will take an example and I will solve it using this iterative Jacobi iterative scheme, so here let us take this system of equations here am having 3 equations with 3 unknowns x 1, x 2, x 3 the iterative equations can be written in this way x 1 K plus 1 will be 1 upon 4, 8 minus

2x 2 K minus 3x 3 K, x 2 K plus 1 equals to minus 1 upon 5 into minus 14 minus 3x 1 K minus 2x 3 K and similarly from the 3rd equation I can write the iterative equations for x 3 that is x 3 K plus 1 equals to 1 upon 8 into 27 plus 2x 1 K minus 3x 2 K. Now if I start with an initial solution let us say x 10 is 0, x 20 is 0, x 30 is 0 then the iteration 1 I will get the value of x 11 as 2 that is coming from the 1st equation here I am putting x 20 is 0, x 20 is 0, so x 11 will become 8 upon 4 so it is coming out as 2.

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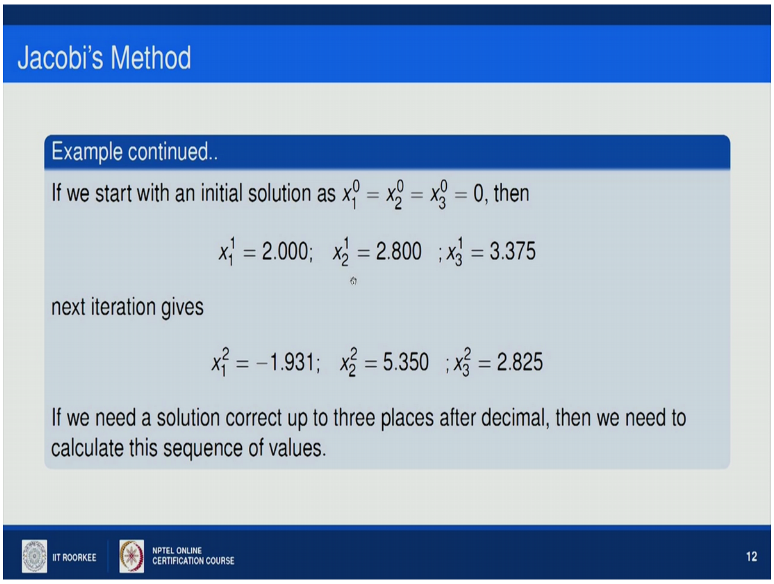

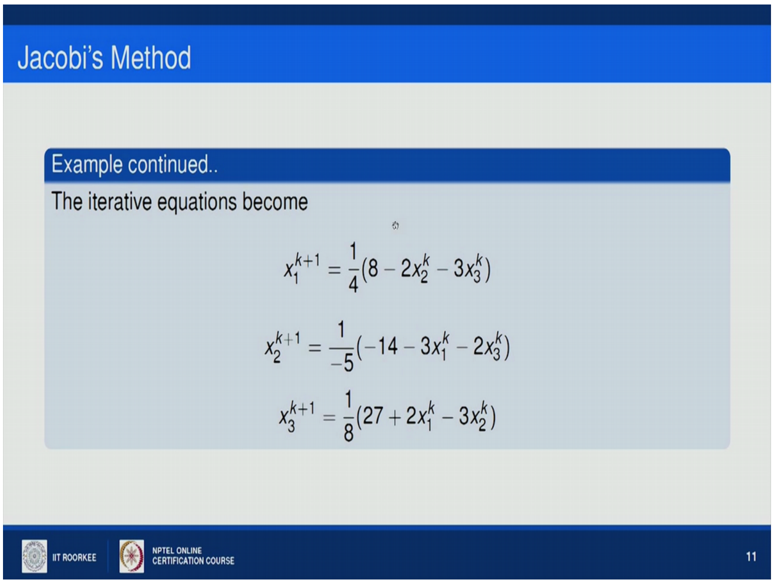

Similarly from the 2nd equation I am getting x 2 1 that is coming out to be 2.8 and x 3 1 equals to 3.375. Now if I these values in the right-hand side of these 3 iterative equations I can get the next iterate of x and that will be for x 1 it will be minus 1.9314 x 2 it will be 5.350

and for x 3 it will be 2.825. If we need a solution correct up to 3 places after decimal, then we need to calculate this sequence of values for further (())(16:17).

