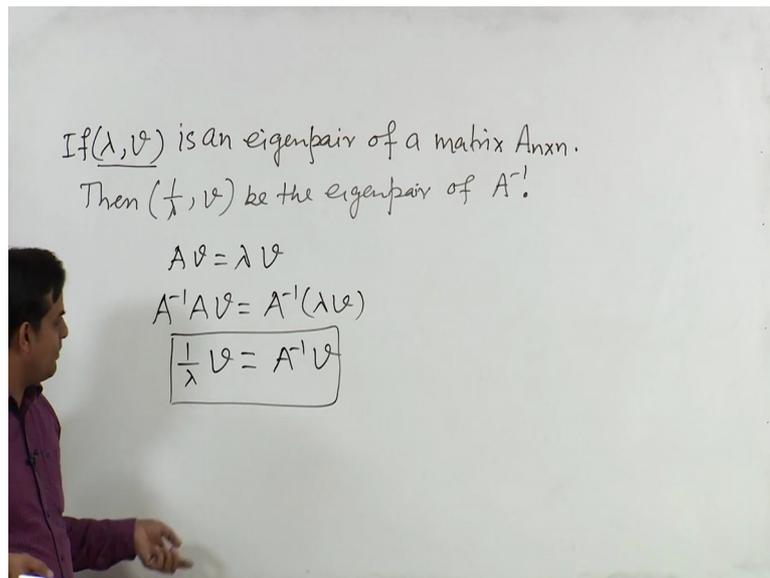


Numerical Methods
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Lecture 15
Inverse Power Method

Hello everyone, so welcome to the last lecture of this module and again in this lecture we are going to introduce another method that is the inverse power method sifted inverse power method. Those are the variants of power method for finding the eigenvalues those are not dominant for a given matrix. In the last lecture, we have talked about power method and we have seen that using the power method, we can find only the dominant eigenvalue. However, we have seen in the previous lecture that if we use method of deflation with power method we can compute the other eigenvalues than dominant for a given matrix. However, in method of deflation together with power method what you have to do? First you find the dominant eigenvalue and eigenvector, then generate a new matrix and then for that matrix again apply power method, which will give you the next dominant eigenvalue. Then make a new matrix again apply the power method and so on.

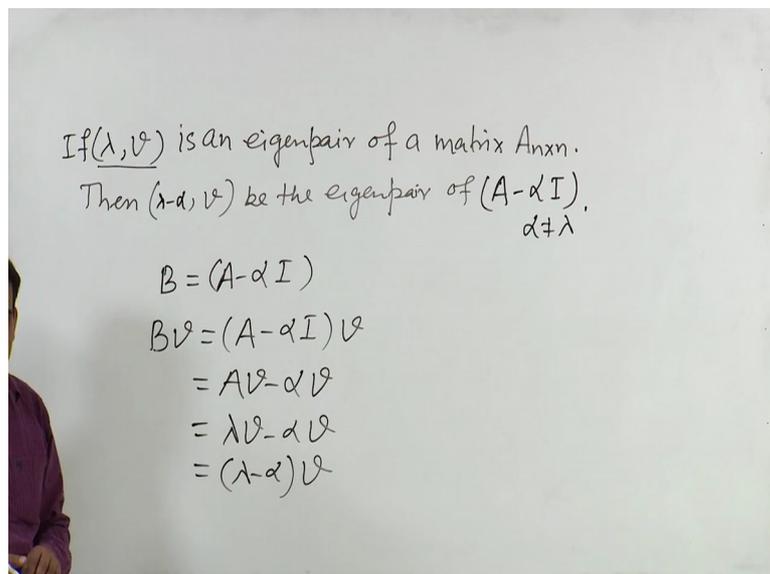
So if I am having a 10 by 10 matrix and suppose I want to find out the 5th eigenvalue of this matrix, which in order of decreasing order. So what I have to do? For doing this, I need to apply 5 times power method to a 10 by 10 matrix and 4 times deflation transformation I need to use. So hence it will be very expensive in terms of computational course. So inverse and shifted inverse power methods give us algorithms for computing the eigenvalues and eigenvectors, those are not dominant directly in one with step by using the power method or process of power method just once.

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So basically these methods are based on the two principles. The first principle is if λ and v is an eigenpair of a square matrix A of order n then, λ^{-1} , which will be basically $1/\lambda$ and v be the eigenpair of matrix A inverse. This is one of the rule and this can be shown very easily as it is the eigenpair for A . So I can write Av equals to λv . If A is invertible then I can multiply both side by A inverse. So inverse into A into v will become, A inverse into λv . What I can do λ is a scalar, I can take out, so I can write and then I can divide the whole thing by $1/\lambda$ both sides, so $1/\lambda v$ will become A inverse v , it means the eigenvalue of A inverse is $1/\lambda$ and corresponding eigenvector is v .

