Integral Equations, Calculus of Variations and their Applications
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Lecture 58
Variational problem with moving boundaries; One sided variation

Hello friends welcome to my lecture on variational problem with moving boundaries, here we shall discuss one sided variations.

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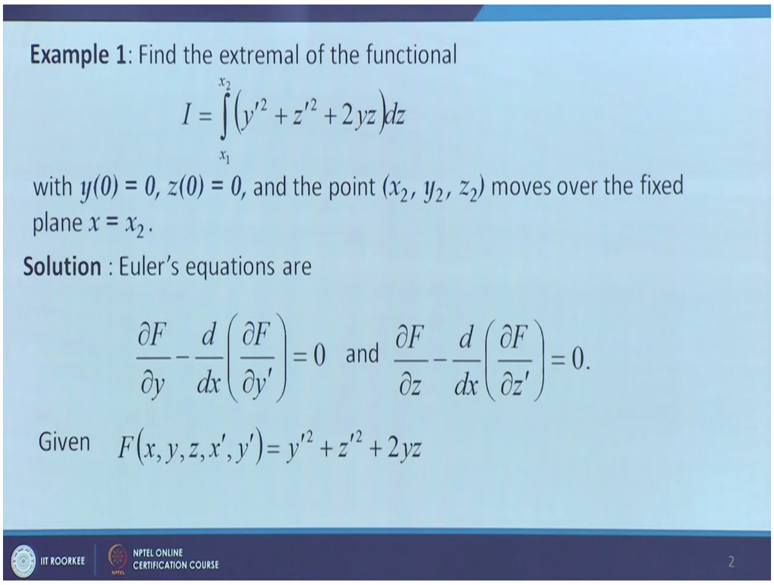

Now first we will begin with the problem where the functional depends on two functions y and z of x of the independent variable x and one arbitrary point x 1, y 1, z 1 boundary point x 1, y 1, z 1 is fixed while the other point x 2, y 2, z 2 lies on the surface. So the surface here we have taken as the plane. So here the point which is fixed is given by y 0 equal to 0, z 0 equal to 0 that is origin. So x is 0, y is 0 and z is 0 and the points x 2, y 2, z 2 moves over the fixed plane x equal to x 2.

Now again when we want to find the extremal of this functional we since the necessary equations Eulers Equations are F y minus d over dx F y dash equal to 0 and F z minus d over dx F z dash equal to 0. So here we are given F x, y, z, x, y dash, z dash equal to y dash square plus z dash square plus 2yz.

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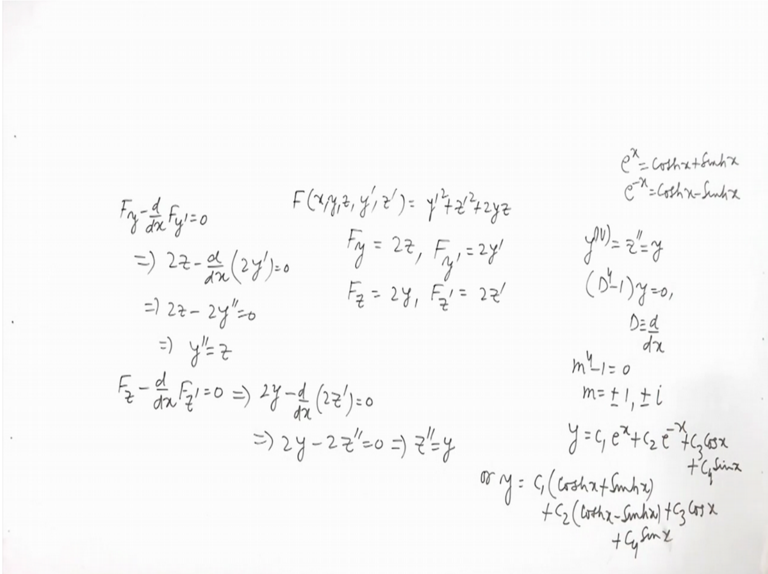

From the equation from F x, y, z, y dash, z dash we get from this equation we get F y because F is dependent on y so F y is equal to 2z, F y dash equal to 2y dash, F z equal to 2y and F z dash equal to 2z dash. So the Eulers Equations F y minus d over dx of F y dash equal to 0 this gives you 2z minus d over dx of F y dash that is 2y dash equal to 0 or I can say 2z minus 2y double dash equal to 0. So this gives you y double dash equal to z.

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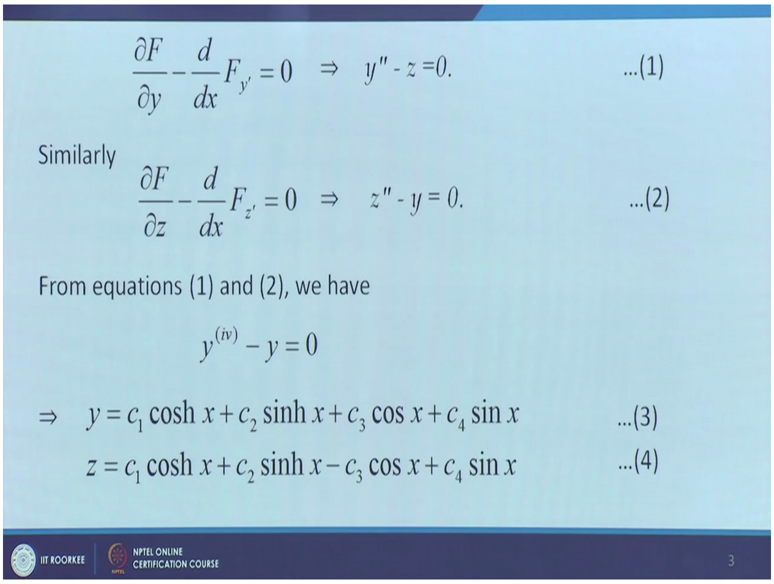