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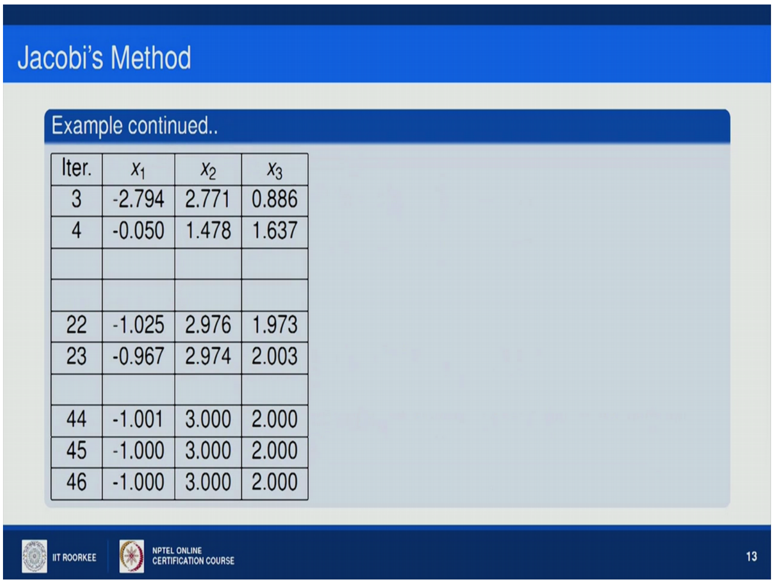

So using the same this table gives me the values of x 1, x 2, x 3 in 3^{rd} iteration then in 4^{th} iteration and you can see the values are changing quite differently in each iteration going in the same manner in 22^{nd} iteration is coming out minus 1.025 for x 2 it is 2.976 for x 3 it is 1.973 and then in the next iteration the values can be seen like this after 43^{rd} equation that is in 44^{th} iteration I am getting the value of x 1 as minus 1.001 x 2 as 3 and x 3 as 2. In 45^{th} iteration I am getting minus 1, 3 and 2 then in 46^{th} iteration am getting minus 1, 3, 2. So in 2 successive iterations I am getting the same value hence my solution converge to minus 1, 3 and 2 and which is the exact solution also of the given linear system. So here we have taken 46 iterations to solve this 3 by 3 system using the Jacobi method and this is the this example is gives an illustration of the Jacobi method.

Now as I told you we have taken 46 iterations to solve a 3 by 3 system just by starting with an initial solution 0, 0, 0 hence I can comment on the convergence of Jacobi method and I can say it is having a slow convergence. So in the next method that is called Gauss Seidel method I will introduce some modification in the Jacobi scheme and then Gauss Seidel method iterative equations will give a bit faster convergence compared to the Jacobi scheme.

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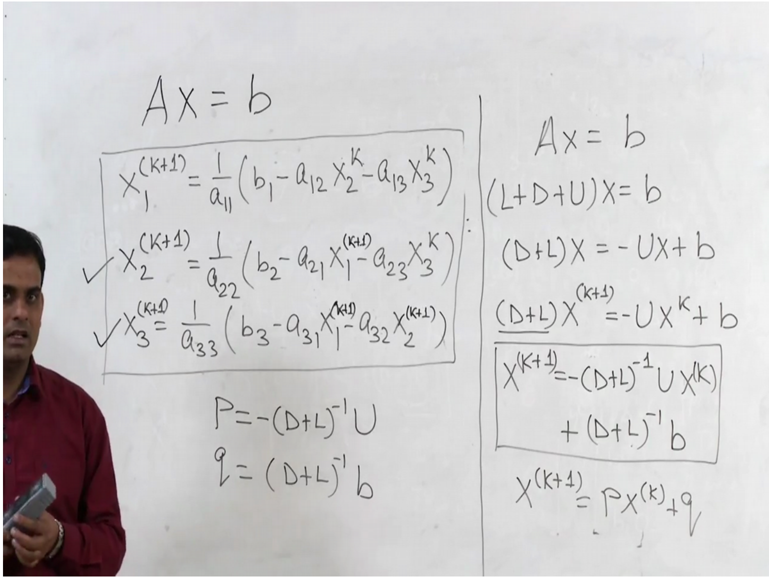

If we see the iterative equation of this Jacobi method here you can see in the 2nd equation I am using the value of x 1 from the previous iteration however before going to the 2nd equation I have find out the value of x 1 at in the current iteration, so here the value of x 1 from the current iteration has available to me but I am using the old value. Similarly if you see the 3rd equation here both the values of x 1 and x 2 are available to us from the current iteration that is at the K plus 1 iteration from the 1st 2 equations but what we are doing, we are still using the values which we obtain in the previous iteration.