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The other result we shift the eigenvalue that is, it is saying keep lambda V is an eigenpair of a matrix A then, lambda minus alpha together with eigenvector V will be the eigenpair of matrix A minus alpha I for the scalar alpha and here, alpha not equals to lambda. So this again we can show, we are having a new matrix B that is A minus alpha I, so BV will become A minus alpha I into V and this will become AV minus alpha V and we know that lambda is an eigenvalue of A, so it will be lambda minus alpha V or I can write lambda minus alpha into V. So it means the eigenvalue of B is lambda minus alpha and the eigenvector is V, which is the same as of A and V is A minus alpha I. So with these two results we will start our shifted inverse power method or inverse power method.

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Shifted inverse power method

Theorem 1
Let (λ, \mathbf{v}) is an eigenpair of \mathbf{A} . If $\alpha \neq \lambda$, then $(\frac{1}{\lambda-\alpha}, \mathbf{v})$ is the eigenpair of the matrix $(\mathbf{A} - \alpha\mathbf{I})^{-1}$.

Theorem 2
Suppose that $\mathbf{A}_{n \times n}$ has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Consider the value λ_j . Then, a constant α can be chosen so that $\mu_1 = \frac{1}{\lambda_j - \alpha}$ is the dominant eigenvalue of $(\mathbf{A} - \alpha\mathbf{I})^{-1}$. Furthermore, if \mathbf{v}_0 is chosen appropriately, then the sequence $\{\mathbf{v}_k = [v_1^{(k)}, v_2^{(k)}, \dots, v_n^{(k)}]\}$ and c_k given by

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So suppose that A as distinct eigenvalues lambda1, lambda2, up to lambda n. Consider the eigenvalue lambda j, suppose I need to calculate or I need to compute this eigenvalue. So then a constant alpha can be chosen so that Mu1 is 1 upon lambda j minus alpha is the dominant eigenvalue of A minus alpha I inverse, further more if we chose V0 carefully then the sequence Vk that is V1k, V2k, Vnk having the components.

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Shifted inverse power method

Theorem 2 cont...

$$\mathbf{y}_k = (\mathbf{A} - \alpha \mathbf{I})^{-1} \mathbf{v}_k \quad (1)$$

$$\mathbf{v}_{k+1} = \frac{1}{c_{k+1}} \mathbf{y}_k \quad (2)$$

where

$$c_{k+1} = x_j^{(k)} \text{ and } x_j^{(k)} = \max_{1 \leq i \leq n} \{|x_i^{(k)}|\} \quad (3)$$

will converge to the dominant eigenpair (μ_1, \mathbf{v}_j) of the matrix $(\mathbf{A} - \alpha \mathbf{I})^{-1}$.
Finally, the corresponding eigenvalue will be given by

$$\lambda_j = \frac{1}{\mu_1} + \alpha \quad (4)$$

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And c_k given by y_k equals to $\mathbf{A} - \alpha \mathbf{I}$ inverse \mathbf{V}_k and \mathbf{V}_{k+1} is $\frac{1}{c_{k+1}} \mathbf{y}_k$. This is the power method only power method for $\mathbf{A} - \alpha \mathbf{I}$ inverse and finally we can calculate the j th eigenvalue that is λ_j as $\frac{1}{\mu_1} + \alpha$ from the μ_1 .

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Shifted inverse power method

Proof:

Suppose the eigenvalues satisfy $\lambda_1 < \lambda_2 < \dots < \lambda_n$. Also, let α be the number such that $\alpha \neq \lambda_j$ but closer to λ_j as compared to other eigenvalues i.e

$$|\lambda_j - \alpha| < |\lambda_i - \alpha|, \quad i = 1, 2, \dots, j-1, j+1, \dots, n \quad (5)$$

Using theorem 1, $(\frac{1}{\lambda_j - \alpha}, \mathbf{v})$ is the eigenpair of matrix $(\mathbf{A} - \alpha \mathbf{I})^{-1}$. From (5),

$$\frac{1}{(\lambda_j - \alpha)} < \frac{1}{(\lambda_i - \alpha)} \text{ for } i \neq j$$

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Now what should be the choice of α ? We cannot choose α just like equals to λ_j , but to be the μ_1 as the dominant eigenvalue. So it should be very large and for this α should be quite close to λ_j . So for example, if I want to find out eigenvalue 4, α should be somewhere for 4.2 or 3.8 or 4.3, 3.7 like that. So proof of this result can be given very easily, suppose the eigenvalue satisfy $\lambda_1 < \lambda_2 < \dots < \lambda_n$. Also, let α be the number such that $\alpha \neq \lambda_j$,

but very close to λ_j as compare to other eigenvalues then, I can write that $\lambda_j - \alpha < \lambda_i - \alpha$ for rest of the i from 1 to $j - 1$ and then $j + 1$ to n .