And the other Eulers Equations F z minus d over dx of F z dash equal to 0 gives us F z is 2y minus d over dx of 2z dash equal to 0. So we get 2y minus 2z double dash equal to 0 or z double dash equal to y. Now from the two equations y double dash equal to z and z double dash equal to y, what we obtain? The fourth order derivative of y is equal to second order derivative of z which is equal to y, so we get D 4 minus 1 y equal to 0 and where D denotes d over dx. Now so auxiliary equation here is M 4 minus 1 equal to 0, we can write the roots of this equation m plus minus 1, plus minus I, so y equal to c 1 e to the power x plus c 2 e to the power minus x plus c 3 cos x plus c 4 sin x.

Now since e to the power x we know e to the power x is cos hyperbolic x plus sin hyperbolic x e to the power minus x is cos hyperbolic x minus sin hyperbolic x. So this equations can also be written in terms of hyperbolic functions, so we can write it as or y equal to c 1 cos hyperbolic x plus sin hyperbolic x plus c 2 times cos hyperbolic x minus sin hyperbolic x plus c 3 cos x plus c 4 sin x c 4 sin x.

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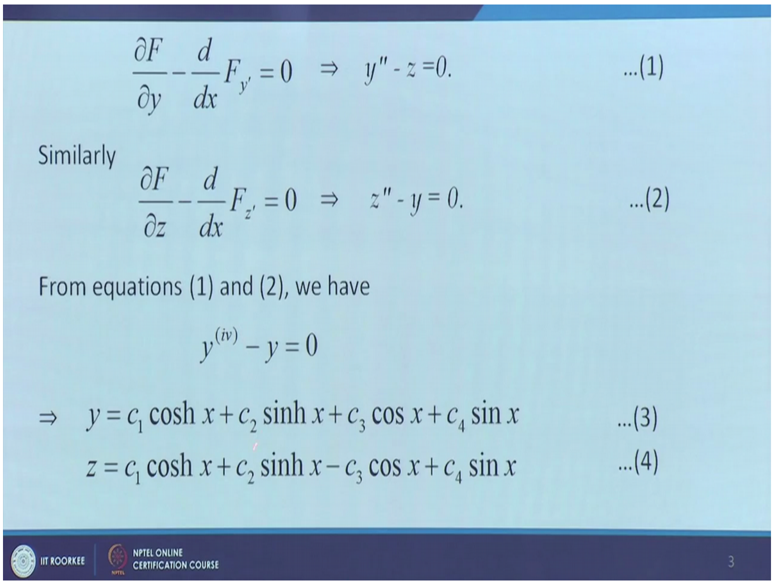

