Numerical Methods

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Lecture 15

Inverse Power Method

Hello everyone, so welcome to the last lecture of this module and again in this lecture we are going to introduce another method that is the inverse power method sifted inverse power method. Those are the variants of power method for finding the eigenvalues those are not dominant for a given matrix. In the last lecture, we have talked about power method and we have seen that using the power method, we can find only the dominant eigenvalue. However, we have seen in the previous lecture that if we use method of deflation with power method we can compute the other eigenvalues than dominant for a given matrix. However, in method of deflation together with power method what you have to do? First you find the dominant eigenvalue and eigenvector, then generate a new matrix and then for that matrix again apply power method, which will give you the next dominant eigenvalue. Then make a new matrix again apply the power method and so on.

So if I am having a 10 by 10 matrix and suppose I want to find out the 5^{th} eigenvalue of this matrix, which in order of decreasing order. So what I have to do? For doing this, I need to apply 5 times power method to a 10 by 10 matrix and 4 times deflation transformation I need to use. So hence it will be very expensive in terms of computational course. So inverse and shifted inverse power methods give us algorithms for computing the eigenvalues and eigenvectors, those are not dominant directly in one with step by using the power method or process of power method just once.

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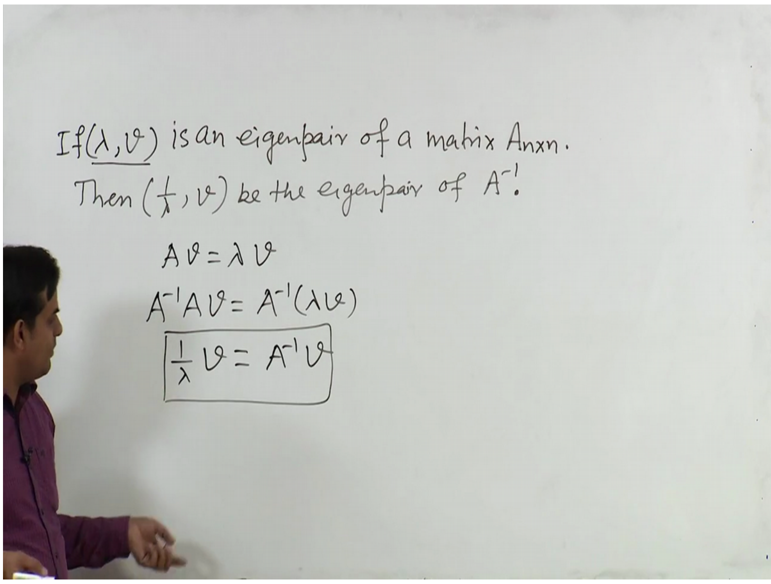

So basically these methods are based on the two principles. The first principle is if lambda and V is an eigenpair of a square matrix A of order n then, lambda inverse, which will be basically 1 upon lambda and V be the eigenpair of matrix A inverse. This is one of the rule and this can be shown very easily as it is the eigenpair for A. So I can write AV equals to lambda V. If A is invertible then I can multiply both side by A inverse. So inverse into A into V will become, A inverse into lambda V. What I can do lambda is a scalar, I can take out, so I can write and then I can divide the whole thing by 1 upon by lambda both sides, so 1 upon lambda V will become A inverse V, it means the eigenvalue of A inverse is 1 upon lambda and corresponding eigenvector is V.

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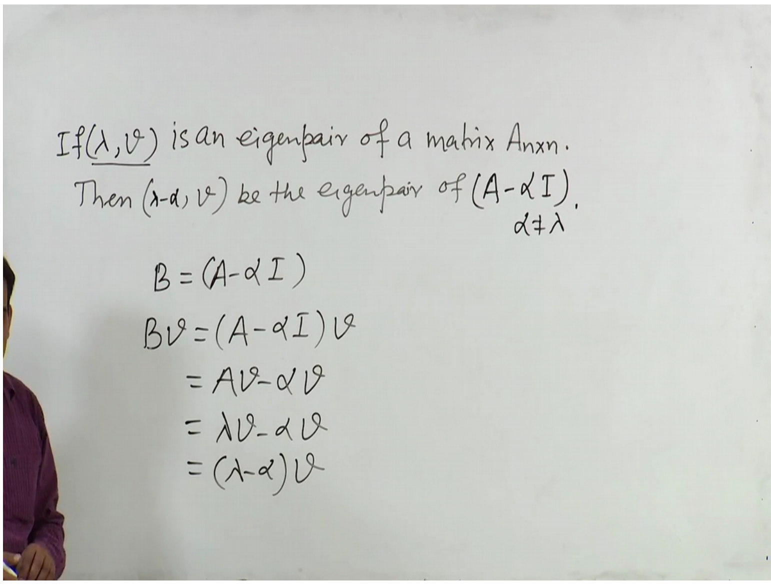

The other result we shift the eigenvalue that is, it is saying keep lambda V is an eigenpair of a matrix A then, lambda minus alpha together with eigenvector V will be the eigenpair of matrix A minus alpha I for the scalar alpha and here, alpha not equals to lambda. So this again we can show, we are having a new matrix B that is A minus alpha I, so BV will become A minus alpha I into V and this will become AV minus alpha V and we know that lambda is an eigenvalue of A, so it will be lambda minus alpha V or I can write lambda minus alpha into V. So it means the eigenvalue of B is lambda minus alpha and the eigenvector is V, which is the same as of A and V is A minus alpha I. So with these two results we will start our shifted inverse power method or inverse power method.