If I modify these values the values of current iteration then I can have a bit more fast convergence. For example like in the 1st equation I will calculate x 1 at K plus 1 iteration and what I will do here I will use this value. From the 1st and 2nd equations I will get the value of x 1 and x 2 respectively in K plus 1 iteration and in the 3rd equation I will use these new values. So basically in a general setting what I am doing I am having a system Ax equals to b again I am writing it L plus D plus U x equals to b and then what I am doing? I am writing this D plus L x equals to minus Ux plus b and from here I am writing my iterative scheme that is D plus L at K plus 1 iteration equals to minus Ux K plus b.

So this equation can be written in a more appropriate form like this just I am pre-multiplying by the inverse of this matrix, so it will become minus D plus L inverse Ux K plus b plus L inverse b. So this particular equation gives the Gauss Seidel scheme in a general setting where my coefficient matrix A is n by n matrix. Here iteration matrix P can be written as minus D plus L inverse into U and the column vector q is given as D plus L inverse into b if I compare this scheme with the abstract equation of iterative scheme.

So this scheme is called Gauss Seidel method for solving the linear system of equations and if I use this method on a 3 by 3 general system a 11 x 1 plus a 12 x 2 plus a 13 x 3 equals to b 1 and 2 equations like we have taken in Jacobi method. Now final system comes out to this particular 3 iterative equations now the only changes as a told you earlier also here we are using the updated value of x 1 and x 2 in 2nd and 3rd iterative equations respectively for finding the values in the current iteration that is K plus 1 iteration. Hence we hope that this scheme will be more faster because we are using the more updated values of the variables in the scheme.

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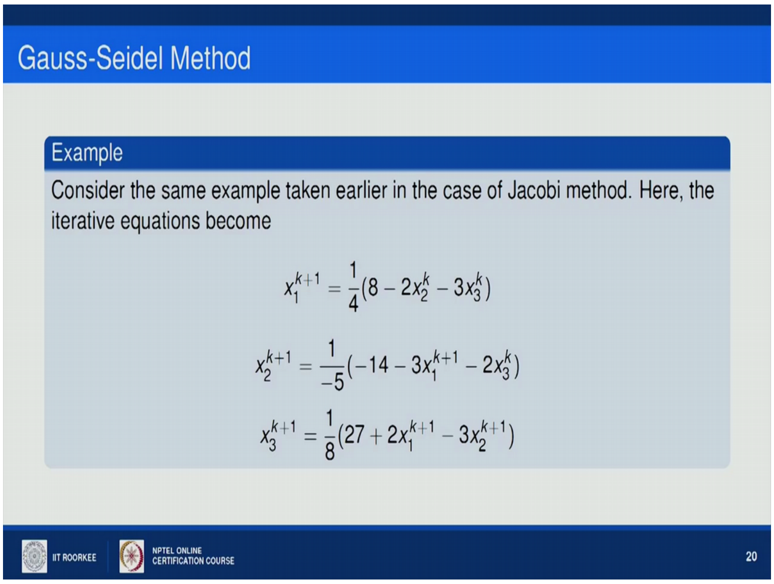