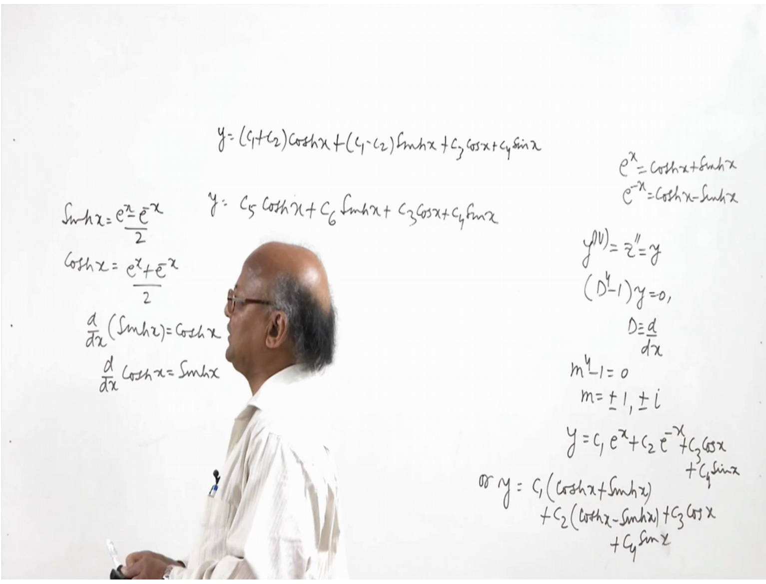

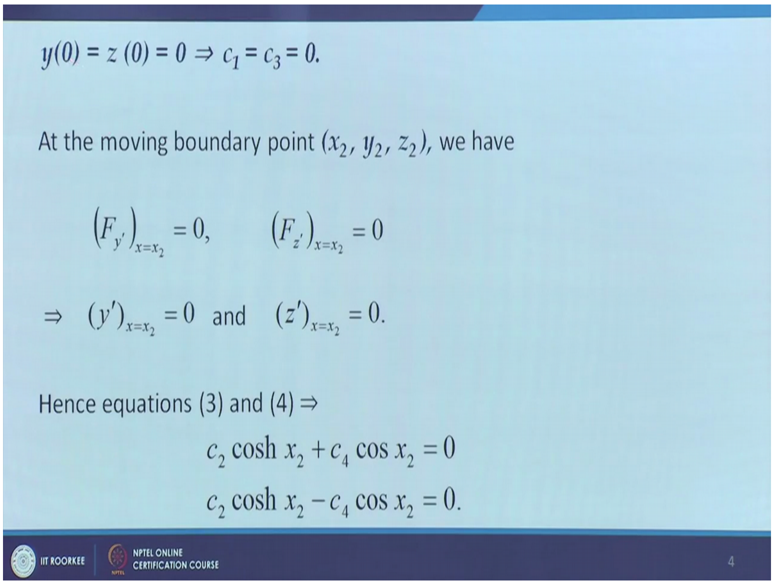

And then collecting the coefficients of cos hyperbolic x and sin hyperbolic x one can write it as y equal to c 1 plus c 2 cos hyperbolic x plus c 1 minus c 2 sin hyperbolic x plus c 3 cos x plus c 4 sin x. Now c 1 plus c 2 and c 1 minus c 2 can be replaced by new arbitrary constants and then we have so using new arbitrary constants we can write y equal to c 5 cos hyperbolic x plus c 6 sin hyperbolic x plus c 3 cos x plus c 4 sin x.

Now what I want to demonstrate is that the roots of the auxiliary equation are plus minus 1, plus minus I, so in this case the solution of the fourth order linear differential equation with constants coefficients D 4 minus 1 y equal to 0 can also be expressed in this form. So I have written it in the form of cos hyperbolic and sin hyperbolic and trigonometric functions sin x cos x. So y I have written as c 1 cos hyperbolic x plus c 2 sin hyperbolic x plus c 3 cos x plus c 4 sin x.

And then z is y double dash is equal to z, so we can differentiate this y twice, when you differentiate y with respect to x we get derivative of cos hyperbolic x derivative of cos hyperbolic x is sin hyperbolic x and derivative of sin hyperbolic x is cos hyperbolic x this we can see easily here. Sin hyperbolic x is e to the power x minus e to the power minus x by 2 and cos hyperbolic x is e to the power x plus e to the power minus x by 2. So d over dx of sin hyperbolic x is cos hyperbolic x and d over dx of cos hyperbolic x is sin hyperbolic x.

So we can differentiate y twice and then we will get z, y double dash is equal to z, so z will be c 1 cos hyperbolic x plus c 2 sin hyperbolic x minus c 3 cos x plus c 4 sin x. Now we are given that y 0 is equal to 0, z 0 equal to 0. So putting y 0 equal to 0 here what we get cos hyperbolic 0 is 1 so we get c 1, sin hyperbolic 0 is 0 and then cos hyperbolic 0 cos 0 is 1 so we get c 1 plus c 3 this term is 0. Say c 1 plus c 3 is 0 and z 0 equal to 0 gives c 1 then we get minus c 3, so c 1 minus c 3 equal to 0.

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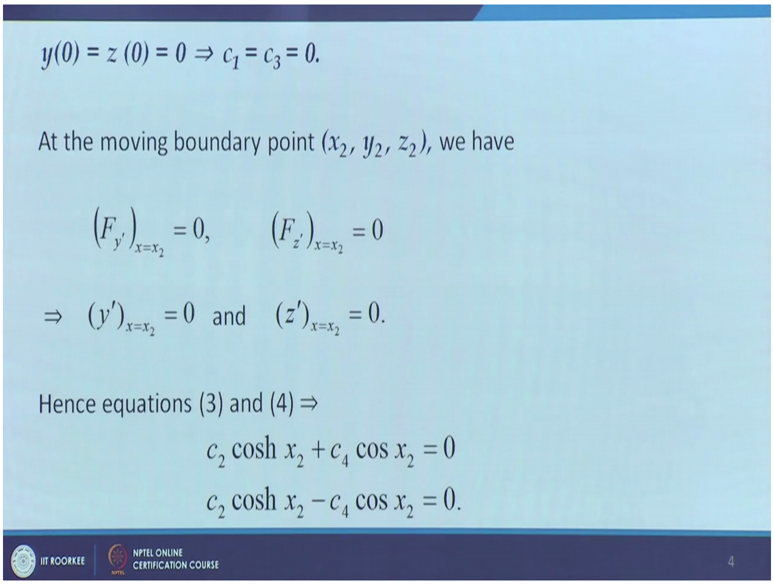

