Numerical Methods
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Lecture 23
Interpolation Part VIII (Divided Difference Interpolation with Examples)

Welcome to the lecture series on numerical methods. Currently we are discussing interpolation. In interpolation we have discussed like finite difference operators and like Newton's forward difference formula, backward difference formula and central difference formula. And in the last lecture we have cover up this Lagrange interpolation formula. So now we will discuss about Newton's divided difference formula.

So before going to Newton's divided difference interpolation first we will discuss about what is divided differences? So divided difference means if you will just have a set of data points like x 0 y 0, x 1 y 1 up to x n y n here. Maybe it is uniformly spaced or non-uniformly spaced but we can use this divided differences here. So suppose these corresponding values of y in terms of x if we are just expressing here as y equals to f of x for x equals to x 0 to x n here.

We can just express this divided difference for this function y equals to f of x as like f of x 0 x 1 for two consecutive points we can just write f of x 1 minus f of x 0 by x 1 minus x 0. Similarly if we want to write for x 1, x 2 for two consecutive points since whatever this tabular points I have written it up here that is starting point is x 0, next point is x 1, corresponding these y values are associated values like y 0, y 1.

So we can just write this first function as also y 1 minus y 0 by x 1 minus x 0 here. And for the second two points like x 1 and x 2 if I want to express in divided difference form I can just write f of x 2 minus f of x 1 by x 2 minus x 1 here which can be written as y 2 minus y 1 by x 2 minus x 1 here.

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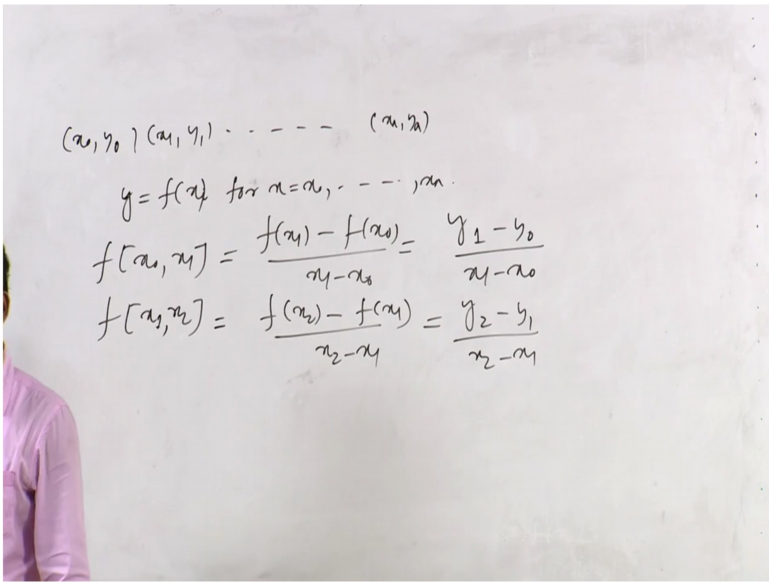

Obviously the question arises that if we are just writing this values of x 0 and x 1 here in argument form or in arguments then how we can just express f of x 0 here? So we can just write obviously f of x 0 equals to f of x 0 as this one and f of x 1 since all are independent points we can just write these arguments as functional values at that points.

And if we are just going up to the last of the interval here that is x n minus 1 to x n here we can just write this divided difference of x n minus 1 to x n as f of x n minus f of x n minus 1 divided by x n minus x n minus 1. And this can be written as also y n minus y n minus 1 divided by x n minus x n minus 1 here.

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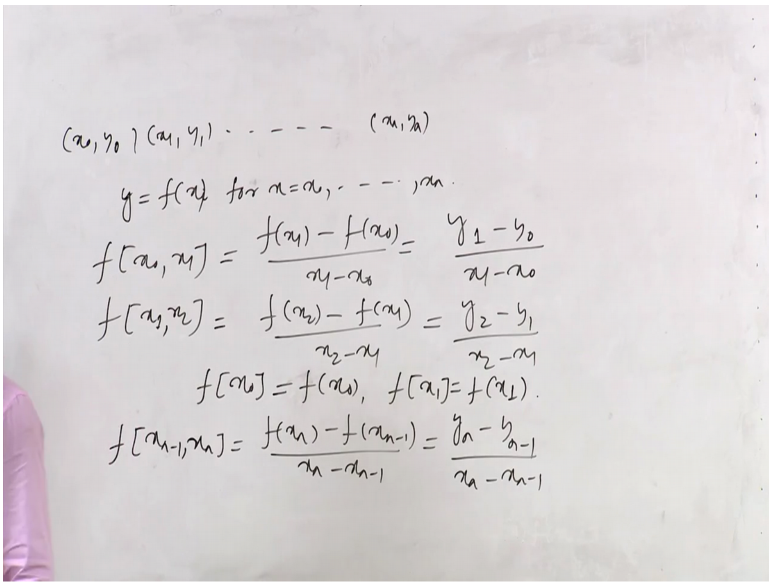

So these are all called first order divided difference here and if you just go for second order divided differences here then we can just express the second order divided difference especially if the points are placed at like x 0 y 0, x 1 y 1 up to x n y n here. At a time if we are just considering three points suppose x 0, x 1 and x2 we can just write this divided difference that in the form of second order divided differences. That as f of x 0, x 1, x 2 as f of x 1, x 2 minus f of x 0, x 1 divided by x 2 minus x 0 here.

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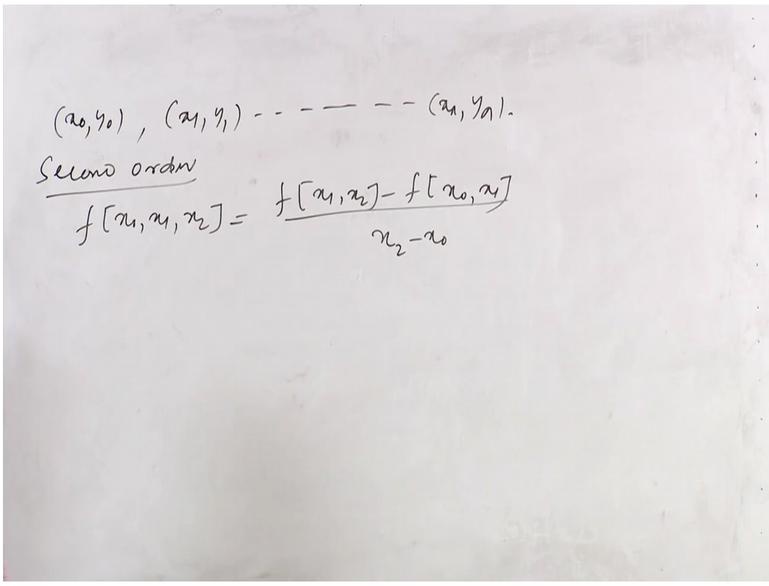

And the third order divided difference since if you will just see here for second order divided difference we are just considering three points. For the first order divided difference we are just considering two points there. So if we will just go for third order divided difference here then we have to consider like four points here x 0, x 1, x 2, x 3 and it can be written in the form of like f of x 1, x 2, x 3 minus f of x 0, x 1, x 2 divided by x 3 minus x 0.

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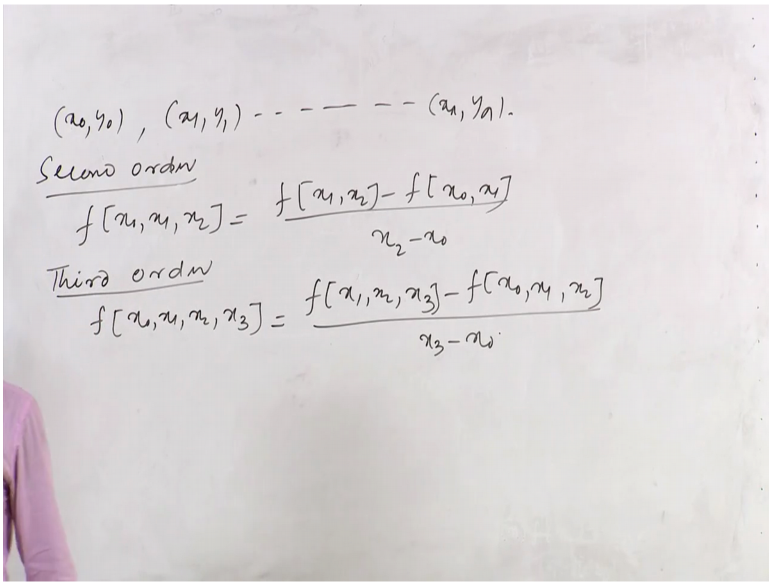

And for nth order divided difference we can just write that one as like nth order if you will have like n plus 1 values here, x 0 x 1 up to x n here. We can just write this one as x 1 to x n minus f of x 0 to x n minus 1 divided by x n minus x 0 here. We can just see that if we are just going for this like first order divided difference we are just using two points and especially that can establish a relationship between these functional values at x 1 and x 0.