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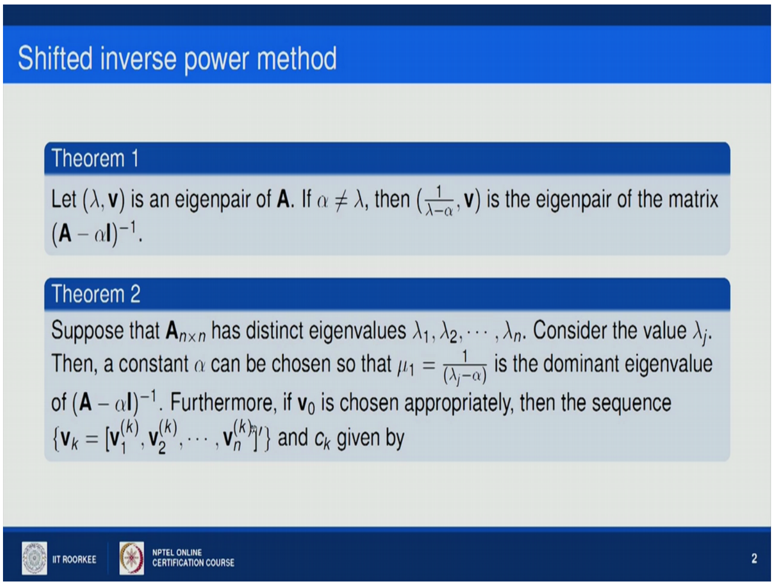

So suppose that A as distinct eigenvalues lambda1, lambda2, up to lambda n. Consider the eigenvalue lambda j, suppose I need to calculate or I need to compute this eigenvalue. So then a constant alpha can be chosen so that Mu1 is 1 upon lambda j minus alpha is the dominant eigenvalue of A minus alpha I inverse, further more if we chose V0 carefully then the sequence Vk that is V1k, V2k, Vnk having the components.

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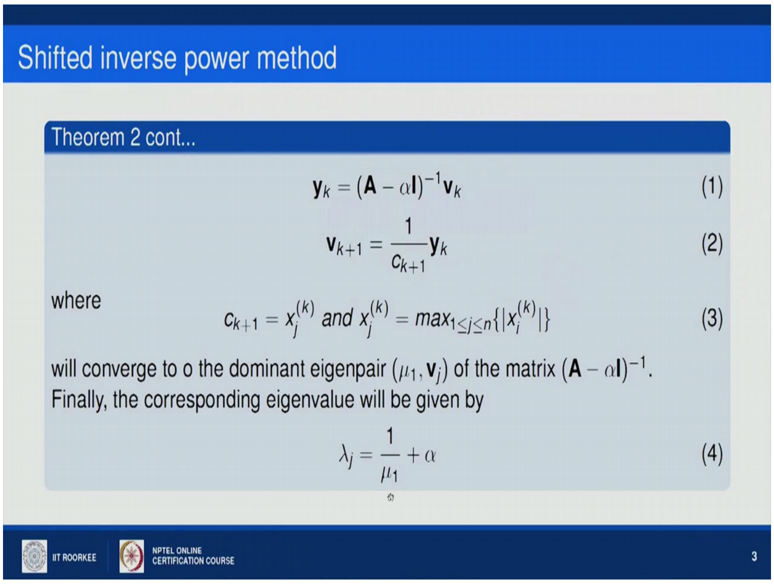

And Ck given by yk equals to A minus alpha I inverse V k and Vk plus 1 is 1 upon Ck plus 1 yk. This is the power method only power method for A minus alpha I inverse and finally we can calculate the jth eigenvalue that is lambda j as 1 upon Mu1 plus alpha from the Mu.

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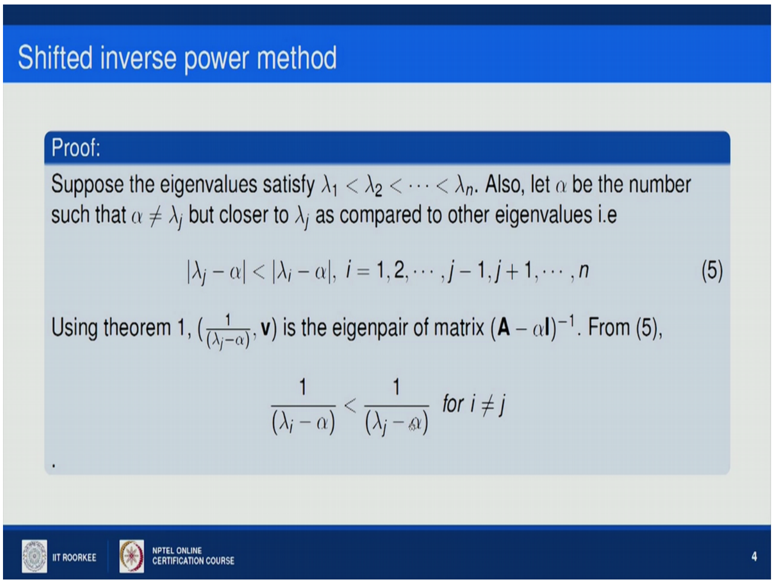

Now what should be the choice of alpha? We cannot choose alpha just like equals to lambda j, but to be the Mu1 as the dominant eigenvalue. So it should be very large and for this alpha should be the should be quite close to lambda j. So for example, if I want to find out eigenvalue 4, alpha should be somewhere for 4.2 or 3.8 or 4.3, 3.7 like that. So proof of this result can be given very easily, suppose the eigenvalue satisfy lambda1 less than lambda2 up to less than lambda n. Also, let alpha be the number such that alpha not equals to lambda j, but very close to lambda j as compare to other eigenvalues then, I can write that lambda j minus alpha less than lambda (j) i minus alpha for rest of the i from 1 to j minus 1and then j plus 1 to n.

Then using the result which I have derive on the board I can say that 1 upon lambda j minus alpha will be the eigenvalue of A minus alpha I inverse and the corresponding eigenvector will remain V, which is the eigenvector of original A to corresponding to the eigenvalue lambda j. So more over we can say that 1 upon lambda i minus alpha will be less than 1 upon lambda j minus alpha and hence, lambda1 upon lambda j minus alpha, which is my Mu1 will be the dominant eigenvalue of the matrix A minus alpha I inverse. So how it will work, suppose I want to find out an eigenvalue of a given matrix let us say, some eigenvalue lambda j. So I will choose 1 alpha close to this eigenvalue and which is not close to rest of the eigenvalues.

Now question arise without knowing about eigenvalues how we will choose alpha, because each and every time I am saying that alpha should be close to lambda j compare to any other lambda i. So how to do it without looking or without knowing about the eigenvalues so this will come from the gershgorin disc just by looking on the given matrix I can say about the range of eigenvalue or in which disc eigenvalue will lie and from there I can get an idea.