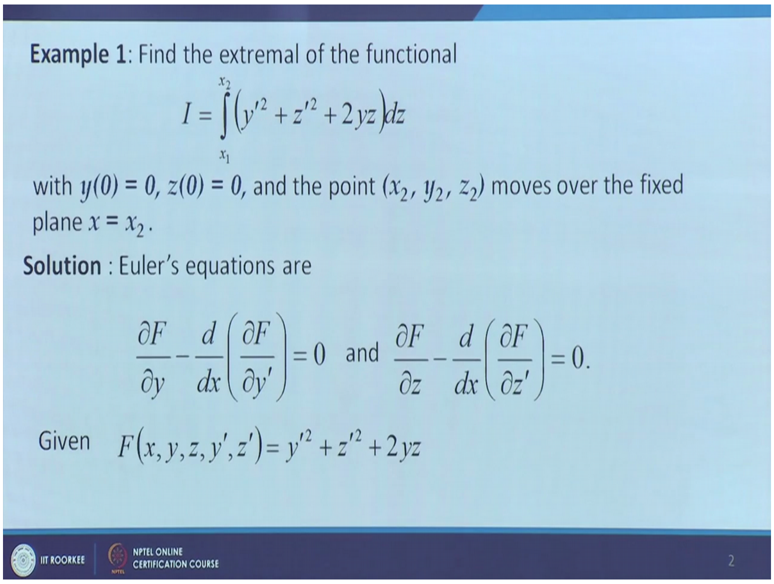

So c 1 plus c 3 equal to 0 and c 1 minus c 3 equal to 0 imply c 1 equal to 0, c 3 equal to 0. Now at the moving boundary point x 2, y 2, z 2 since the boundary point x 2, y 2, z 2 lies on the surface we have the other conditions F y dash at x equal to x 2 0, F z dash at x equal to x 2 0. So F y dash here if we find F y dash F y dash is 2 y dash and F z dash is 2 z dash. So F y dash at x equal to x 2 0 means y dash at x equal to x 2 0 and F z dash at x equal to x 2 0 implies z dash at x equal to x 2 0.

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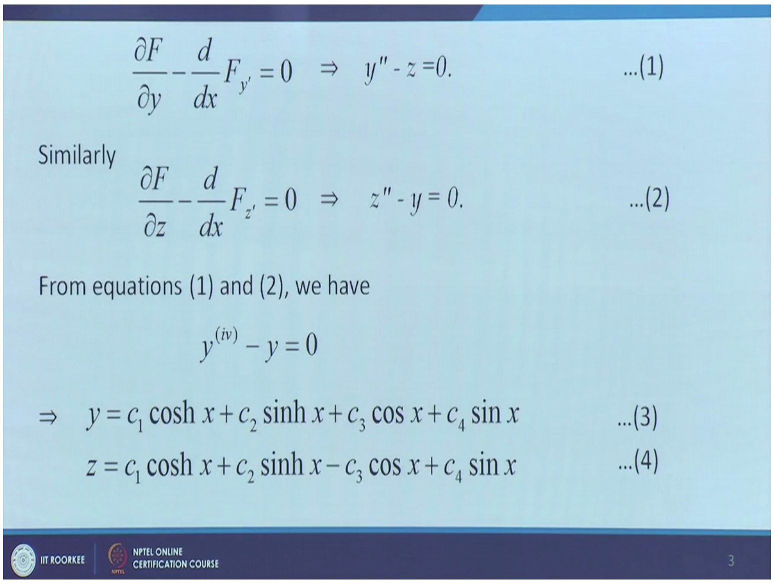

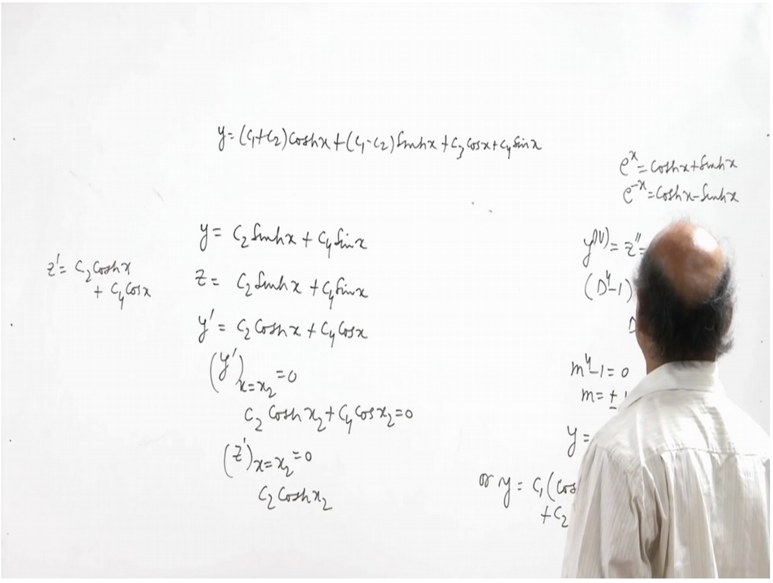

Hence what we get is we have the following equation c 1 equal to c 3 equal to 0 gives you y dash from this equation y equal to c 1 cos hyperbolic x we have found c 1 equal to 0, c 3 equal to 0, so this reduces to c 2 sin hyperbolic x and c 4 sin x this is y and z has become c 1 c 3 0, so c 2 sin hyperbolic x plus c 4 sin x. Now y dash at x equal to x 2 0, so y dash is equal to c 2 cos hyperbolic x plus c 4 cos x, y dash at x equal to x 2 is equal to 0 gives you c 2 cos hyperbolic x 2 plus c 4 cos x 2 is equal to 0 and z dash z dash is equal to c 2 cos hyperbolic x plus c 4 cos x. So z dash at x equal to x 2 is equal to 0 gives you c 2 cos hyperbolic x 2, okay.

So y 4 minus y equal to 0 gives you D 4 minus 1 y equal to 0 where D is d over dx. Now the auxiliary equation is M 4 minus 1 equal to 0, so M is equal to plus minus 1, plus minus I, and so we can write y equal to c 1 e to the power x plus c 2 e to the power minus x plus c 3 cos x plus c 4 sin x, y double dash minus z equal to 0 and z double dash minus y equal to 0 together give us the fourth derivative of y minus y equal to 0 because fourth derivative of y here is equal to second derivative of z and second derivative of z is equal to y, so fourth derivative of y fourth order derivative of y minus y is equal to 0.