And if we are just going for the second order divided difference here we can just find that this establishes a relationship between like if again if you just take this differences here it can be written in the form of f of x 2 minus f of x 1 divided by x 2 minus x 1 minus f of x 1 minus f of x 0 divided by x 1 minus x 0 divided by x 2 minus x 0 here. This means that this establishes a relationship between x 2 x 1 and x 1 x 0.

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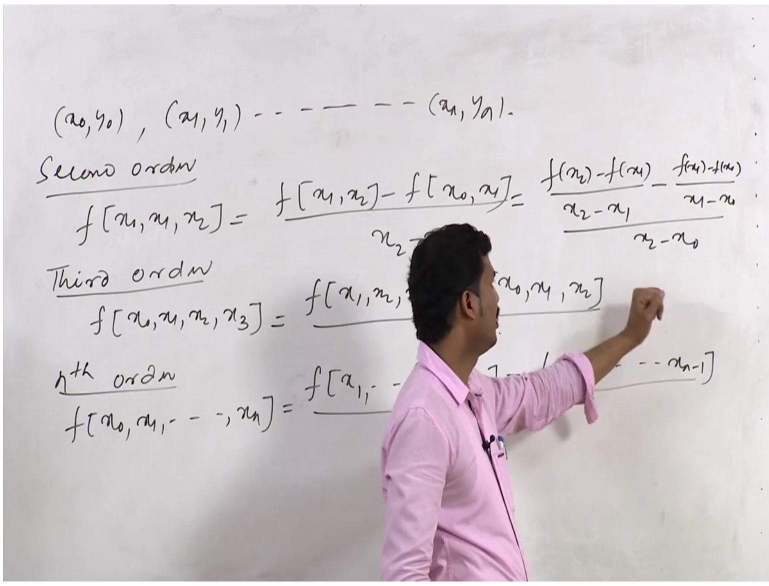

And if we just go for this third order divided difference here, it establishes a relationship between like x 0, x 1, x 2 and x 3 combinedly here. The advantage of this method is that in a Lagrange interpolation method usually whenever an extra point is added there so we have do all the computations newly. This means that all of these products again we have to consider in a modified form that the extra point can be multiplied with the rest of the factors there.

But here if you will just see this extra point can be added in a uniform way that extra multiplication is not needed. If we are just seeing this differences here, these arguments are like equal if the divided differences may still have a meaning here. This means that if the arguments are equal like suppose x 0 if we want to write like f of x 0 x 0 there. This means that if we are just writing f of x 0 x 1 suppose where x 1 can be written as x 0 plus epsilon where epsilon is very small.

This means that we can just consider epsilon tends to 0 here. So then we can just express f of x 0 x 1 as f of x 1 minus f of x 0 divided by x 1 minus x 0. Obviously this is the like first order this divided difference formula represents in this form here.

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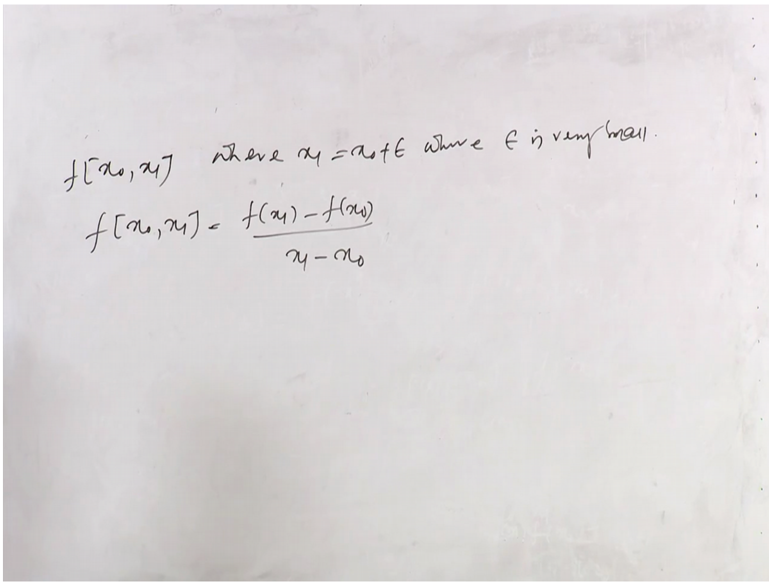

But if you will just replace this x 1 here in the form of x 0 plus epsilon where epsilon is very small we can just write this statement as in the form of limit epsilon tends to 0, f of x 0 plus epsilon minus f of x 0 divided by epsilon here. And obviously this is nothing but f dash of x 0 here. So we can just say that here if the arguments are equal then the divided difference still produces some value here.

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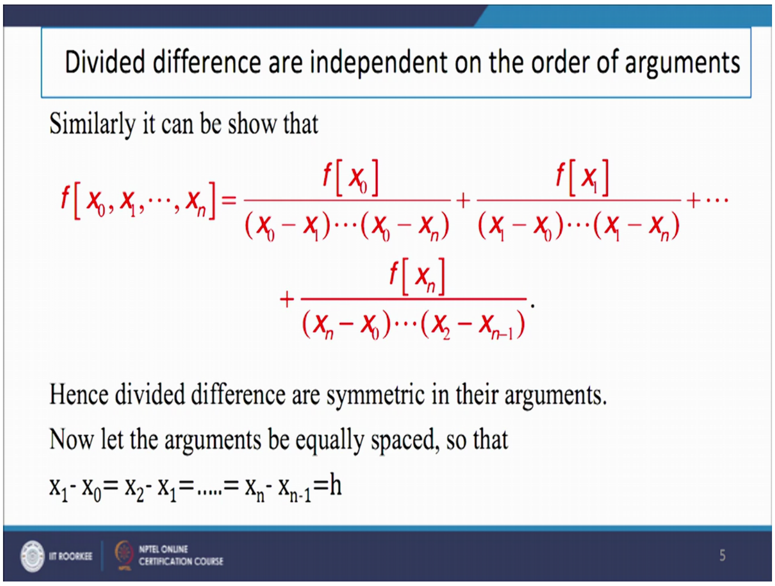

And it is in the form like if we are just considering here f of x 0 x 0 where x 1 can be represented in the form of x 0 plus epsilon and epsilon is very small then it can just represent the derivative of this function at that point only. So if suppose f of x is a differentiable and if we will have this r plus 1 argument suppose. Like we can just write f of x 0 x 0 up to r plus 1 times. We can just write this one as f to the power r of x 0 by r factorial. Since if you will just see here two points are there, so that is why we are just writing this as f of x 0 x 0 as f dash x 0 here.

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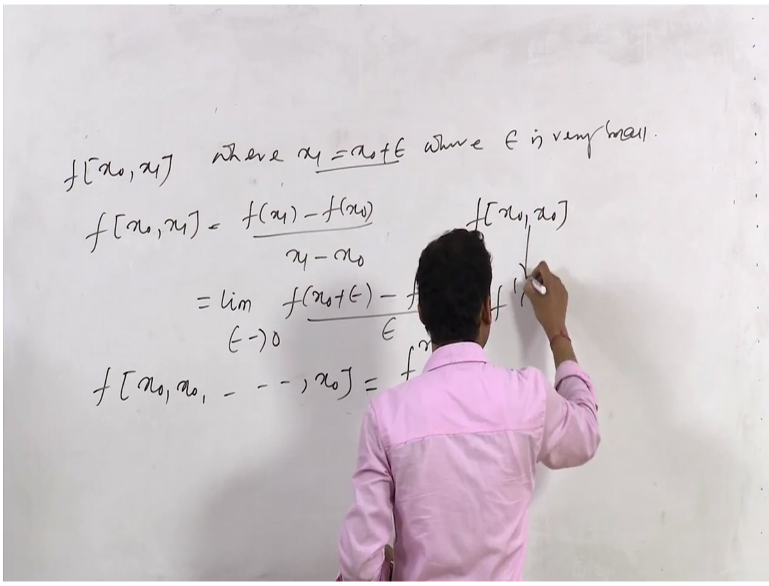

Similarly we can just consider that this should be r plus 1 points to get this derivative in the form order of r here.