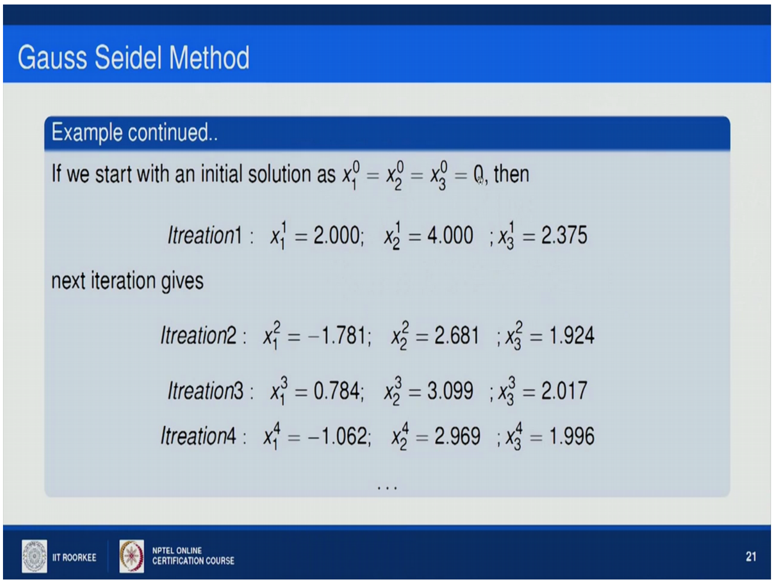

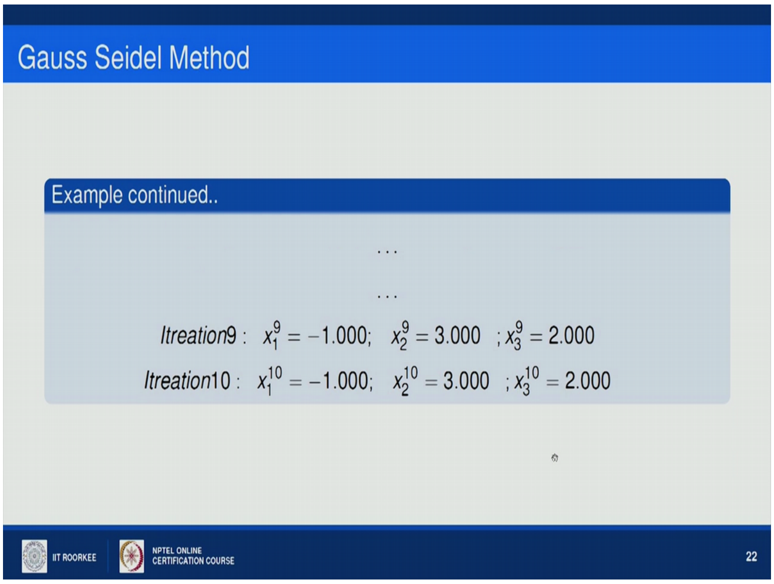

Let us take an example of this method so the same example which we have taken in the earlier case that is in the case of Jacobi method there we used or we need to use 46 iteration for finding the exact solution I mean for the convergence of the solution here let us see how many iterations we need using the Gauss Seidel method. So for that particular example the 3 iterative equations that is for x 1, x 2 and x 3 can be written in this way so you can note down this particular term and these 2 terms here we are using the updated values as I have told you in the derivation of this particular method. Then if I start with an initial solution that is x 1 0 equals to x 2 0 equals to x 3 0 equals to 0.

So initially I am taking a 0 vector as the initial solution then in the 1st iteration I am getting the value, so here I need to put x 2 equals to 0 and x 3 equals to 0, so what will happen once I will put the 00 I will get x 1 1 as 8 upon 4 that will be 2. Now in the 2nd equation I will not use your x 1 equals to 0 because if I will use your x 1 equals to 0 that will be something that will be exactly like Jacobi. Here I am using this updated value of x 1 that is 2 so this will become x 2 1 equals to minus 1 upon 5 minus 14 minus 3 into 2 that is minus 6 minus 2 into 0.

So here x 2 1 will become minus 20 upon minus 5 that is 4 so x 2 1 will become 4 and finally x 3 1 will become 1 upon 8 27 plus 2 into 2 please note that again I am using the updated value of x 1 in Jacobi method I have used here 0 that is the value from the previous iteration. So 2 into 2 4 minus 3 into 4 12 so 27 plus 4 minus 12 so it will be 27 minus 8 so 19 by 8 and that will be coming out something like 2.375. Now in the next iteration I will get the value of x 2 in the 2nd iteration x 1 as minus 1.781 x 2 equals to 2.681 and x 3 equals to 1.924.

When these are the values in the 3rd iteration then we are having values in 4th iteration and so on continuing this calculation in 9 iteration am getting the value of x 1 is minus 1 x 2 is 3 and x 3 is 2. In the 10th iteration I am getting x 1 is minus 1 x 2 is 3 and x 3 is 2 so hence my iterative scheme converges in 10 iteration itself while in case of Jacobi method it was taking 45 iterations but here in Gauss Seidel method it converge in 10 iterations hence the claim which we made that by using these 2 updates Gauss Seidel method will be something more faster than the Jacobi method is verified by this example.

Now in this particular lecture we discussed the 2 iterative schemes for solving linear system of equations and we have seen that later one is quite faster than the earlier one hence an iterative scheme is good if it is more faster, so the effort should be made for developing an iterative scheme by taking care of its convergence or rate of convergence, how fast it converge to the exact solution? In the next lecture I will talk about it and then I will introduce one more technique to you that is basically called successive over relaxation.

In this particular technique I will use a relaxation parameter I will find out the optimal value of that relaxation parameter and then based on that optimal value in terms of convergence speed. Once I will get the optimal value I will get my iterative system and using that particular iterative system I will write the iterative equations. Further in the next lecture I will also introduce about the conditions for convergence of an iterative scheme, so thank you very much for this lecture.