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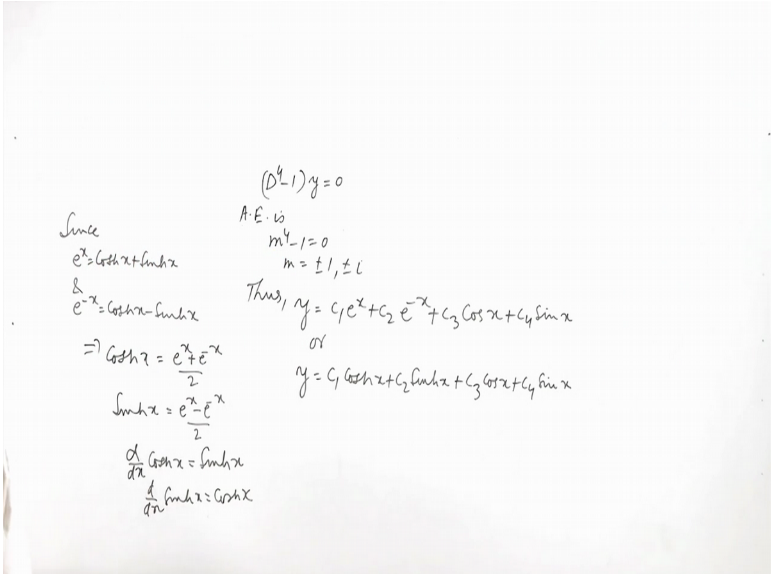

And this can be written as D 4 minus 1 y equal to 0. Now the auxiliary equation is therefore M 4 minus 1 equal to 0 so which gives the roots plus minus 1, plus minus I, and thus y is equal to c 1 e to the power x plus c 2 e to the power minus x plus c 3 cos x plus c 4 sin x. Now since e to the power x is equal to cos hyperbolic x plus sin hyperbolic x and e to the power minus x is cos hyperbolic x minus sin hyperbolic x.

This solution of the equation D 4 minus 1 y equal to 0 can be (written as) also written as y equal to c 1 cos hyperbolic x plus c 2 sin hyperbolic x plus c 3 cos x plus c 4 sin x that is in terms of the hyperbolic function and the trigonometric functions. So we have written the solution of this equation in terms of the hyperbolic and cosine sin function, so y equal to c 1 cos hyperbolic x plus c 2 sin hyperbolic x and c 3 cos x plus c 4 sin x and y double dash is equal to z so when we differentiate this y twice we can see that derivative of cos hyperbolic x cos hyperbolic x from here is e to the power x plus e to the power plus e to the power minus x by 2 and sin hyperbolic x is e to the power x minus e to the power minus x by 2.

So this gives you d over dx cos hyperbolic x is sin hyperbolic x and d over dx of sin hyperbolic x equal to cos hyperbolic x.

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$$\frac{\partial F}{\partial y} - \frac{d}{dx} F_{y'} = 0 \quad \Rightarrow \quad y'' - z = 0. \qquad \text{...(1)}$$

Similarly

$$\frac{\partial F}{\partial z} - \frac{d}{dx} F_{z'} = 0 \quad \Rightarrow \quad z'' - y = 0. \qquad \text{...(2)}$$

From equations (1) and (2), we have

$$y^{(iv)} - y = 0$$

$$\Rightarrow \quad y = c_1 \cosh x + c_2 \sinh x + c_3 \cos x + c_4 \sin x \qquad \text{...(3)}$$

$$z = c_1 \cosh x + c_2 \sinh x - c_3 \cos x - c_4 \sin x \qquad \text{...(4)}$$

$y(0) = z(0) = 0 \Rightarrow c_1 = c_3 = 0.$

At the moving boundary point (x_2, y_2, z_2), we have

$$\left(F_{y'}\right)_{x=x_2} = 0, \qquad \left(F_{z'}\right)_{x=x_2} = 0$$

$$\Rightarrow \quad (y')_{x=x_2} = 0 \quad \text{and} \quad (z')_{x=x_2} = 0.$$

Hence equations (3) and (4) $\Rightarrow$

$$c_2 \cosh x_2 + c_4 \cos x_2 = 0$$

$$c_2 \cosh x_2 - c_4 \cos x_2 = 0.$$

So by taking the derivative of y with respect to x twice we get the expression for z c 1 cos hyperbolic x plus c 2 sin hyperbolic x this is minus c 3 cos x this gives us c 4 sin x gives after twice twice differentiating we give minus c 4 sin x. So now let us apply the condition y 0 equal to 0 and z 0 equal to 0 when you put y at x equal to 0 as 0, what we get is c 1 plus c 3 equal to 0 and z 0 equal to 0 gives us c 1 minus c 3 equal to 0.