Then using the result which I have derive on the board I can say that $\lambda_j - \alpha$ will be the eigenvalue of $A - \alpha I$ inverse and the corresponding eigenvector will remain V , which is the eigenvector of original A to corresponding to the eigenvalue λ_j . So more over we can say that $\lambda_j - \alpha$ will be less than $\lambda_i - \alpha$ and hence, $\lambda_j - \alpha$ will be the dominant eigenvalue of the matrix $A - \alpha I$ inverse. So how it will work, suppose I want to find out an eigenvalue of a given matrix let us say, some eigenvalue λ_j . So I will choose α close to this eigenvalue and which is not close to rest of the eigenvalues.

Now question arise without knowing about eigenvalues how we will choose α , because each and every time I am saying that α should be close to λ_j compare to any other λ_i . So how to do it without looking or without knowing about the eigenvalues so this will come from the gershgorin disc just by looking on the given matrix I can say about the range of eigenvalue or in which disc eigenvalue will lie and from there I can get an idea.

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The slide is titled "Shifted inverse power method". It contains the following text:

So far we have discussed shifted inverse power method with a fixed shift. The algorithm for this is given as

Algorithm shifted inverse power method with fixed shifts

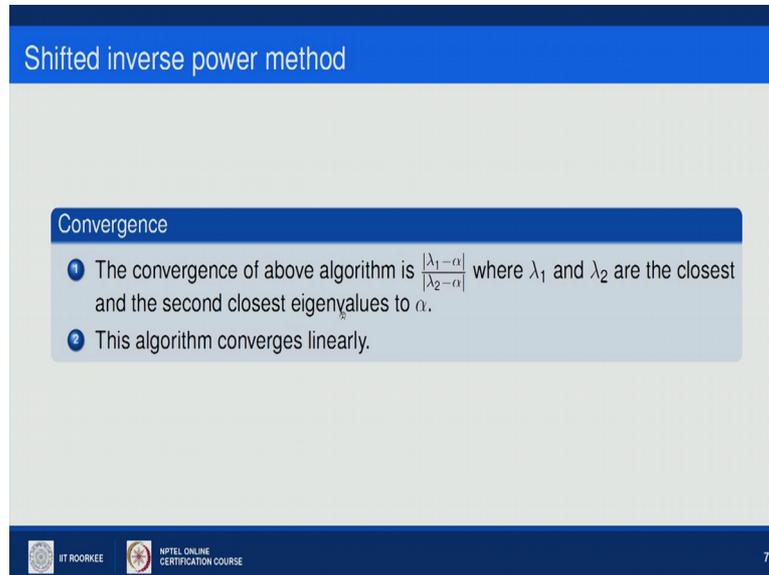
Choose an initial $\mathbf{v}_0 \neq 0$.
 For $k = 0, 1, 2, \dots$
 Compute $\mathbf{y}_k = (\mathbf{A} - \alpha \mathbf{I})^{-1} \mathbf{v}_k$ and $c_{k+1} = x_j^{(k)}$, $x_j^{(k)} = \max_{1 \leq j \leq n} \{|x_j^{(k)}|\}$
 Set $\mathbf{v}_{k+1} = \frac{1}{c_{k+1}} \mathbf{y}_k$

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So the algorithm should be like this, first of all you have to choose initial \mathbf{V}_0 , which should be a non-zero then for k equals to 0,1, 2 you will find out \mathbf{y}_k and \mathbf{y}_k will be $(A - \alpha I)^{-1} \mathbf{V}_k$ from here c_{k+1} will be the largest component of vector \mathbf{y}_k in terms of

absolute value and then you can define your V_{k+1} as 1 upon C_{k+1} into y_k . So the shifted inverse power method with this with this fixed shift α is nothing just power method where the matrix A is replaced with a new matrix $A - \alpha I$ inverse.