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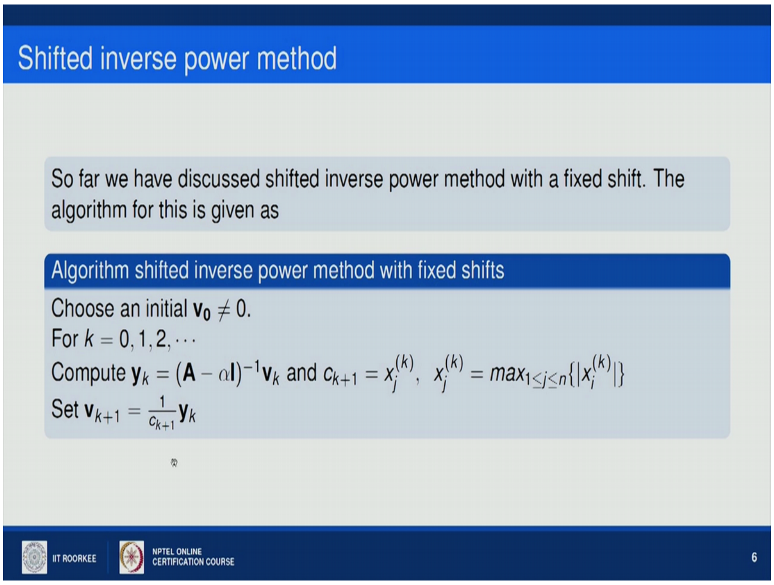

So the algorithm should be like this, first of all you have to choose initial V0, which should be a non-zero then for k equals to 0,1, 2 you will find out yk and yk will be A minus alpha I inverse Vk from here Ck plus 1 will be the largest component of vector yk in terms of absolute value and then you can define your Vk plus 1 as 1 upon Ck plus 1 into yk. So the shifted inverse power method with this with this fixed shift alpha is nothing just power method where the matrix A is replaced with a new matrix A minus alpha I inverse.

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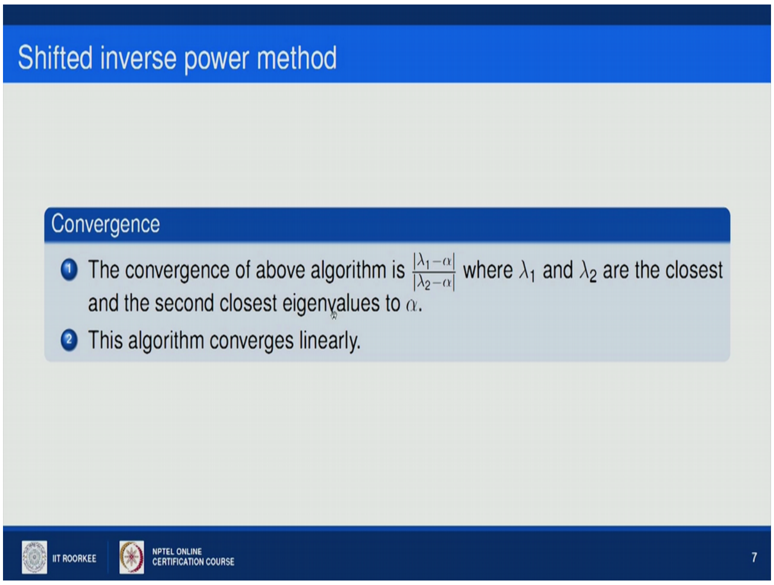

The convergence of this algorithm is given by this lambda1 minus alpha upon lambda2 minus alpha, where lambda1 and lambda2 are the closest and the second closest eigenvalue to alpha. So for example, if you are choosing a matrix having eigenvalue 5, 8 and 10 and I am choosing alpha is 4. So it will be convergence will be 1 upon that is 5 minus 4 upon 8 minus 4. So 1 upon 2, so it will be linear okay, so it will be always linear. It will be something like always between 0 and 1, okay or up to 1.

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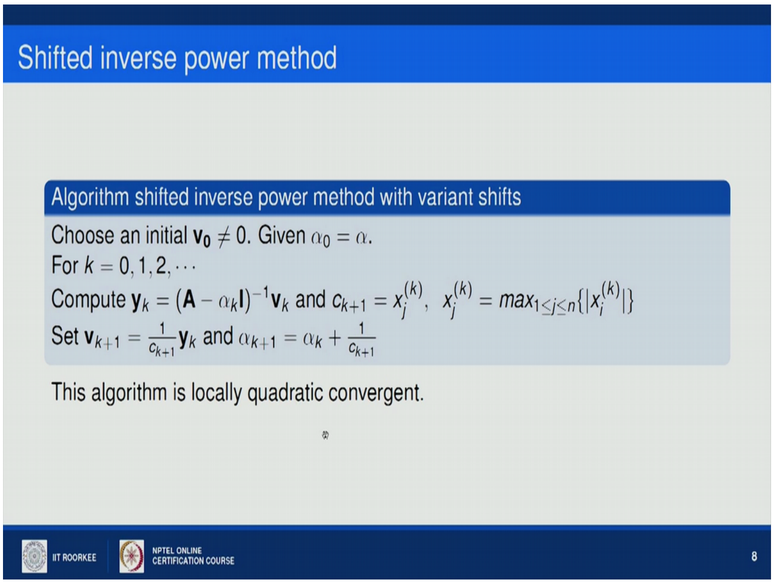

Now we can use this shifted inverse power method with variable shift also. Variable shift means we can update our alpha also in the earlier algorithm we have a fixed alpha chosen it in the beginning and we are using that; however here we can update our alpha also to improve the convergence of a given method. So here the algorithm will be like that you take a non-zero vector V0 and a initial alpha that is let us say alpha not, then compute yk, yk will be A minus alpha kI inverse into Vk and Ck plus 1 will be the maximum component in terms of absolute value of yk like if V over for a 3 by 3 matrix yk is coming 1 minus 2 minus 4.

So here Ck will become minus 4 and then you set your Vk plus 1 as 1 upon Ck plus 1 into yk and here, the same time you will update your alpha, alpha will becomes alpha k plus 1 upon Ck plus 1 and tis method is locally quadratic convergent having the quadratic or second order convergence locally.