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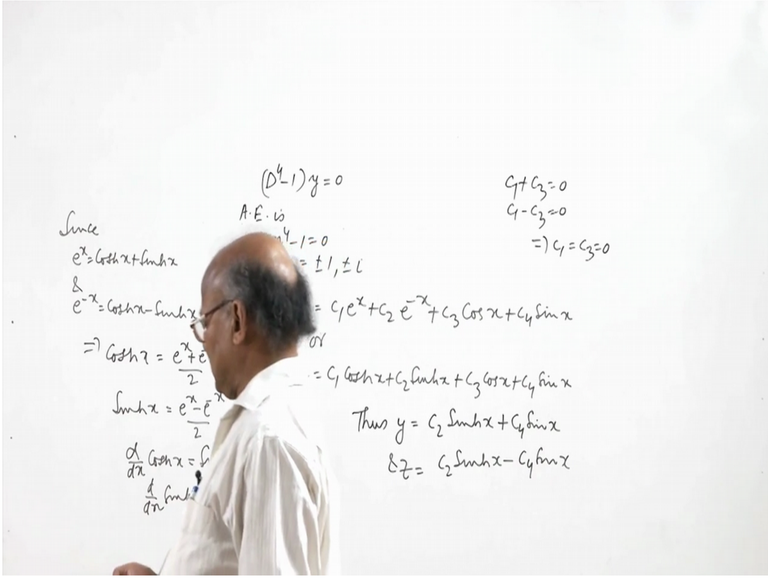

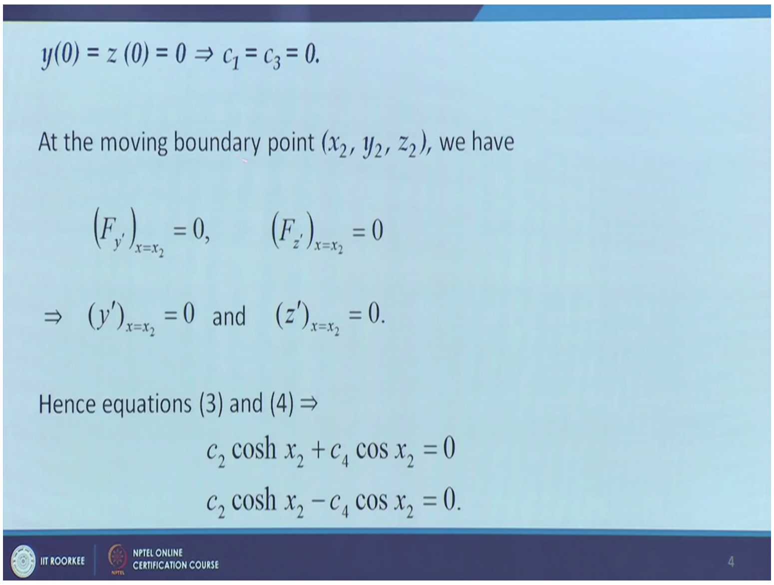

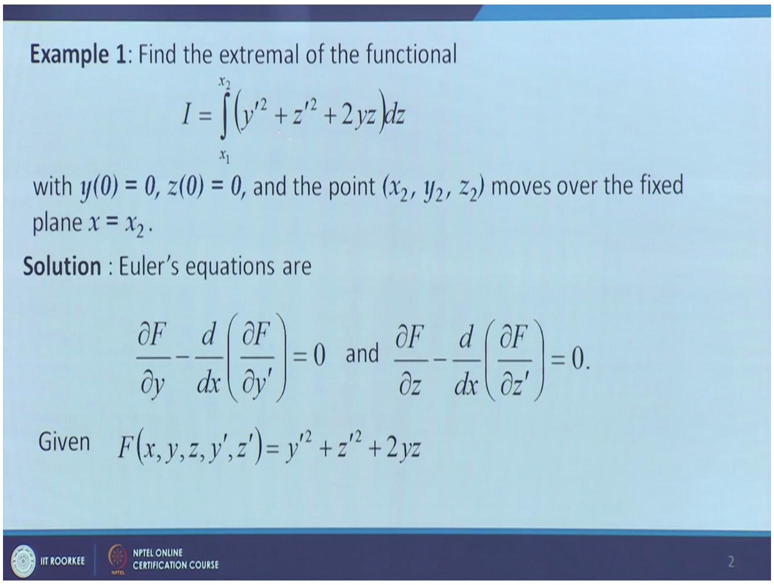

So the two equations c 1 plus c 3 equal to 0 and c 1 minus c 3 equal to 0 imply that c 1 equal to c 3 equal to 0. And so then the expressions of y and z reduce to thus y becomes c 2 sin hyperbolic x plus c 4 sin x and z is equal to c 1 and c 3 are 0, so c 2 sin hyperbolic x minus c 4 sin x. Now let us apply the condition F y dash at x equal to x 2 is equal to 0 which is which are the conditions which we get by the moving boundary point x 2, y 2, z 2 which lies on the surface surface of the plane.

So F y dash x equal to x 2 gives y dash at x equal to x 2 equal to 0 because (F y F x, y, y dash) F x, y, z, y dash, z dash is this y dash square plus z dash square plus 2yz. So F y dash gives 2 y dash, F z dash gives 2 z dash. So we will get here 2 y dash at x equal to x 2 0 or y dash at x equal to x 2 0 and 2 z dash at x equal to x 2 equal to 0 gives z dash at x equal to x 2 is 0.