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The slide is titled "Shifted inverse power method". It contains a section titled "Convergence" with two bullet points:

- 1 The convergence of above algorithm is $\frac{|\lambda_1 - \alpha|}{|\lambda_2 - \alpha|}$ where λ_1 and λ_2 are the closest and the second closest eigenvalues to α .
- 2 This algorithm converges linearly.

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The convergence of this algorithm is given by this $\lambda_1 - \alpha$ upon $\lambda_2 - \alpha$, where λ_1 and λ_2 are the closest and the second closest eigenvalue to α . So for example, if you are choosing a matrix having eigenvalue 5, 8 and 10 and I am choosing α is 4. So it will be convergence will be 1 upon that is $5 - 4$ upon $8 - 4$. So 1 upon 2 , so it will be linear okay, so it will be always linear. It will be something like always between 0 and 1, okay or up to 1.

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The slide is titled "Shifted inverse power method" in a blue header. Below the title, there is a blue box containing the text "Algorithm shifted inverse power method with variant shifts". The main content of the slide is on a light gray background and includes the following text: "Choose an initial $\mathbf{v}_0 \neq \mathbf{0}$. Given $\alpha_0 = \alpha$. For $k = 0, 1, 2, \dots$ Compute $\mathbf{y}_k = (\mathbf{A} - \alpha_k \mathbf{I})^{-1} \mathbf{v}_k$ and $c_{k+1} = x_j^{(k)}$, $x_j^{(k)} = \max_{1 \leq j \leq n} \{|x_j^{(k)}|\}$ Set $\mathbf{v}_{k+1} = \frac{1}{c_{k+1}} \mathbf{y}_k$ and $\alpha_{k+1} = \alpha_k + \frac{1}{c_{k+1}}$. This algorithm is locally quadratic convergent." At the bottom of the slide, there are logos for "IIT ROORKEE" and "NITEL ONLINE CERTIFICATION COURSE", and the number "8" in the bottom right corner.

Now we can use this shifted inverse power method with variable shift also. Variable shift means we can update our alpha also in the earlier algorithm we have a fixed alpha chosen in the beginning and we are using that; however here we can update our alpha also to improve the convergence of a given method. So here the algorithm will be like that you take a non-zero vector \mathbf{V}_0 and a initial alpha that is let us say alpha not, then compute \mathbf{y}_k , \mathbf{y}_k will be $\mathbf{A} - \alpha_k \mathbf{I}$ inverse into \mathbf{V}_k and c_{k+1} will be the maximum component in terms of absolute value of \mathbf{y}_k like if \mathbf{V} over for a 3 by 3 matrix \mathbf{y}_k is coming 1 minus 2 minus 4.

So here c_k will become minus 4 and then you set your \mathbf{V}_{k+1} as $\frac{1}{c_{k+1}} \mathbf{y}_k$ and here, the same time you will update your alpha, alpha will becomes $\alpha_{k+1} = \alpha_k + \frac{1}{c_{k+1}}$ and tis method is locally quadratic convergent having the quadratic or second order convergence locally.

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Shifted inverse power method

Algorithm shifted inverse power method with Rayleigh Quotient

Choose an initial $\mathbf{v}_0 \neq 0$ with $\|\mathbf{v}_0\|_2 = 1$.
Compute $\alpha_0 = \mathbf{v}_0^T \mathbf{A} \mathbf{v}_0$
For $k = 0, 1, 2, \dots$
Compute $\mathbf{y}_k = (\mathbf{A} - \alpha_k \mathbf{I})^{-1} \mathbf{v}_k$
Set $\mathbf{v}_{k+1} = \frac{1}{\|\mathbf{y}_k\|_2} \mathbf{y}_k$ and $\alpha_{k+1} = \mathbf{v}_{k+1}^T \mathbf{A} \mathbf{v}_{k+1}$.

This algorithm is locally cubically convergent for symmetric matrix \mathbf{A} .

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We can apply the shifted inverse power method with Rayleigh Quotient also and that is like choose an initial value \mathbf{V}_0 that is not equals to 0 such a way that, this is having the unit length then compute alpha not, which will be the Rayleigh Quotient of this vector \mathbf{V}_0 and that will be \mathbf{V}_0 transpose \mathbf{A} into \mathbf{V}_0 . Now for k equals to 0, 1,2 compute \mathbf{y}_k . So \mathbf{y}_k will become \mathbf{A} minus alpha $k\mathbf{I}$ inverse into \mathbf{V}_k . Set \mathbf{V}_k plus 1 as 1 upon \mathbf{y}_k 2 \mathbf{y}_k and then alpha k plus 1 can be updated as \mathbf{V}_k plus 1 transpose \mathbf{A} into \mathbf{V}_k plus 1. So in each iteration again updating my \mathbf{V} and here I am updating my alpha by the definition of Rayleigh Quotient and this method is having (cubic) this method is cubic convergent in case of symmetric matrices.