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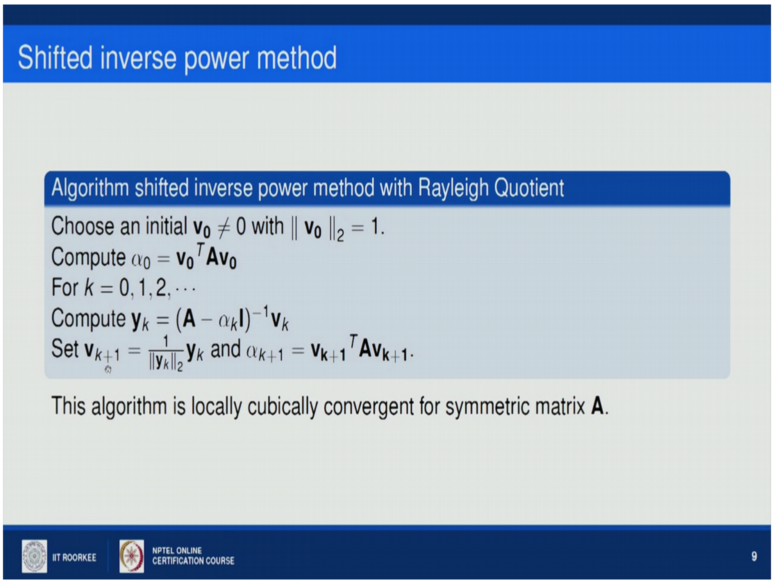

We can apply the shifted inverse power method with Rayleigh Quotient also and that is like choose an initial value V0 that is not equals to 0 such a way that, this is having the unit length then compute alpha not, which will be the Rayleigh Quotient of this vector V0 and that will be V0 transpose A into V0. Now for k equals to 0, 1,2 compute yk. So yk will become A minus alpha kI inverse into Vk. Set Vk plus 1 as 1 upon yk 2 yk and then alpha k plus 1 can be updated as Vk plus 1 transpose A into Vk plus 1. So in each iteration again updating my V and here I am updating my alpha by the definition of Rayleigh Quotient and this method is having (cubic) this method is cubic convergent in case of symmetric matrices.

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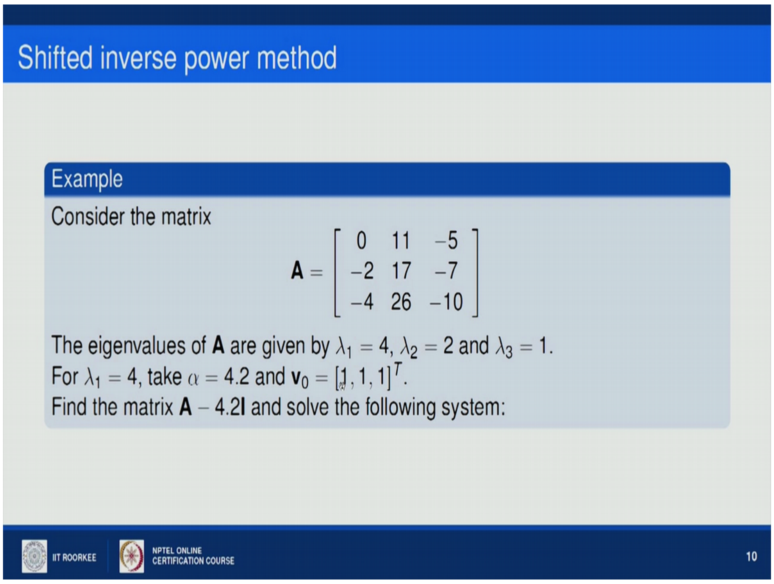

So let us take an example of separate inverse power method for finding the eigenvalue one of the eigenvalue of a 3 by 3 matrix and here, we will use the fixed alpha version of the inverse shifted inverse power method hence, with fix shift. So the eigenvalue of this matrix is (4, 2, 1) the dominant eigenvalue is 4. So suppose I take alpha 4.2, so if I take alpha 4.2 my shifted inverse power method will converge to eigenvalue 4 and corresponding eigenvector. So for lambda1 equals to 4, I can define my A minus alpha will become A minus 4.2 1 and then I will apply power method on this A minus 4.2 I with initial value 1,1, 1. So this I have taken in this way.

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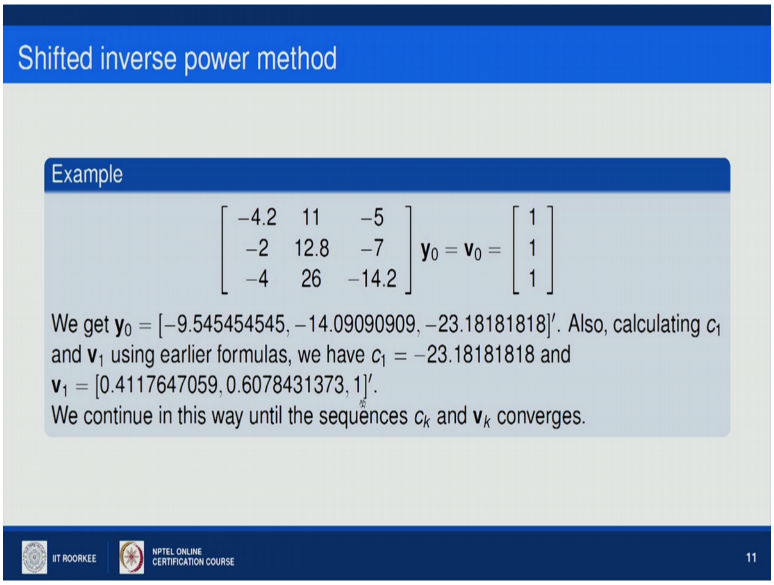

And then using this we continue in this way until the sequence Ck and Vk converges. So y0 is this value then C1 comes -23.18181818 and then V1 is this particular vector.