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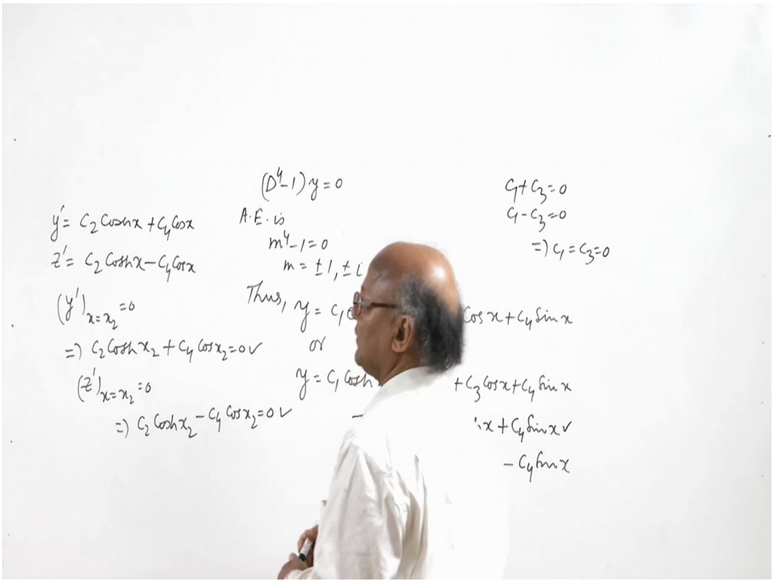

And so here y dash will be 0 at x equal to x 2 and therefore we differentiate this equation with respect to x and put x equal to x 2 equal to 0, so we get y dash is equal to c 2 cos hyperbolic x plus c 4 cos x and z dash is equal to c 2 cos hyperbolic x minus c 4 cos x. So y dash at x equal to x 2 equal to 0 implies c 2 cos hyperbolic x 2 plus c 4 cos x 2 equal to 0, z dash at x equal to x 2 equal to 0 gives us c 2 cos hyperbolic x 2 minus c 4 cos x 2 equal to 0, so we get this equation and this equation.

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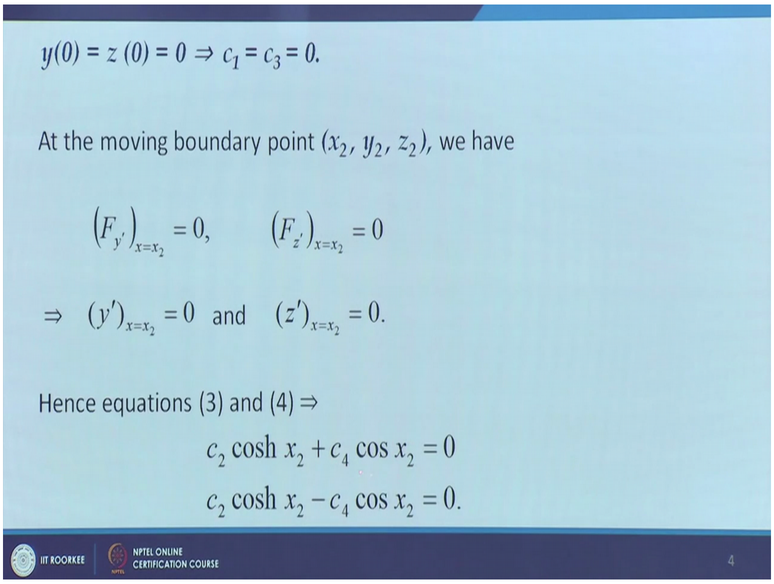

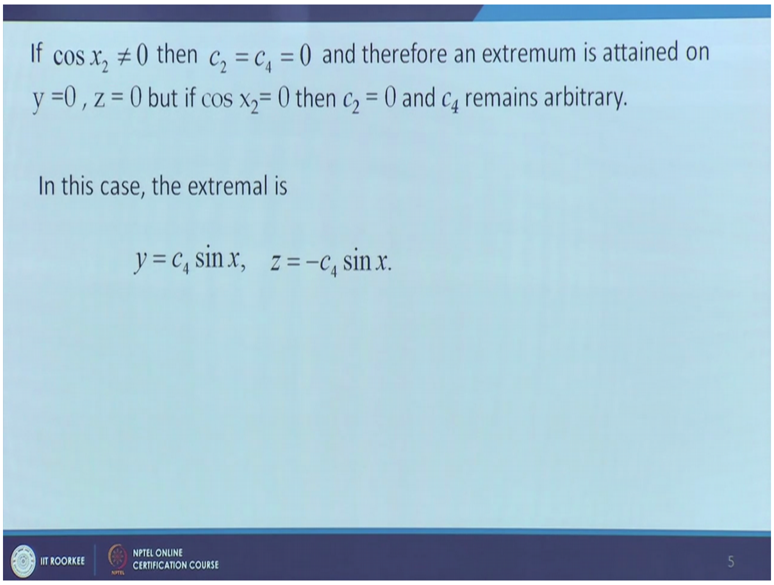

Now these equations we get. So here if cos x 2 is not equal to 0 if cos x 2 is not equal to 0 then what we will get is?