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Shifted inverse power method

Example

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}$$

The eigenvalues of \mathbf{A} are given by $\lambda_1 = 4$, $\lambda_2 = 2$ and $\lambda_3 = 1$.
For $\lambda_1 = 4$, take $\alpha = 4.2$ and $\mathbf{v}_0 = [1, 1, 1]^T$.
Find the matrix $\mathbf{A} - 4.2\mathbf{I}$ and solve the following system:

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So let us take an example of separate inverse power method for finding the eigenvalue one of the eigenvalue of a 3 by 3 matrix and here, we will use the fixed alpha version of the inverse shifted inverse power method hence, with fix shift. So the eigenvalue of this matrix is (4, 2, 1) the dominant eigenvalue is 4. So suppose I take alpha 4.2, so if I take alpha 4.2 my shifted inverse power method will converge to eigenvalue 4 and corresponding eigenvector. So for lambda1 equals to 4, I can define my A minus alpha will become A minus 4.2 I and then I will apply power method on this A minus 4.2 I with initial value 1,1, 1. So this I have taken in this way.

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Shifted inverse power method

Example

$$\begin{bmatrix} -4.2 & 11 & -5 \\ -2 & 12.8 & -7 \\ -4 & 26 & -14.2 \end{bmatrix} \mathbf{y}_0 = \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We get $\mathbf{y}_0 = [-9.545454545, -14.09090909, -23.18181818]^T$. Also, calculating c_1 and \mathbf{v}_1 using earlier formulas, we have $c_1 = -23.18181818$ and $\mathbf{v}_1 = [0.4117647059, 0.6078431373, 1]^T$.

We continue in this way until the sequences c_k and \mathbf{v}_k converges.

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And then using this we continue in this way until the sequence C_k and V_k converges. So y_0 is this value then C_1 comes -23.18181818 and then V_1 is this particular vector.

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Shifted inverse power method

Example

So, after 8 iterations we have $\mu_1 = -5$ which is the dominant eigenvalue of $(\mathbf{A} - 4.2\mathbf{I})^{-1}$ and \mathbf{v}_k converges to $\mathbf{v}_1 = \left[\frac{2}{5}, \frac{3}{5}, 1\right]^T$.
Hence, the eigenvalue is given by

$$\lambda_1 = \frac{1}{\mu_1} + \alpha = \frac{1}{-5} + 4.2 = 4$$

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After 8 iterations we have μ_1 equals to minus 5, which is the dominant eigenvalue of $\mathbf{A} - 4.2\mathbf{I}$ inverse and then \mathbf{v}_k converges to \mathbf{v}_1 that is 2 by 5, 3 by 5 and 1. So hence the eigenvalue is given by 1 upon μ_1 plus alpha that is minus 1 upon 5 plus 4.2 that is 4, which verify our claim that for a given alpha it will converge to the closest eigenvalue.

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Shifted inverse power method

Example

For $\lambda_2 = 2$, take $\alpha = 2.1$ and $\mathbf{v}_0 = [1, 1, 1]^T$.
Find the matrix $\mathbf{A} - 2.1\mathbf{I}$ and solve the following system:

$$\begin{bmatrix} -2.1 & 11 & -5 \\ -2 & 14.9 & -7 \\ -4 & 26 & -12.1 \end{bmatrix} \mathbf{y}_0 = \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We get $\mathbf{y}_0 = [11.05263158, 21.57894737, 42.63157895]^T$. Also, calculating c_1 and \mathbf{v}_1 using earlier formulas, we have $c_1 = 42.63157895$ and $\mathbf{v}_1 = [0.2592592593, 0.5061728395, 1]^T$.

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Shifted inverse power method

Example

Similarly as for λ_1 , we will continue in this way until the sequences c_k and \mathbf{v}_k converges.

So, after 6 iterations we have $\mu_1 = -10$ which is the dominant eigenvalue of $(\mathbf{A} - 2.1\mathbf{I})^{-1}$ and \mathbf{v}_k converges to $\mathbf{v}_2 = [\frac{1}{4}, \frac{1}{2}, 1]^T$.

Hence, the eigenvalue is given by

$$\lambda_2 = \frac{1}{\mu_1} + \alpha = \frac{1}{-10} + 2.1 = 2.$$

In the same way, we can proceed for λ_3 .