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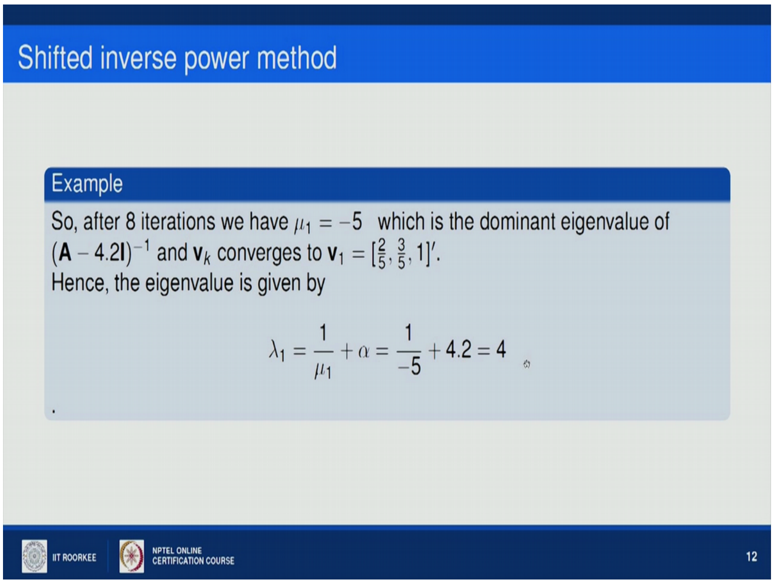

After 8 iterations we have Mu1 equals to minus 5, which is the dominant eigenvalue of A minus 4.2 I inverse and then Vk converges toV1 that is 2 by 5, 3 by 5 and 1. So hence the eigenvalue is given by 1 upon Mu1 plus alpha that is minus 1 upon 5 plus 4.2 that is 4, which verify our claim that for a given alpha it will converge to the closest eigenvalue.

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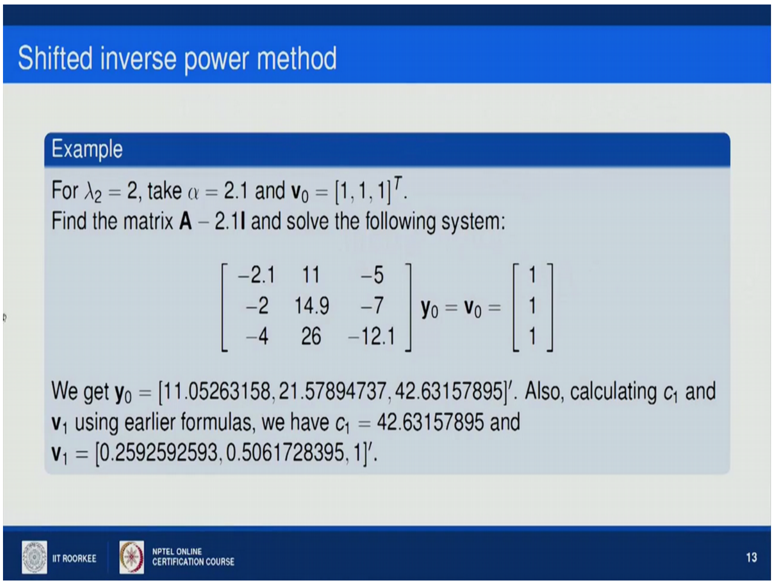

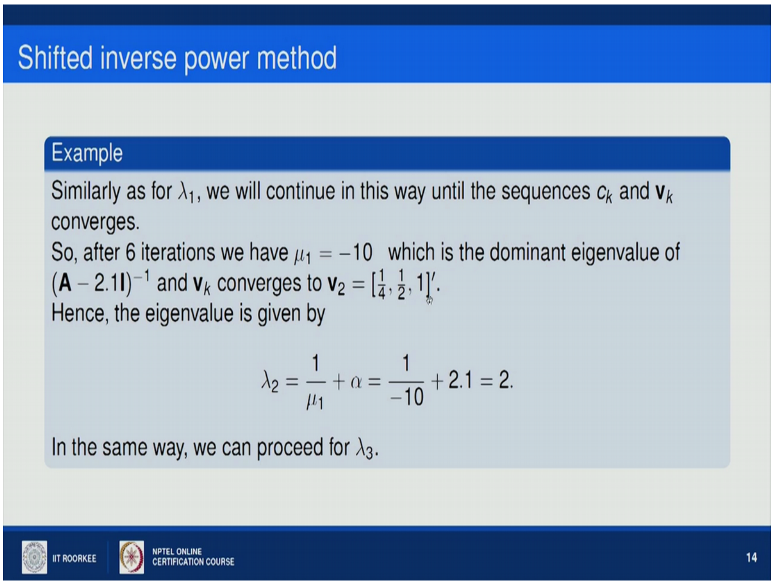

If I take alpha equals to 2.1 and I apply the same process, it converges to eigenvalue 2 with corresponding eigenvector 1 by 4, 1 by 2 and 1.

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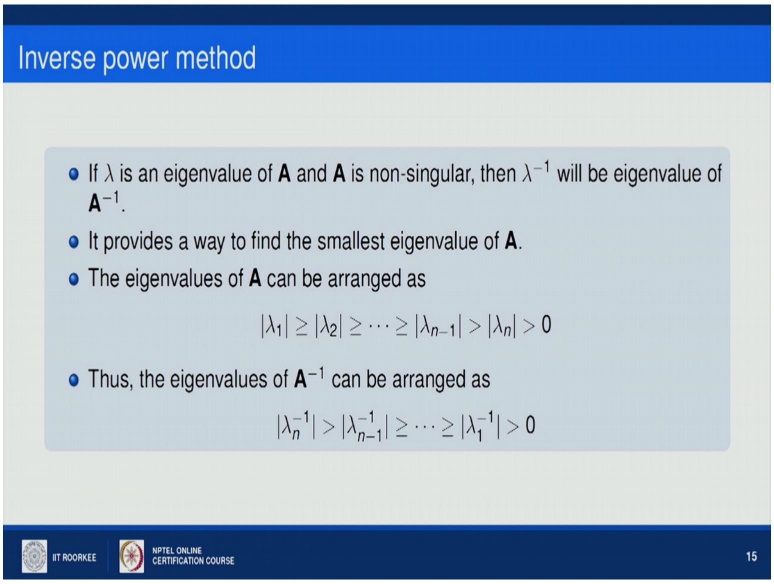

So far we were talking about shifted inverse power method. Let us take a and in shifted inverse power method we need to calculate inverse and like that. Let us take a other variant of this shifted inverse power method just inverse power method and this we can use for finding the smallest eigenvalue of a given matrix and the corresponding eigenvector and here we are using the result that if lambda and V be the eigenpair of a matrix A then 1 upon lambda and V will be the eigenpair of A inverse.