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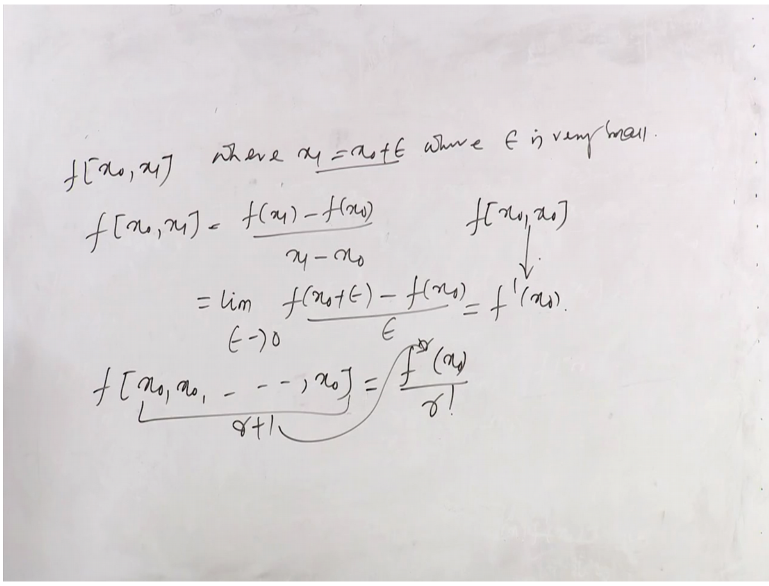

So if you will just go for this divided difference table here perfectly we can just find that this tabular values like i equals to 0, 1, 2, 3, 4 and its corresponding values suppose x 0, x 1, x 2, x 3, x 4 then we can just evaluate this first order divided difference, second order divided difference, third order divided difference to get any of the approximated value within that range.

This means that suppose like our earlier computation if the value is asked to compute within any of the interval suppose then we have to use this divided difference in such a fashion that this value can be computed in easy form. So that is why if you will just consider like i as the values as 0, 1, 2, 3, 4 here, the corresponding x i values are x 0, x 1, x 2, x 3, x 4 here. And corresponding these y values we can just write y as y 0, y 1, y 2, y 3, y 4 here.

Or we can just write this one as f of x 0, f of x 1, f of x 2, f of x 3, f of x 4. And if you will take this first order divided difference here sometimes also this divided difference is written in this form also. And the first order divided difference if you will just see that is just giving you f of x 0 x 1 here. This means that the difference of these two divided by the difference of these two here.

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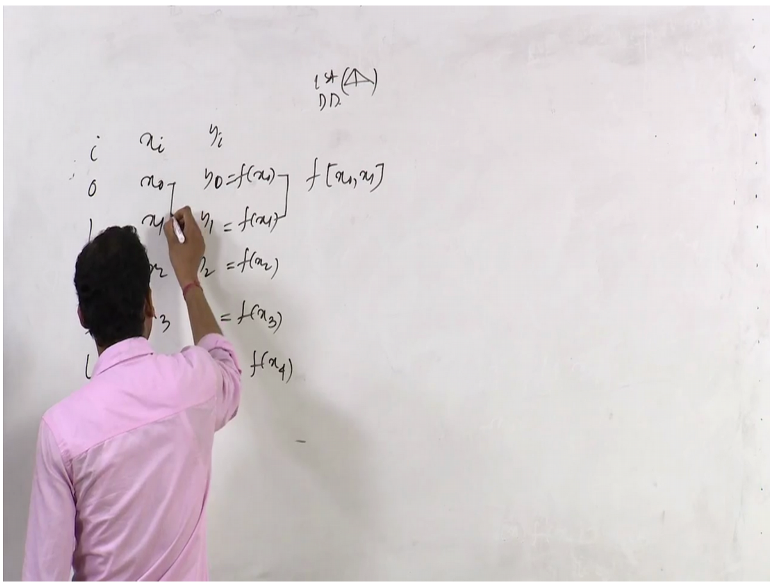

And if you will just take this divided difference of x 1 x 2 here, that will just give you f of x 2 minus f of x 1 divided by x 2 minus x 1 here. Similarly if you just consider this divided difference of like x 2 x 3 here then we can just consider that one as f of x 3 minus f of x 2 divided by x 3 minus x 2. Similarly we can just find this divided difference of like x 3 x 4 as f of x 4 minus f of x 3 divided by x 4 minus x 3.

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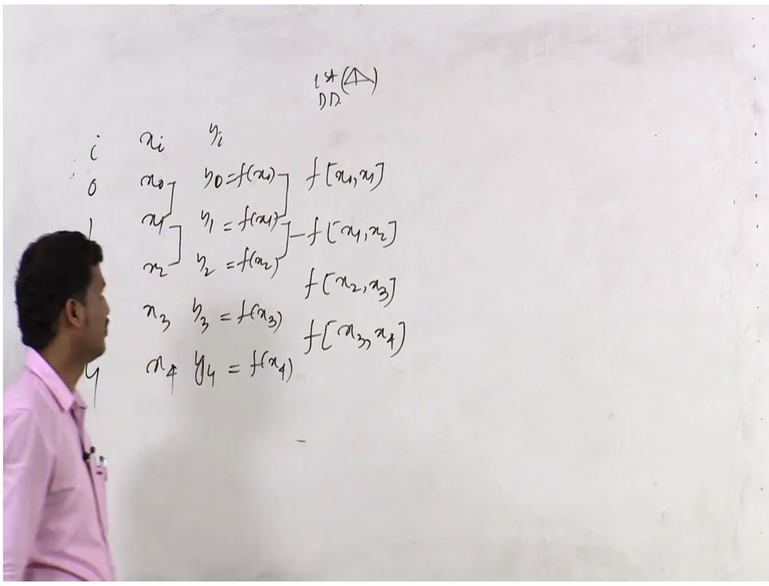

And then second order divided difference you can just find it out here that is in the form of like f of x 0, x 1, x 2. And then we can just write this one as f of x 1, x 2, x 3 here. And next divided difference we can just write x 2, x 3, x 4 here. Similarly the third order divided difference if you just see you can just write this one as f of x 0, x 1, x 2, x 3 and the third order divided difference for this function if you will just write, this can be started from x 1 to x 3, x 4 here.

And the last divided difference if we want to write this can be (re) represented as fourth divided difference which can be written as f of x 0, x 1, x 2, x 3, x 4 here.

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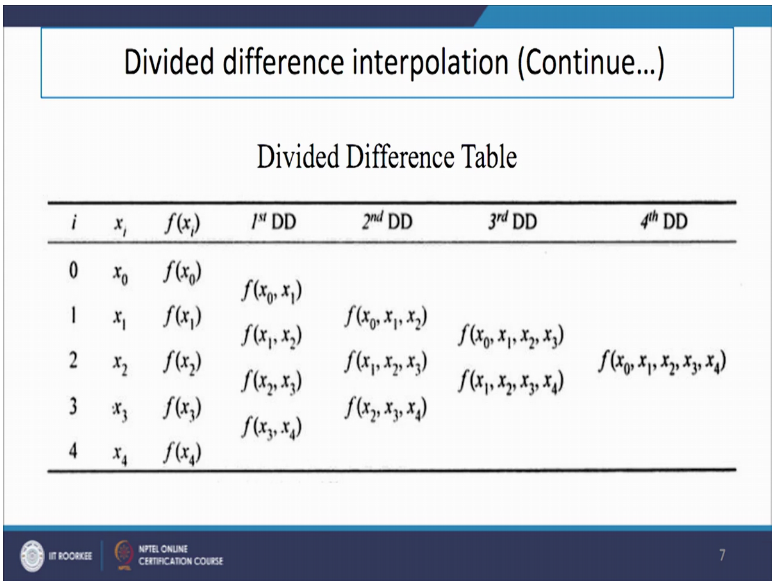

So if the tabular value is just given like x i and y i so we can just find this divided difference table and based on this divided difference table data we can just evaluate this polynomial at any point within this interval using any of the formula like Newton's divided difference formula or any specified formula which can be based on this divided differences. So next we can just show that this divided difference arguments are independent.