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If I take alpha equals to 2.1 and I apply the same process, it converges to eigenvalue 2 with corresponding eigenvector 1 by 4, 1 by 2 and 1.

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Inverse power method

- If λ is an eigenvalue of \mathbf{A} and \mathbf{A} is non-singular, then λ^{-1} will be eigenvalue of \mathbf{A}^{-1} .
- It provides a way to find the smallest eigenvalue of \mathbf{A} .
- The eigenvalues of \mathbf{A} can be arranged as

$$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_{n-1}| > |\lambda_n| > 0$$
- Thus, the eigenvalues of \mathbf{A}^{-1} can be arranged as

$$|\lambda_n^{-1}| > |\lambda_{n-1}^{-1}| \geq \dots \geq |\lambda_1^{-1}| > 0$$



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So far we were talking about shifted inverse power method. Let us take a and in shifted inverse power method we need to calculate inverse and like that. Let us take a other variant of this shifted inverse power method just inverse power method and this we can use for finding the smallest eigenvalue of a given matrix and the corresponding eigenvector and here we are using the result that if lambda and V be the eigenpair of a matrix A then 1 upon lambda and V will be the eigenpair of A inverse.

So if lambda is an eigenvalue or lambda is the dominant dominant eigenvalue of a given matrix A then 1 upon lambda will be the dominant eigenvalue of then lambda inverse will be

the eigenvalue of A inverse and hence, $1/\lambda$ will be the smallest eigenvalue of the matrix A . So if we apply the power method on A inverse what we can get we can get the smallest eigenvalue of the matrix A .

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Inverse power method

- The inverse power method has advantage over power method that it can approximate any eigenvalue λ_i , $i = 1, 2, \dots, n$.
- Consider $\mathbf{y}_0 (\neq 0) \in \mathbb{R}^n$. \mathbf{y}_0 can be expressed as linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.
- Applying power method on \mathbf{A}^{-1} , we have

$$\mathbf{z}_{k+1} = \mathbf{A}^{-1} \mathbf{y}_k \quad (6)$$

$$\mathbf{y}_{k+1} = \frac{\mathbf{z}_{k+1}}{m_{k+1}} \quad (7)$$

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So the inverse power method has advantage over power method that it can approximate any eigenvalue. Consider \mathbf{y}_0 which is a non-zero eigenvector vector in \mathbb{R}^n and \mathbf{y}_0 can be expressed as linear combination of eigenvectors of A and then applying power method on A inverse we can get $\mathbf{z}_{k+1} = \mathbf{A}^{-1} \mathbf{y}_k$ and $\mathbf{y}_{k+1} = \frac{\mathbf{z}_{k+1}}{m_{k+1}}$.

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Inverse power method

- Which gives an approximation to the dominant eigenvalue of \mathbf{A}^{-1} in modulus i.e. the smallest eigenvalue of \mathbf{A} in modulus.
- However, we don't need to find \mathbf{A}^{-1} to find smallest eigenvalue of \mathbf{A} .

We write (6) as

$$\mathbf{A} \mathbf{z}_{k+1} = \mathbf{y}_k \quad (8)$$

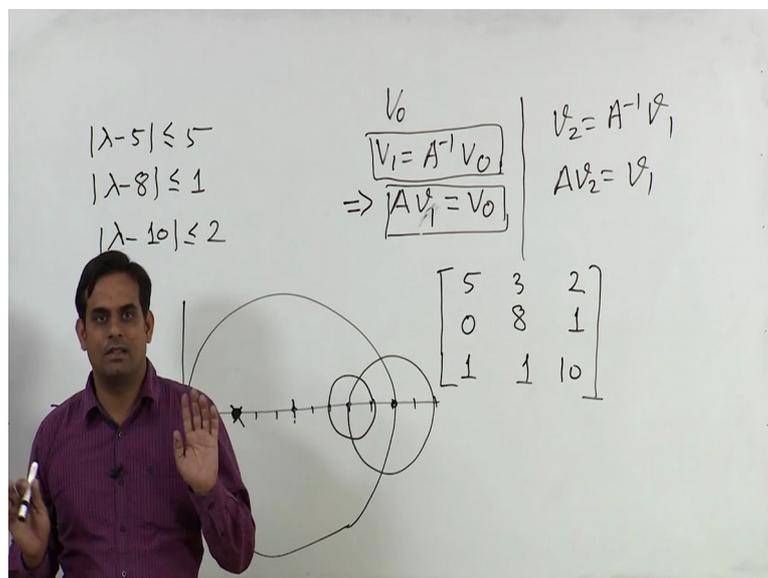
(8) is the system of linear equations which is solved to find \mathbf{z}_{k+1} . Also, normalization is done according to (7).
The eigenvalue can be given in the same way as in the power method.