So if lambda is an eigenvalue or lambda is the dominant dominant eigenvalue of a given matrix A then 1 upon lambda will be the dominant eigenvalue of then lambda inverse will be the eigenvalue of A inverse and hence, 1 upon lambda will be the smallest eigenvalue of the matrix A. So if we apply the power method on A inverse what we can get we can get the smallest eigenvalue of the matrix A.

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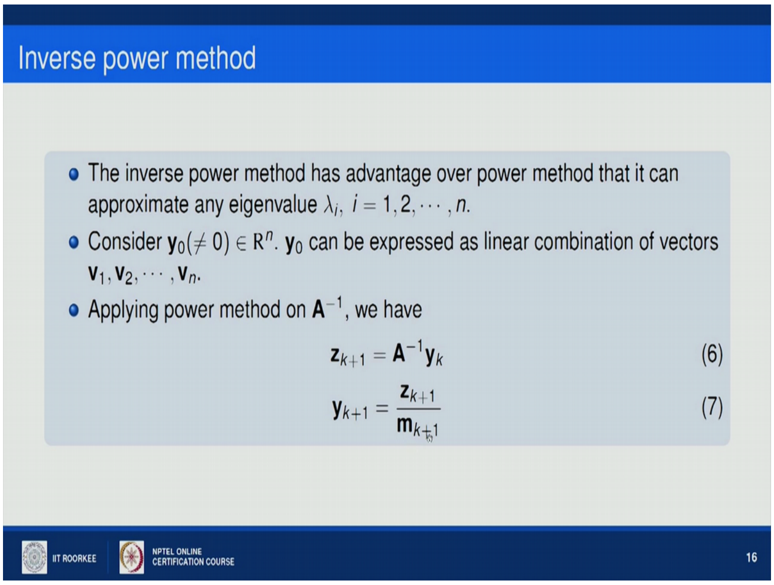

So the inverse power method as advantage over power method that it can approximate any eigenvalue. Consider y not which is a non-zero eigenvector vector in Rn and y0 can be expressed as linear combination of eigenvectors of A and then applying power method on A inverse we can get Zk plus 1 equals to A inverse yk and yk plus 1 equals to Zk plus 1 upon mk plus 1.

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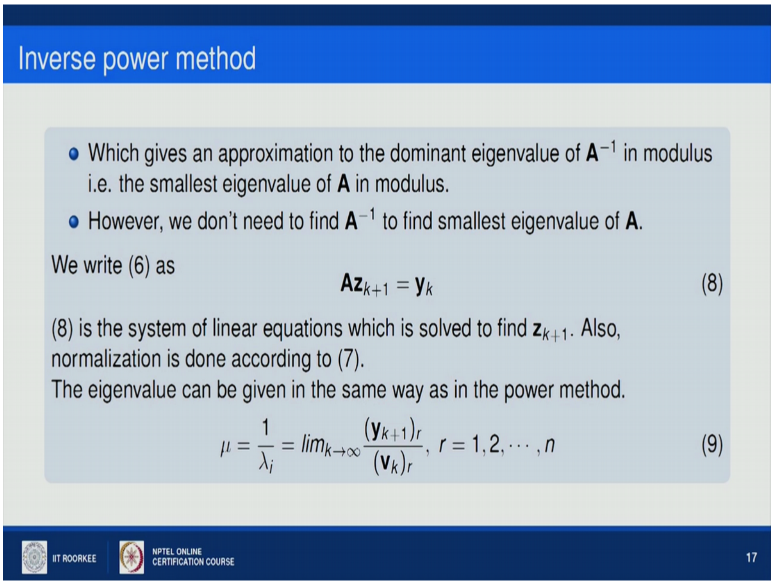

So in this way which gives the approximation to the dominant eigenvalue of A inverse in modulus that is the smallest eigenvalue of A in modulus. However, here we do not need to find A inverse to find smallest eigenvalue of A, because if you are having a 10 by 10 matrix, so finding the inverse of a 10 by 10 matrix is computationally expensive and I will not prefer, suppose I want to find out the smallest eigenvalue of a 10 by 10 matrix A. So what is the inverse power method I need to calculate A inverse and the dominant eigenvalue of A inverse will be the smallest eigenvalue of A, but we need to find out A inverse. Here, we do not require to find out A inverse at all.

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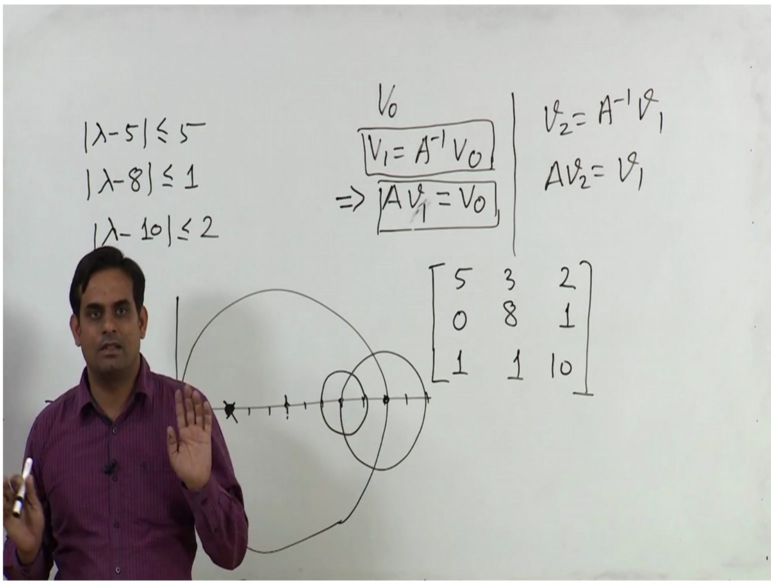

What we can? (We are) what we are? We are starting with a V0 and what we are doing? We are finding V1 as A inverse into V0. So what I will do? Here, I will use multiply both side by A. So my AV1 will become V0. So instead of finding V1 with this iterative process or from this multiplication of a matrix with a column vector, I will be having a linear system of equation and V0 is known to you, A is known to you. So you can find out V1directly from here without using A inverse. Then in the next equation, your V2 will be A inverse into V1. So what you can say you can solve this system AV2 equals to V1 and from here, you will find out the next iteration of the inverse power method that is your V2 and so on. So here no need to calculate A inverse at any stage; however we need to solve a linear system of equation in each and every stage.