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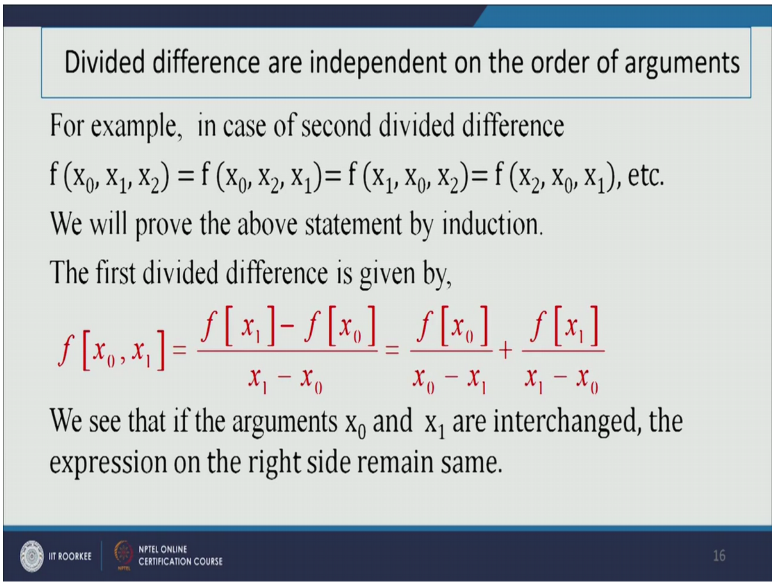

This means that if we are just considering these arguments here, the arguments can be written in the form of like suppose pre arguments if we are just writing or the second order arguments, these arguments can be written in the form of like x 0, x 1, x 2 here. This can be written also as f of x 0, x 2, x 1. And this can also be written in the form of like f of x 1, x 0, x 2.

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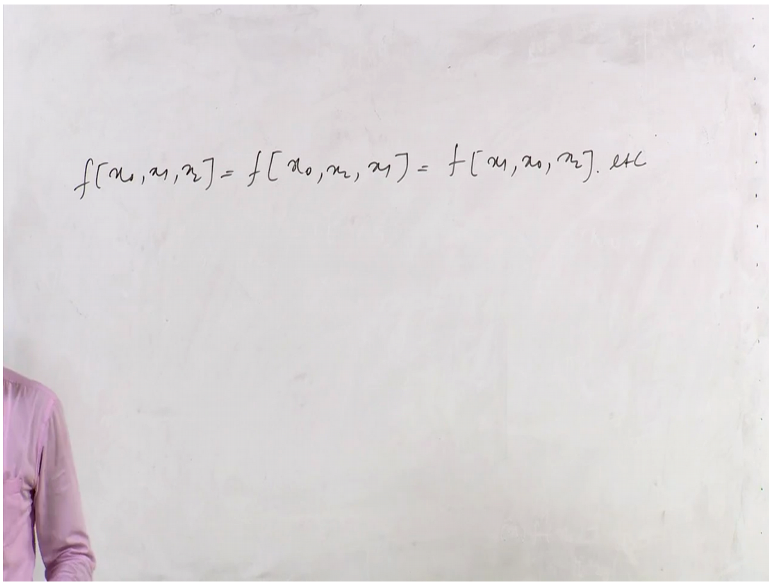

So this can be proved easily since if you will just see for first order divided difference either we can just write f of x 0 x 1, it can be written as f of x 1 minus f of x 0 divided by x 1 minus x 0 or it can be written as f of x 0 minus f of x 1 divided by x 0 minus x 1 here. This is nothing but f of x 1 x 0 here.

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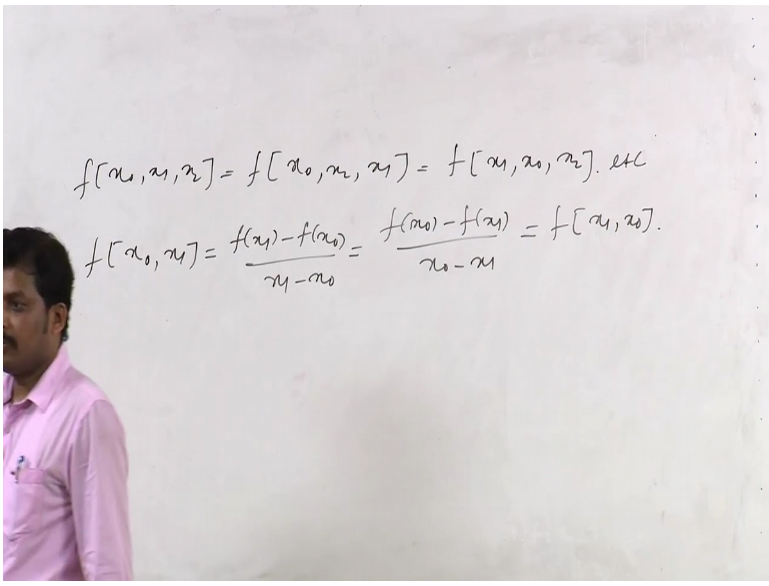

So similarly we can just show this independent order of arguments are like independent in like divided difference of order 2, order 3. We can just show that one also. So if you will just for go for this like second order divided difference we can just find that f of x 0, x 1, x 2, it can be written in the form of like f of x 1, x 2 minus f of x 0, x 1 divided by x 2 minus x 0 here.

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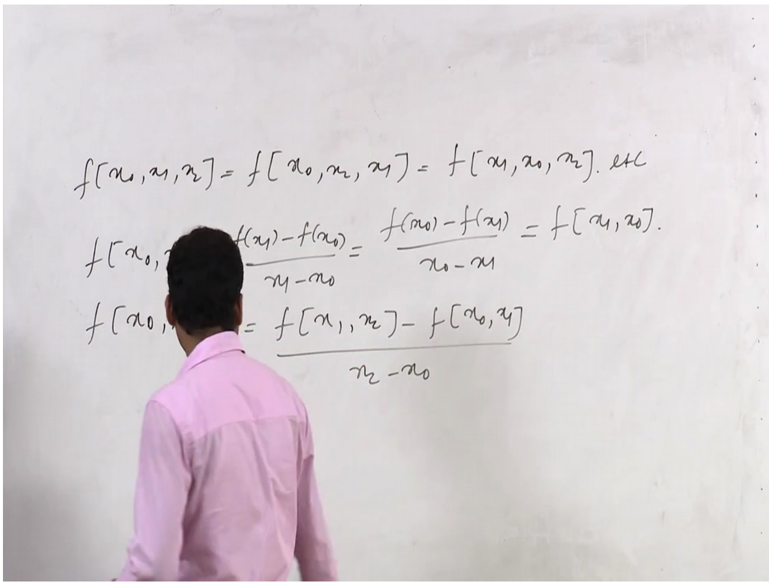

And if I will just take here 1 by x 2 minus x 0, then it can be written in the form of like f of x 2 minus f of x 1 divided by x 2 minus x 1 minus f of x 1 minus f of x 0 divided by x 1 minus x 0 here.

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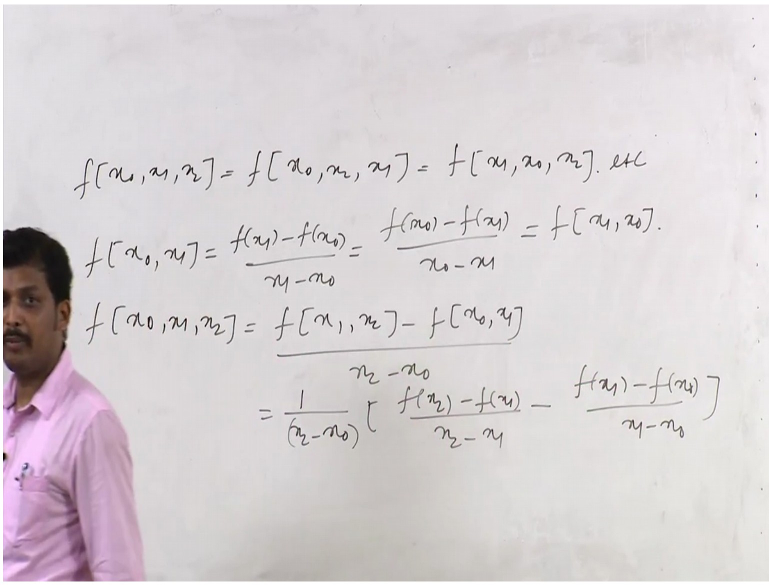

So if you all just take common of these terms here I can just show that 1 minus x 2 minus x 0 this can be written in the form of like f of x 1 minus f of x 2 by x 1 minus x 2 minus f of x 0 minus f of x 1 by x 0 minus x 1 there. And obviously if you will just interchange the signs, we can just write this complete statement as in the form of here f of x 0 divided x 0 minus x 1, x 0 minus x 2 plus f of x 1 divided by x 1 minus x 0, x 1 minus x 2 plus f of x 2 divided by x 2 minus x 0, x 2 minus x 1 here.