$$\mu = \frac{1}{\lambda_i} = \lim_{k \rightarrow \infty} \frac{(\mathbf{y}_{k+1})_r}{(\mathbf{y}_k)_r}, \quad r = 1, 2, \dots, n \quad (9)$$

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So in this way which gives the approximation to the dominant eigenvalue of A inverse in modulus that is the smallest eigenvalue of A in modulus. However, here we do not need to find A inverse to find smallest eigenvalue of A, because if you are having a 10 by 10 matrix, so finding the inverse of a 10 by 10 matrix is computationally expensive and I will not prefer, suppose I want to find out the smallest eigenvalue of a 10 by 10 matrix A. So what is the inverse power method I need to calculate A inverse and the dominant eigenvalue of A inverse will be the smallest eigenvalue of A, but we need to find out A inverse. Here, we do not require to find out A inverse at all.

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What we can? (We are) what we are? We are starting with a V_0 and what we are doing? We are finding V_1 as A inverse into V_0 . So what I will do? Here, I will use multiply both side by A. So my AV_1 will become V_0 . So instead of finding V_1 with this iterative process or from this multiplication of a matrix with a column vector, I will be having a linear system of equation and V_0 is known to you, A is known to you. So you can find out V_1 directly from here without using A inverse. Then in the next equation, your V_2 will be A inverse into V_1 . So what you can say you can solve this system AV_2 equals to V_1 and from here, you will find out the next iteration of the inverse power method that is your V_2 and so on. So here no need to calculate A inverse at any stage; however we need to solve a linear system of equation in each and every stage.

(Refer Slide Time: 21:11)

Inverse power method

Example

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

We have inverse of above matrix as

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Using $\mathbf{y}^{k+1} = \mathbf{A}^{-1}\mathbf{v}^{(k)}$, $k = 0, 1, \dots$

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So let us take an example of this, we are taking this matrix. So A is (2, -1, 0) (-1, 2, -1) (0, -1, 2). Here, we are doing it by calculating A inverse, but we can do it without calculating A inverse also.

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Inverse power method

Example

Let $\mathbf{v}^{(0)} = [1, 1, 1]^T$. We have

$$\mathbf{y}^{(1)} = [1.5, 2, 1.5]^T, \mathbf{v}^{(1)} = [0.75, 1, 0.75]^T$$
$$\mathbf{y}^{(2)} = [1.25, 1.75, 1.25]^T, \mathbf{v}^{(2)} = [0.7143, 1, 0.7143]^T$$
$$\mathbf{y}^{(3)} = [1.2143, 1.7143, 1.2143]^T, \mathbf{v}^{(3)} = [0.7083, 1, 0.7083]^T$$
$$\mathbf{y}^{(4)} = [1.2083, 1.7083, 1.2083]^T, \mathbf{v}^{(4)} = [0.7073, 1, 0.7073]^T$$

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So if we are doing it with A inverse and starting with and starting with an initial solution 1, 1, 1 we are getting our first approximation as y1 that is 1.52 to 1.5 and here, if I divide it by 2, it is 1 upon 2 into 2.751 and 0.75 transpose. So first approximation of the eigenvalue is coming 2 and eigenvector is this one.

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Inverse power method

Example

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

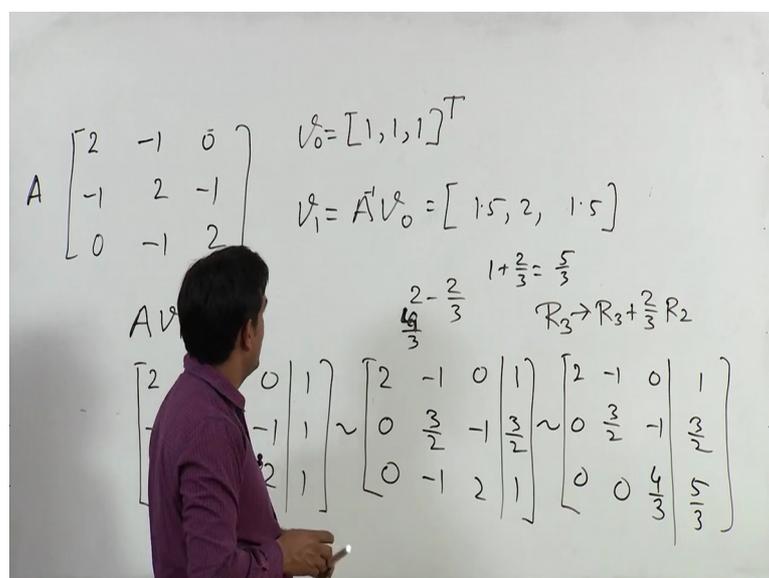
We have inverse of above matrix as

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Using $\mathbf{y}^{k+1} = \mathbf{A}^{-1}\mathbf{v}^{(k)}$, $k = 0, 1, \dots$