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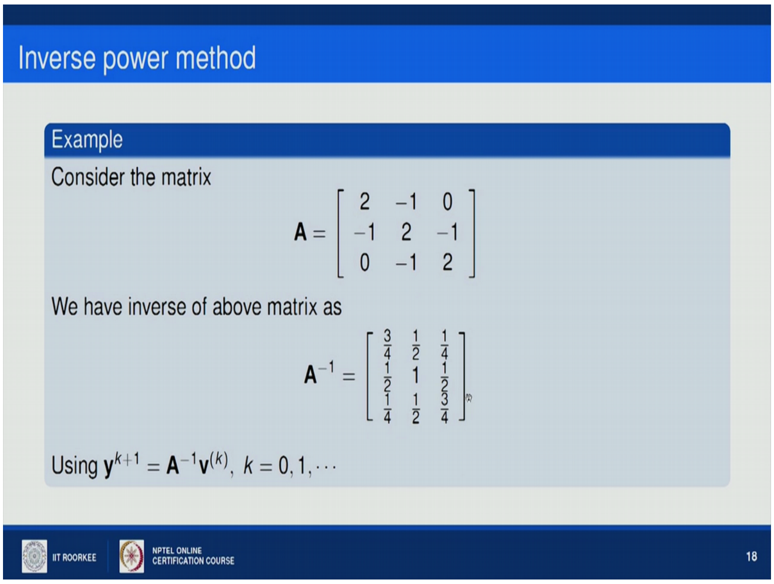

So let us take an example of this, we are taking this matrix. So A is (2, -1, 0) (-1, 2, -1) (0, -1, 2). Here, we are doing it by calculating A inverse, but we can do it without calculating A inverse also.

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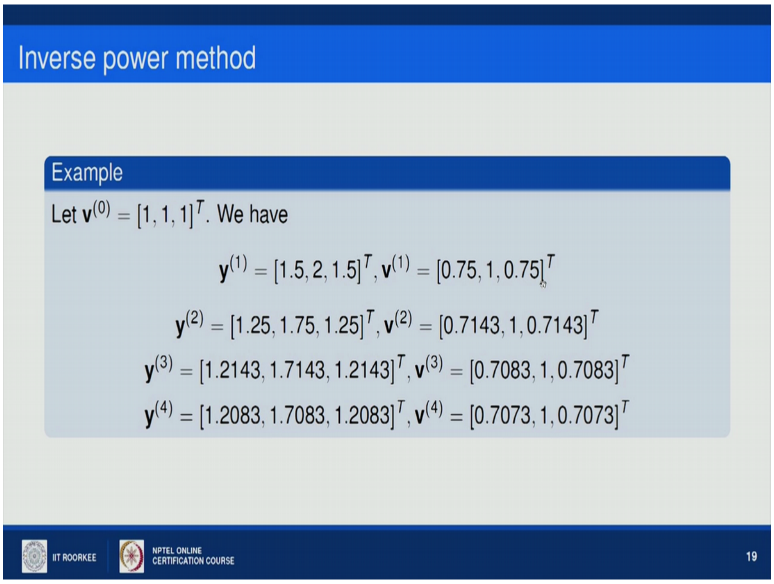

So if we are doing it with A inverse and starting with and starting with an initial solution 1, 1, 1 we are getting our first approximation as y1 that is 1.52 to 1.5 and here, if I divide it by 2, it is 1 upon 2 into 2.751 and 0.75 transpose. So first approximation of the eigenvalue is coming 2 and eigenvector is this one.

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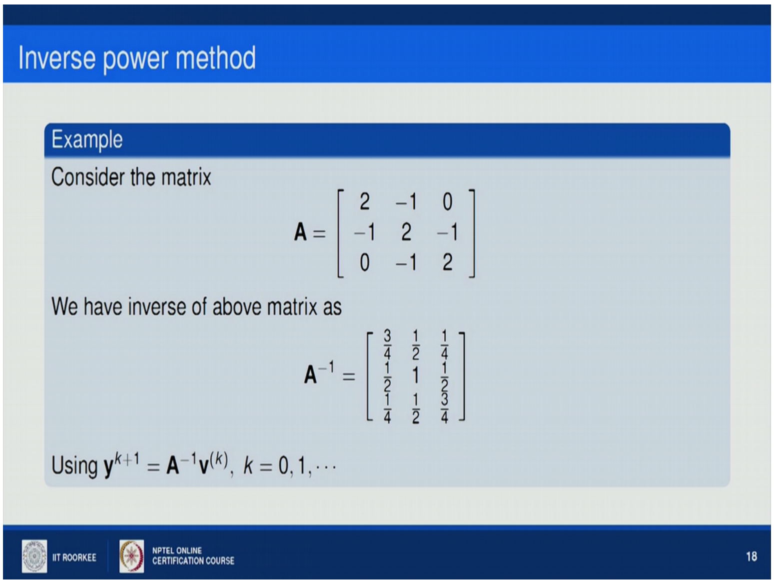