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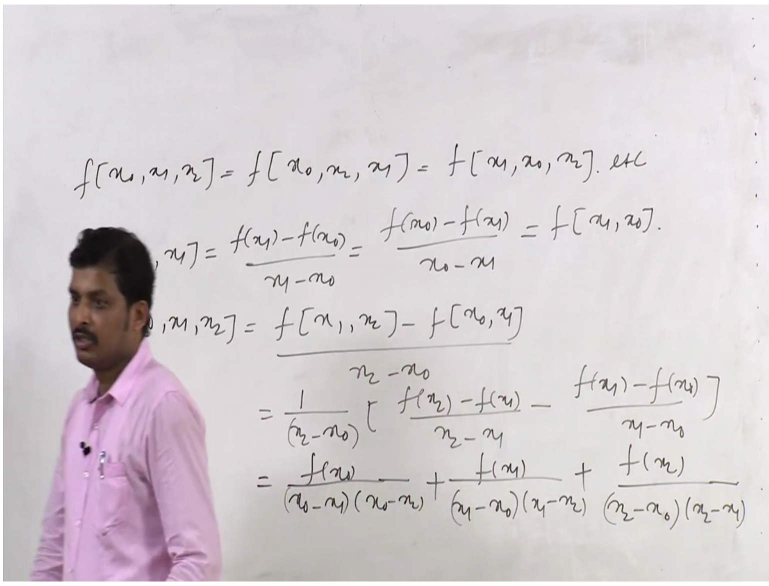

So again interchanging of arguments it does not affect the solution process also here. Similarly it can be shown that if we will just consider a nth ordered divided difference it can be written in the form of like f of x 0 x 1 to x n suppose, n plus 1 terms are there.

So that is why we can just say that this is nth order divided difference and it can be written in the form of like f of x 0 divided by x 0 minus x 1 to x 0 minus x n plus f of x 1 divided by x 1 minus x 0, x 1 minus x 2 up to x 1 minus x n plus f of x n divided by x n minus x 0, x n minus x 1 up to x n minus x n minus 1 here.

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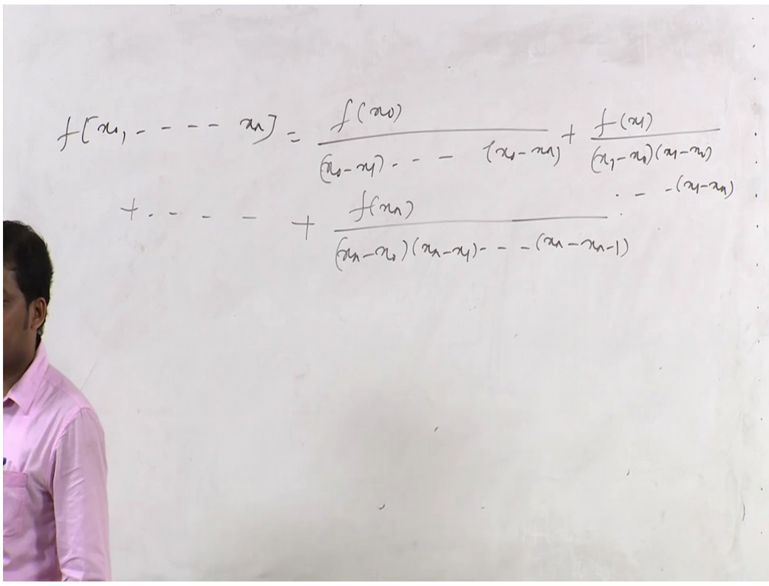

Hence we can just say that these divided differences are symmetric in their arguments. Now let the arguments be equally spaced suppose. Like x 1 minus x 0 equals to x 2 minus 1. Suppose this space size is h here. If it is equally spaced we can just say that all of this differences like we can just express x 0 minus x 1 as h here, so it can be expressed as h to the power n here. Similarly f of x 1 by h to the power n, so likewise we can just express the total arguments here.

So if we want to express it in divided difference (form) formula suppose then we can just use this one in a form that as f of x equals to f of x 0 plus x minus x 0, f of x 0 x 1 plus x minus x 0, x minus x 1, f of x 0 x 1 x 2 plus up to x minus x 0, x minus x 1, x minus x n minus 1, f of x 0 x 1 to x n.

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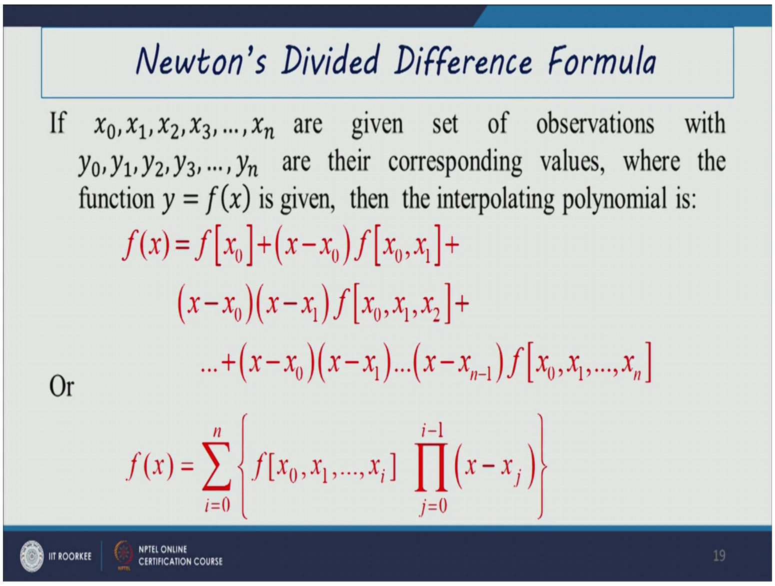

So if we want to prove this formula that Newton's divided difference interpolation formula here the statement is, if x 0 x 1 to x n are given set of observations for y 0, y 1, y 2 up to y n are there corresponding values where the function y equals to f of x is given then the interpolating polynomial is f of x equals to f of x 0 plus x minus x 0, f of x 0 x 1 plus x minus x 0, x minus x 1, f of x 0, x 1, x 2. So likewise we can just write this one as x minus x 0, x minus x 1 to x minus x n minus 1, f of x 0 to x n here.

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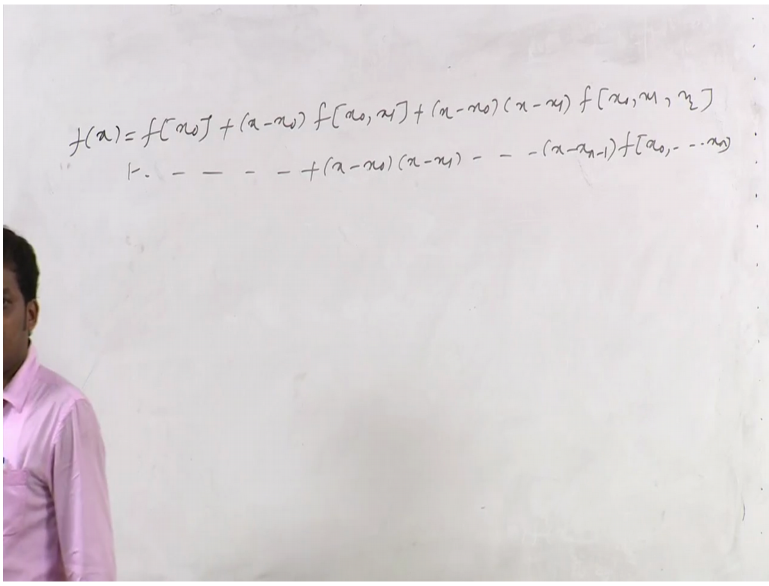

So especially this immediate next term it can be said to be as the remainder term where another x value it is required to get this last term there. Since if you will just see here for the remainder term or the next immediate term if you will just see this next another term here that is x minus x n here and it also requires another extra point to get this remainder term for the series to truncate over there.

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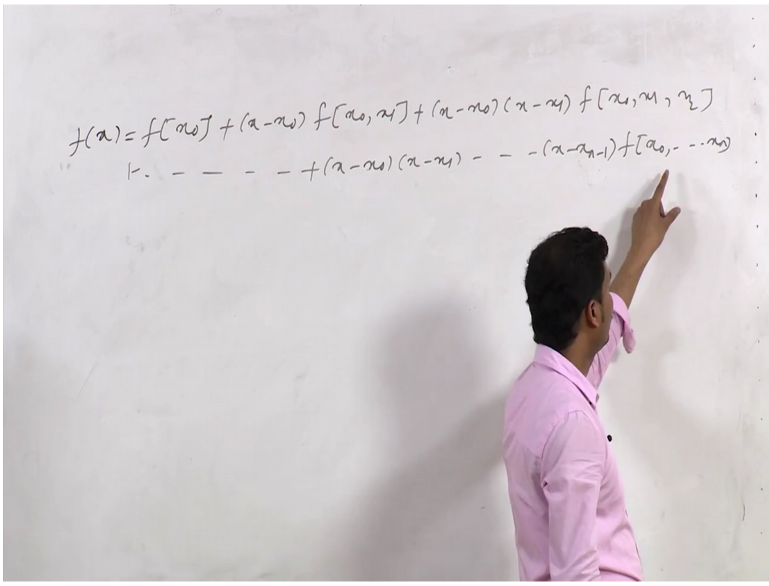

So if you will just see it can be written in the product form also here that is f of x equals to i equals to 0 to n, f of x 0 x 1 to x i that can be written as also product of j equals to 0 to i minus 1, x minus x j. And if you just go for the proof of this divided difference formula here, especially we can just write f of x x 0 this can be written in the form of like f of x minus f of x 0 divided by x minus x 0. Either way you can just write that one since this order of arguments are independent or symmetrical in nature.