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However, here we are doing it with A inverse, but if I do it without finding A inverse then the system can be solved with less computational effort, for example my original matrix is (2,-1,0) (-1,2,-1) and then we are having (0,-1,2). So this is A, V not is 1,1, 1 transpose and I am finding V1 as A into V0, which is coming -1.5, 2, -1.5, I think so which is coming 1.5, 2 and 1.5. Now here what I am doing, sorry it is A inverse into V0, so doing this I need to calculate inverse of tis matrix, but if I solve this system A into V1 equals to V0 then my augmented matrix becomes (2, -1, 0) (-1,2,-1) (0,-1,2) and then (1,1,1).

And after solving this, let us say solve it with Gauss elimination. So (2, -1, 0) and then R2 will be replaced by R2 plus 1 by 2 times R1. So this will be 0, 2,-1 by 2 will be 3 by 2 minus 1, 1. 1 plus 1 by 2 will be 3 by 2, then this is already 0, -1, 2, 1 and then this can be change

into just by R3 replaced by R3 plus 2 by 3 R2 by using this row operation elementary row operation then I will get 2,-1, 0, 1, 0, 3 by 2, -1, 3 by 2 and then 0, 0, 2,-2 by 3, 4 by 3 and then this will become 1 plus 2 by 3. So 1 plus 2 by 3 will be 5 by 3 and from here we will get this vector V1.

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Inverse power method

Example

Let $\mathbf{v}^{(0)} = [1, 1, 1]^T$. We have

$$\mathbf{y}^{(1)} = [1.5, 2, 1.5]^T, \mathbf{v}^{(1)} = [0.75, 1, 0.75]^T$$

$$\mathbf{y}^{(2)} = [1.25, 1.75, 1.25]^T, \mathbf{v}^{(2)} = [0.7143, 1, 0.7143]^T$$

$$\mathbf{y}^{(3)} = [1.2143, 1.7143, 1.2143]^T, \mathbf{v}^{(3)} = [0.7083, 1, 0.7083]^T$$

$$\mathbf{y}^{(4)} = [1.2083, 1.7083, 1.2083]^T, \mathbf{v}^{(4)} = [0.7073, 1, 0.7073]^T$$

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So continuing this I will calculate y2 and then V2, y3 V3, y4 V4.

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Inverse power method

Example

After 4 iterations, we have

$$\mu = \frac{(\mathbf{y}^{(4)})_r}{(\mathbf{v}^{(3)})_r} = (1.7059, 1.7083, 1.7059)$$

Hence, $\mu \approx 1.71$ and $\lambda = \frac{1}{\mu} \approx 0.5848$.
 Since, $|\mathbf{A} - 0.5848\mathbf{I}| \approx 0$, $\lambda = 0.5848$ will be the required eigenvalue. And, the corresponding eigenvector is $[0.7073, 1, 0.7073]^T$.
 The smallest eigenvalue of \mathbf{A} is $2 - \sqrt{2} = 0.5858$.

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And after 4 iterations we observe that my system is converging to Mu equals to 1.71 and lambda equals to 1 upon Mu that is 0.5848. Since, A minus 0.5848I will be 0, so here lambda equals to 0.5848 will be the required eigenvalue that is the smallest eigenvalue and the

corresponding eigenvector is $0.707, 1$ and 0.7073 . The smallest eigenvalue of A will be $2, -\sqrt{2}$ that is again true which is as we have computed numerically.

So in this lecture we have learned the two variants of power method that is the shifted inverse power method and inverse power method for finding the eigenvalues other than dominant for a given matrix. This ends the module 3 of this course and in this module, we have learned various methods for calculating power calculating eigenvalues and eigenvectors like, we started with Jacobi method, then we have learn power method, power method with fixed shift, inverse power method with variable shift, inverse power method with Rayleigh Quotient and finally, the classical inverse power method for finding the smallest eigenvalue of a given matrix. In the next lecture, we will talk about interpolation till the end bye. Thank you very much.