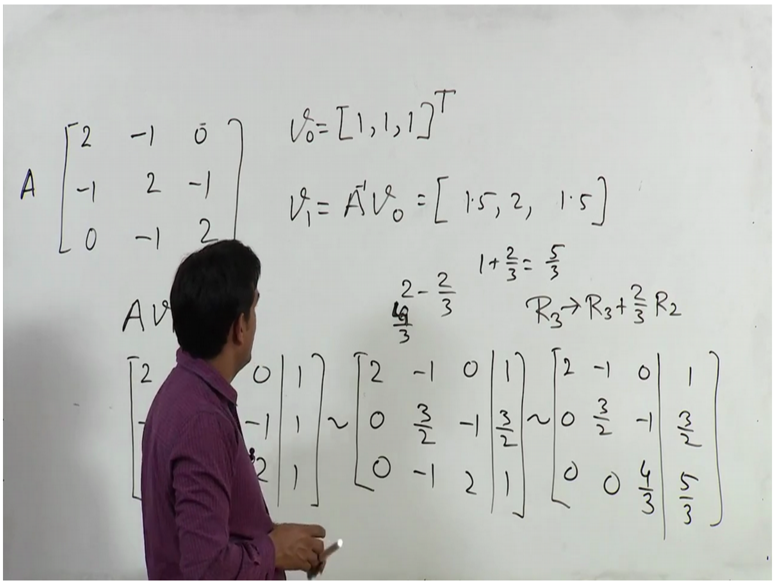

However, here we are doing it with A inverse, but if I do it without finding A inverse then the system can be solved with less computational effort, for example my original matrix is (2,-1,0) (-1,2,-1) and then we are having (0,-1,2). So this is A, V not is 1,1, 1 transpose and I am finding V1 as A into V0, which is coming -1.5, 2, -1.5, I think so which is coming 1.5, 2 and 1.5. Now here what I am doing, sorry it is A inverse into V0, so doing this I need to calculate inverse of tis matrix, but if I solve this system A into V1 equals to V0 then my augmented matrix becomes (2, -1, 0) (-1,2,-1) (0,-1,2) and then (1,1,1).

And after solving this, let us say solve it with Gauss elimination. So (2, -1, 0) and then R2 will be replaced by R2 plus 1 by 2 times R1. So this will be 0, 2,-1 by 2 will be 3 by 2 minus 1, 1. 1 plus 1 by 2 will be 3 by 2, then this is already 0, -1, 2, 1 and then this can be change into just by R3 replaced by R3 plus 2 by 3 R2 by using this row operation elementary row operation then I will get 2,-1, 0, 1, 0, 3 by 2, -1, 3 by2 and then 0 ,0, 2,-2 by 3, 4 by 3 and then this will become 1 plus 2 by 3. So 1 plus 2 by 3 will be 5 by 3 and from here we will get this vector V1.

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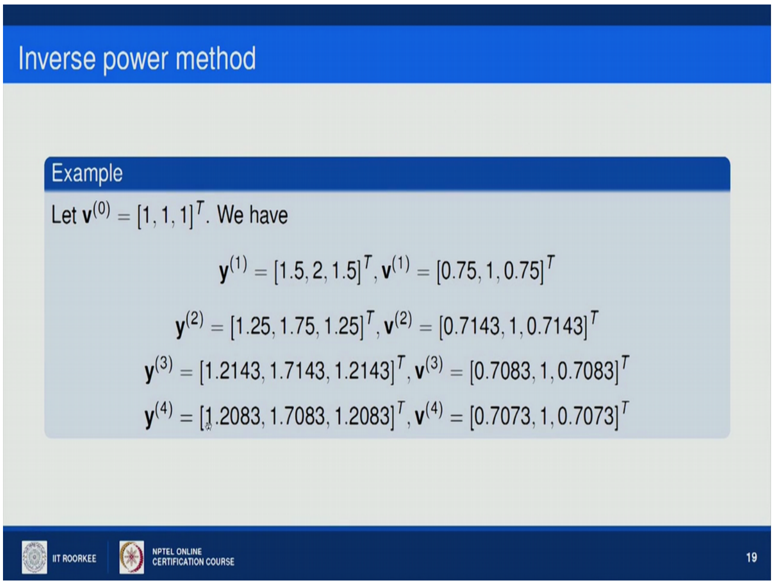

So continuing this I will calculate y2 and then V2, y3 V3, y4 V4.

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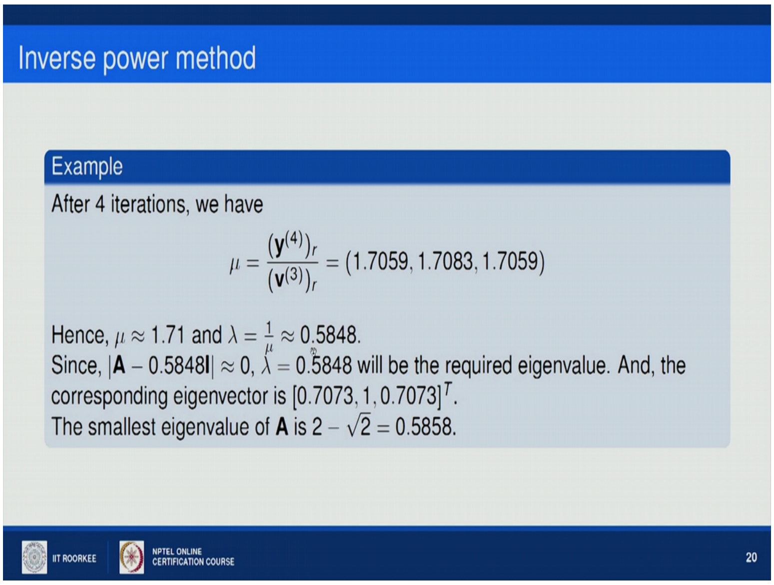

And after 4 iterations we observe that my system is converging to Mu equals to 1.71 and lambda equals to 1 upon Mu that is 0.5848. Since, A minus 0.5848I will be 0, so here lambda equals to 0.5848 will be the required eigenvalue that is the smallest eigenvalue and the corresponding eigenvector is 0.707, 1 and 0.7073. The smallest eigenvalue of A will be 2, -root 2 that is again true which is as we have computed numerically.

So in this lecture we have learned the two variants of power method that is the shifted inverse power method and inverse power method for finding the eigenvalues other than dominant for a given matrix. This ends the module 3 of this course and in this module, we have learned various methods for calculating power calculating eigenvalues and eigenvectors like, we started with Jacobi method, then we have learn power method, power method with fixed shift, inverse power method with variable shift, inverse power method with Rayleigh Quotient and finally, the classical inverse power method for finding the smallest eigenvalue of a given matrix. In the next lecture, we will talk about interpolation till the end bye. Thank you very much.