So from these two if you want to separate we can just write f of x equals to f of x 0 plus x minus x 0, f of x x 0 here. Similarly if you will just write or if you will just add extra more point here like x, x 0, x 1 here we can just write this one as f of x 0, x 1 or we can just write as f of x, x 0 here minus f of, since we are just saying that this is independent of order of arguments we can write in any forms here, so f of x 0, x 1 here divided by x minus x 1.

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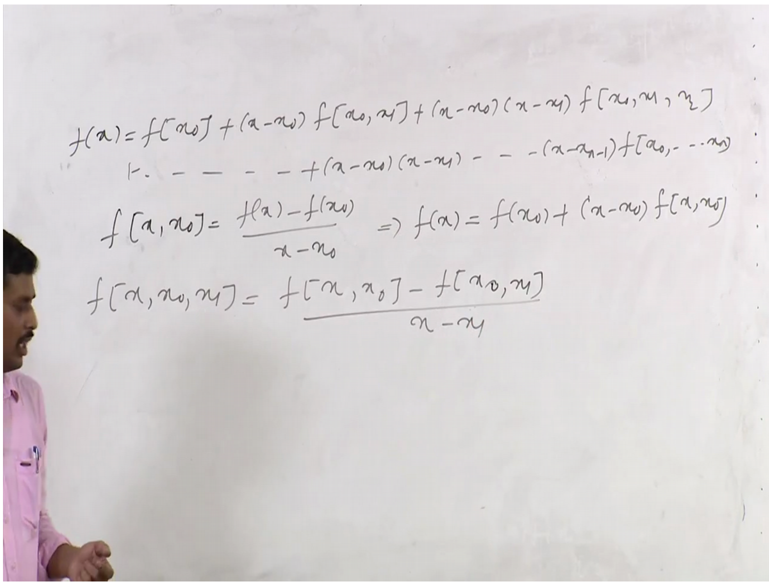

And if we just separate this f of x x 0 here this can be written in the form of f of x 0, x 1 plus x minus x 1 into f of x x 0, x 1 here. So if you just replace this function like f of x x 0 in this expansion here then we can just obtain this expression of f of x as f of x equals to f of x 0 plus x minus x 0 into f of x 0, x 1 plus x minus x 0, x minus x 1, f of x x 0 x 1 here.

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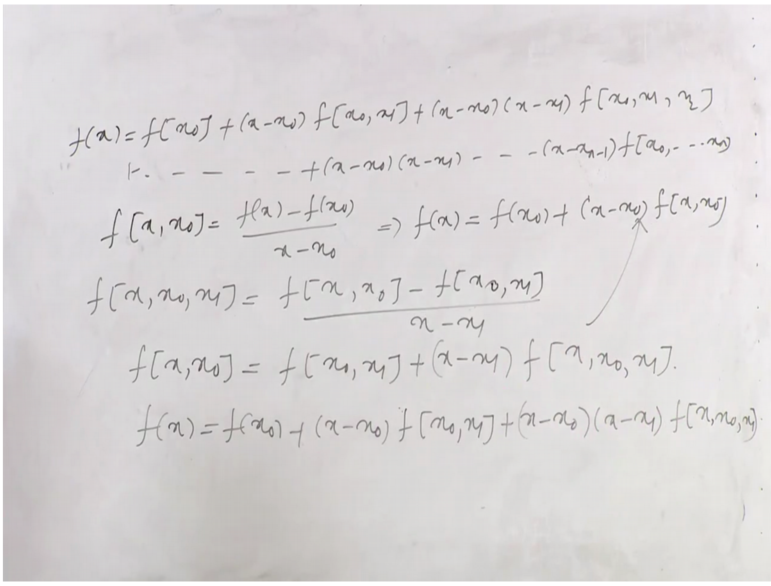

So likewise if you will just add one more point , finally we can just obtain this formula that can be given as f of x as f of x 0 plus x minus x 0, f of x 0 x 1 plus x minus x 0, x minus x 1, f of x 0, x 1, x 2 plus up to x minus x 0, x minus x 1 up to x minus x n minus 1, f of x 0, x 1 to x n plus r of x term or this complete expansion I can just write here that one as r of x term here.

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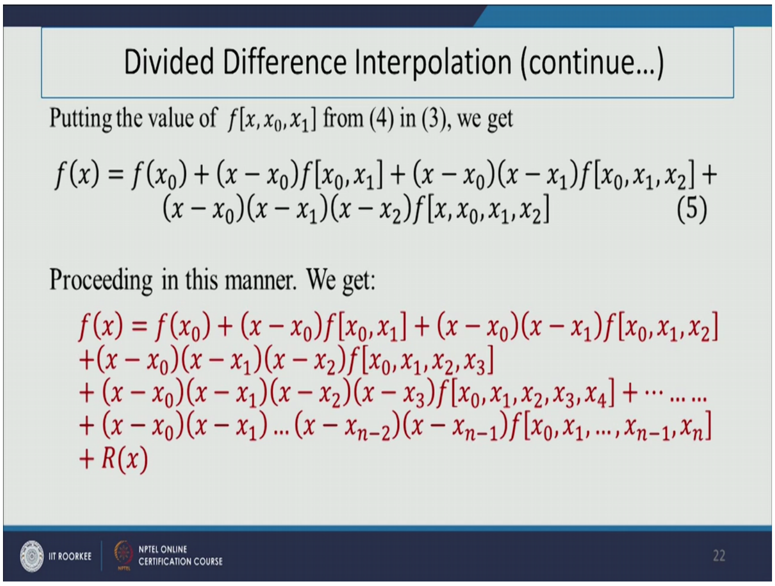

So if I will just write this r of x term so like our earlier error term expansion I can just write this error term as r of x here as x minus x 0, x minus x 1 to x minus x n and last term can be written as x x 0 to x n here. And obviously in a modified form it can be written as x minus x

0, x minus x 1 to x minus x n, f to the power n plus 1 zeta by n plus 1 factorial here where zeta should be lies between x 0 to x n here.

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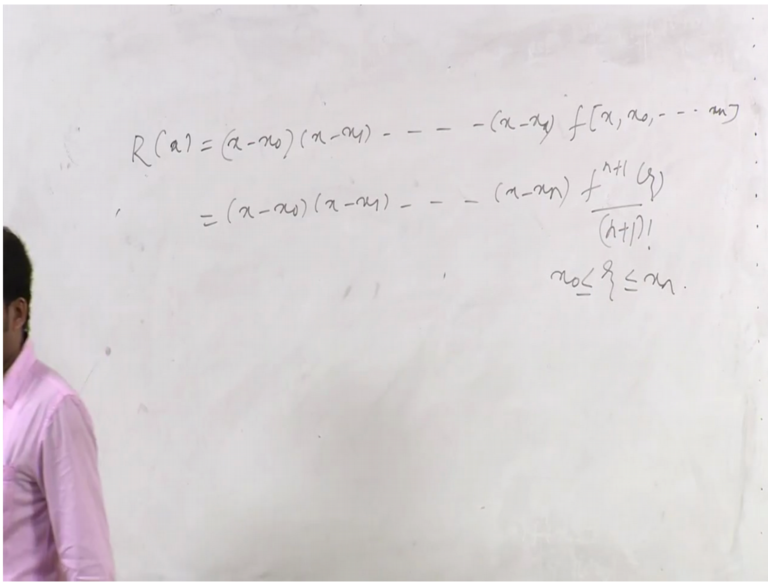

So Newton's divided difference formula also converts to Newton's forward difference formula for equidistance tabular points. So based on this interpolation formula we will just solve 1 problem that is using this like tabular values that has been given as x as minus 1, 0, 3, 6, 7 and f of x is given as 3 , minus 6, 39, 822, 1611 here.

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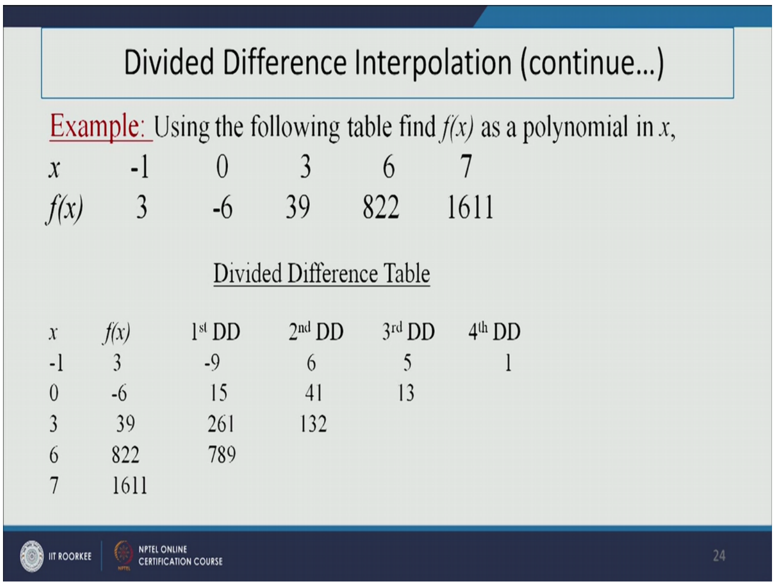

If you will just use this divided difference that as like the tabular values as x and f of x that as minus 1, 0, 3, then 6, 7 and the f of x values are like 3, minus 6, 39, 822, 1611. So if you just find the first divided difference here this means that minus 6, minus 3 by 0 minus of minus 1 here. So it can just give the value of minus 9 here. Since minus 6 minus 3 if I will just write here minus 6 minus 3 divided by like 0 minus of minus 1 here. So that is nothing but minus 9.

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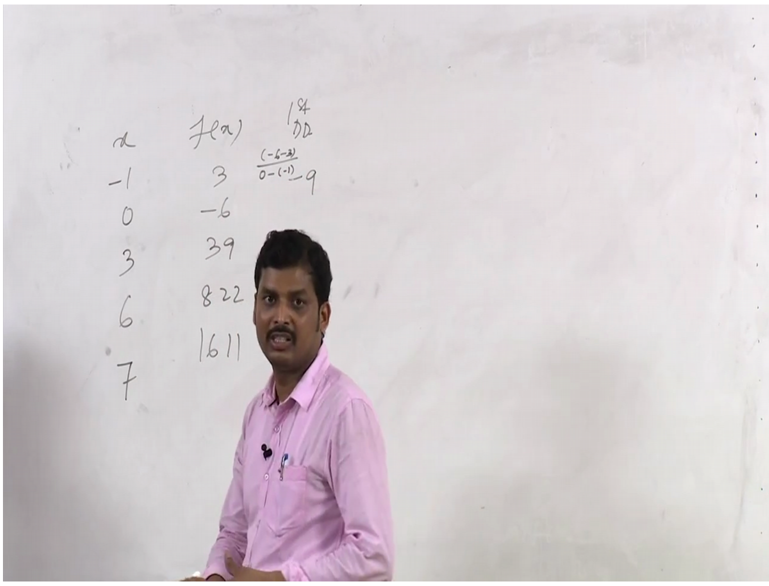

Similarly I can just find the difference like 39, minus of minus 6 by 3 minus 0 here and that will just give you value of 15 here. And if you will just take this next difference here I can just get that one as 261 here. This means that difference between these two here divided by difference of these two here.

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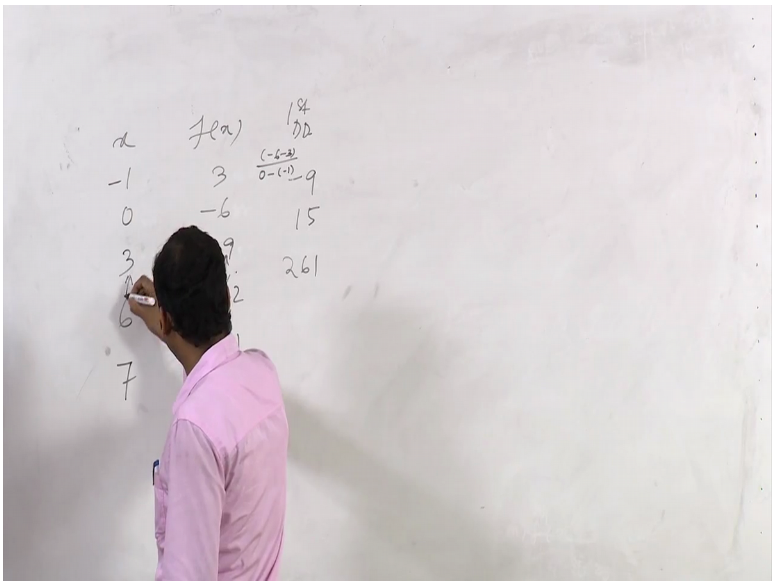

Similarly if you all just take the difference of 1611 minus 822 divided by 7 minus 6 here, I can just obtain this value as 789 here. Next we will just go for second order divided difference here that is 15 minus of minus 9 divided by 3 minus of minus 1 here. So for this one we have to consider like 3 minus of minus 1 here.

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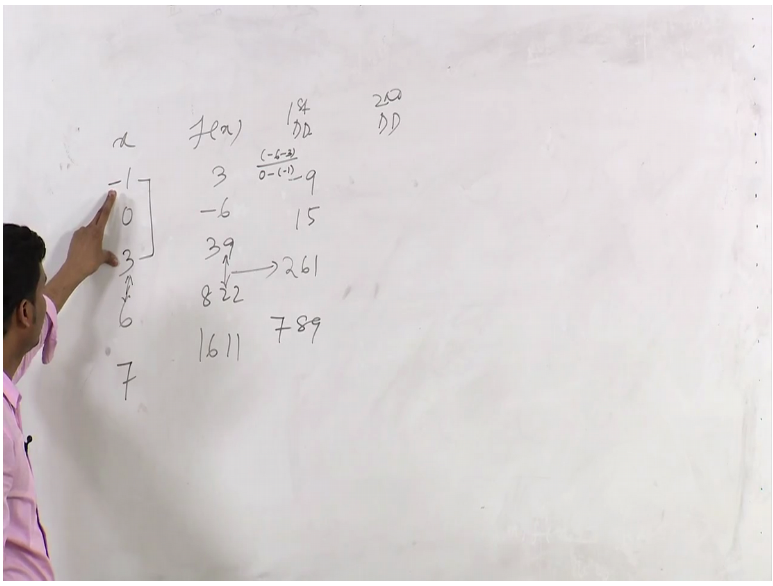

Second if I will just consider this divided difference of these two here, 261 minus 15 here, I have to consider the difference of 6 minus 0 here.

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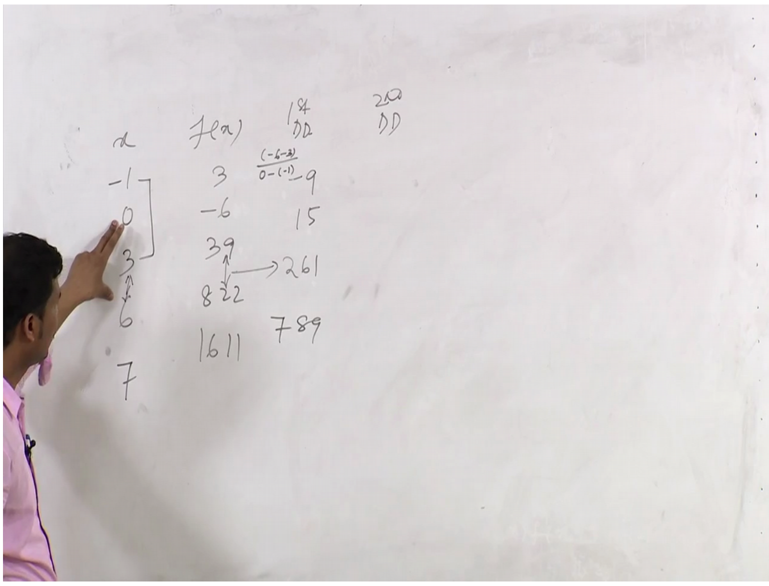

So the next one if I will just consider the difference of (se) 789 minus 261 here, I will just consider the difference of 7 minus 3 here.

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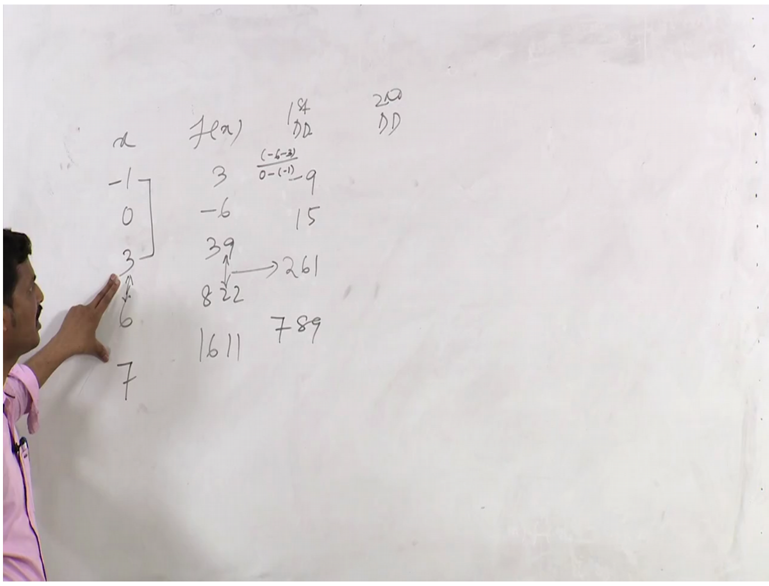

So this divided difference is (())(27:03) the values like 6 here, then 41, then 132 here. And again if I will just take the difference 41 minus 6 here, I have to take the difference of 6 minus of minus 1 here.

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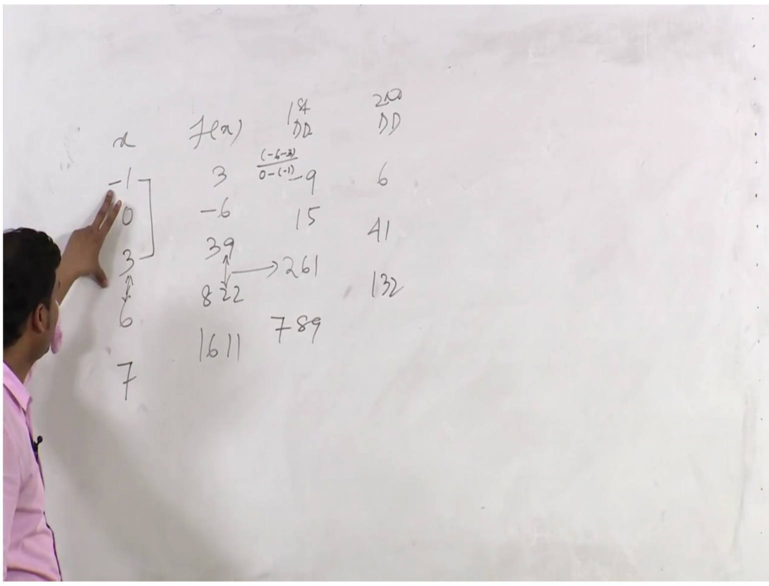

And if I will just take the difference of 132 minus 41 here, I will just take the difference of 7 minus 0 here. So these values are like 5 and 13 here. This is third order divided difference and fourth order divided difference that is 13 minus 5, 7 minus of minus 1 here. That is nothing but the value as 1 here.

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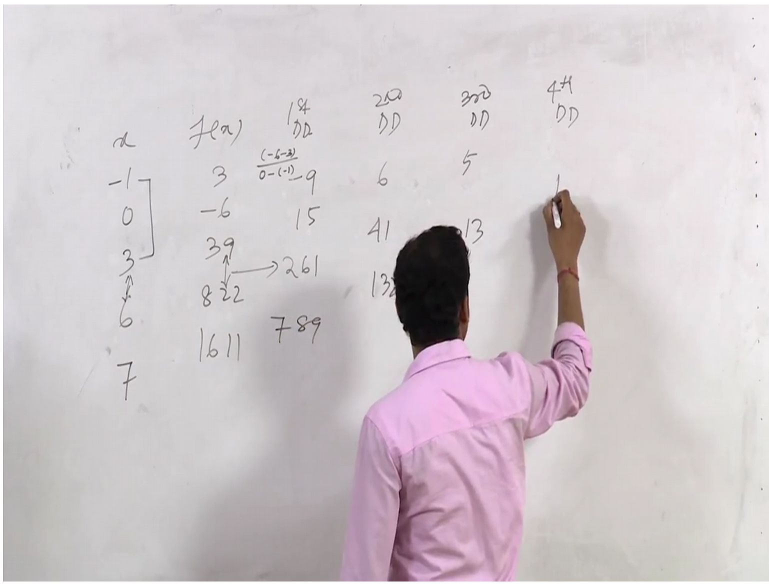

So if you will just put in the formula we can just obtained that one as f of x equals to first value as f of x 0 plus x minus x 0, f of x 0 x 1 plus x minus x 0, x minus x 1, f of x 0, x 1, x 2 plus x minus x 0, x minus x 1, x minus x 2, f of x 0, x 1, x 2, x 3 values. So if you all just put all these values then it can just provide us like one more point I have to add it out here. Since x 0 value is this one here, x 1, x 2, x 3, x 4 here.

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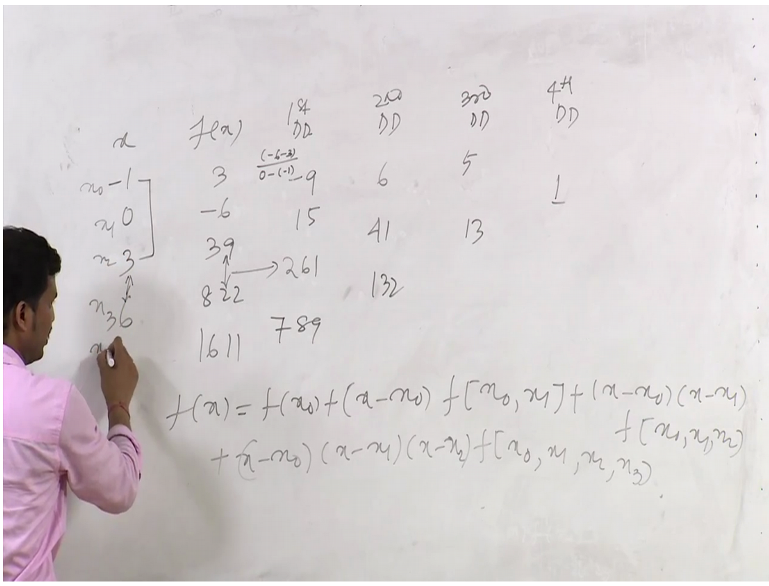

So I have to add one more term here that is x minus x 0, x minus x 1, x minus x 2, x minus x 3, f of x 0, x 1, x 2, x 3, x 4 here.

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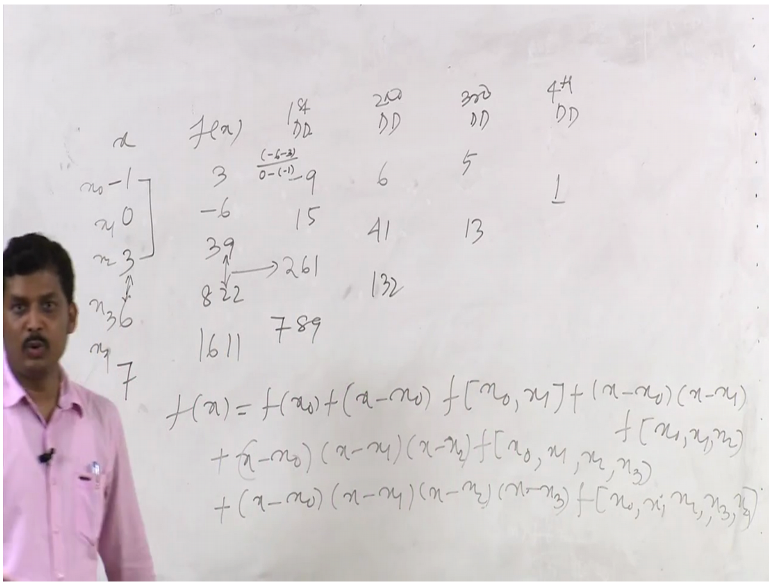

So if you all just put all these values then I can just obtain this polynomial of order 4 here and the final polynomial as x to the power 4 minus 3 x cube plus 5 x square minus 6 here. So if I will just use this divided difference I can just deal any type of like interpolation formula to get this solutions. So next class maybe I will just continue for this some advanced interpolation formula. So that I will just discuss in the next lecture, thank you for listen this lecture.