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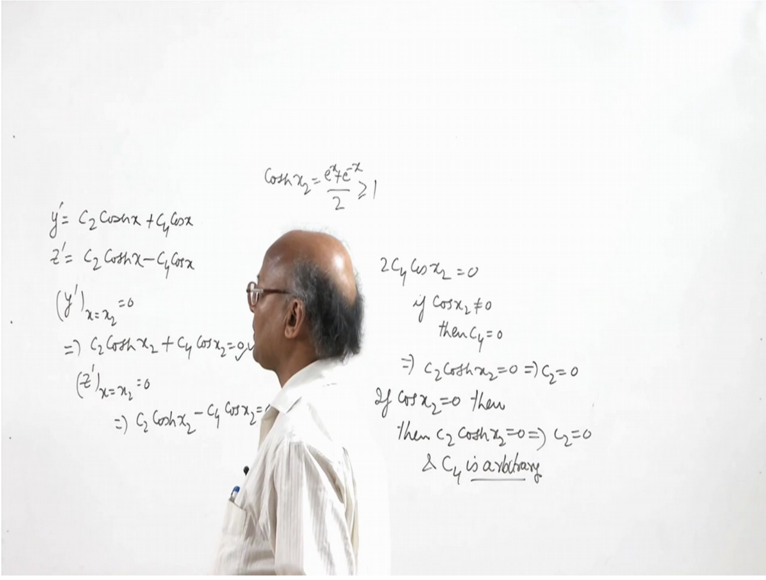

Let us subtract let us subtract this equation from this equation, so do we get 2c 4 cos x 2 is equal to 0, so if cos x 2 is not equal to 0 then c 4 is equal to 0 and c 4 is equal to 0 implies c 2 cos hyperbolic x 2 equal to 0. Now cos hyperbolic x 2 is equal to e to the power x plus e to the power minus x by 2, so it is always greater than or equal to 1 and therefore this is never 0, so this implies c 2 is equal to 0.

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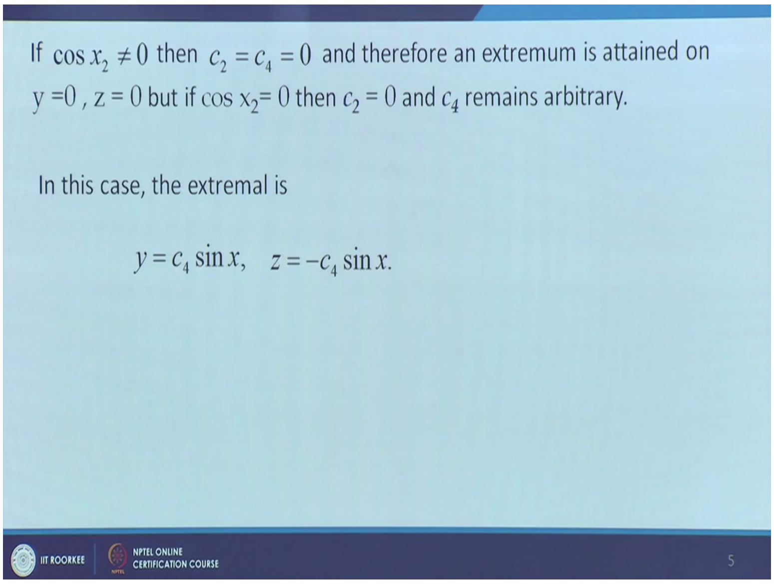

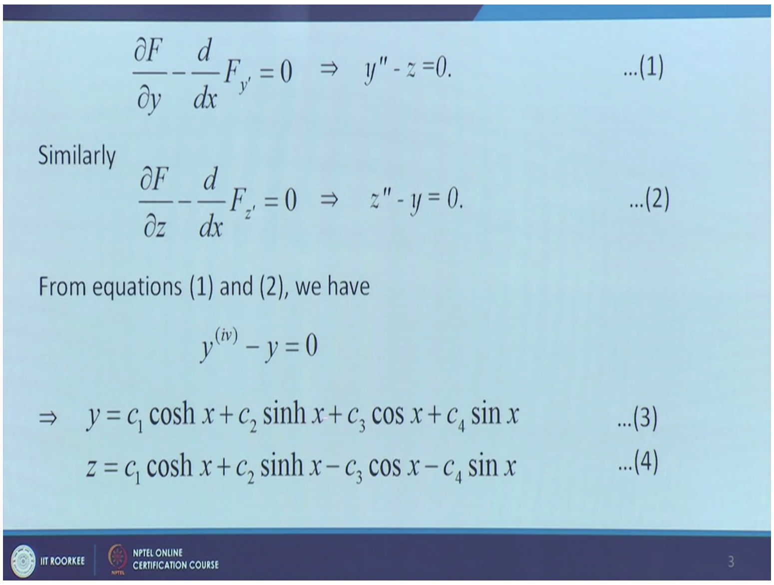

Now c 2 and c 4 is equal to 0 so therefore extremum is attained on y equal to 0, z equal to 0. Now because y is equal to y is equal to c 1 cos hyperbolic x c 2 sin hyperbolic x plus c 3 cos x plus c 4 sin x, c 1 c 3 are 0 already and we now get c 2, c 4 or 0s. So y is equal to 0 similarly z equal to 0. So the extremal is attained at y equal to 0, z equal to 0. But if cos x 2 is equal to 0 if cos x 2 is equal to 0 then what we will get if cos x 2 is equal to 0 then c 2 cos hyperbolic x 2 is equal to 0 then c 2 cos hyperbolic x 2 equal to 0 implies c 2 equal to 0 and c 2 equal to 0 implies c 4 cos x 2, so c 4 cos x 2 equal to 0 and is true for arbitrary c 4, okay. So c 2 equal to 0 and c 4 is arbitrary.

So in this case y will be equal to we have c 1, c 3 equal to 0, c 2 c 2 is equal to 0 so y equal to c 4 sin x and z is equal to minus c 4 sin x. So the extremal will occur here y equal to c 4 sin x, z is equal to minus c 4. So this is the solution of this problem where we have taken one point fixed and the other point varies on the plane.

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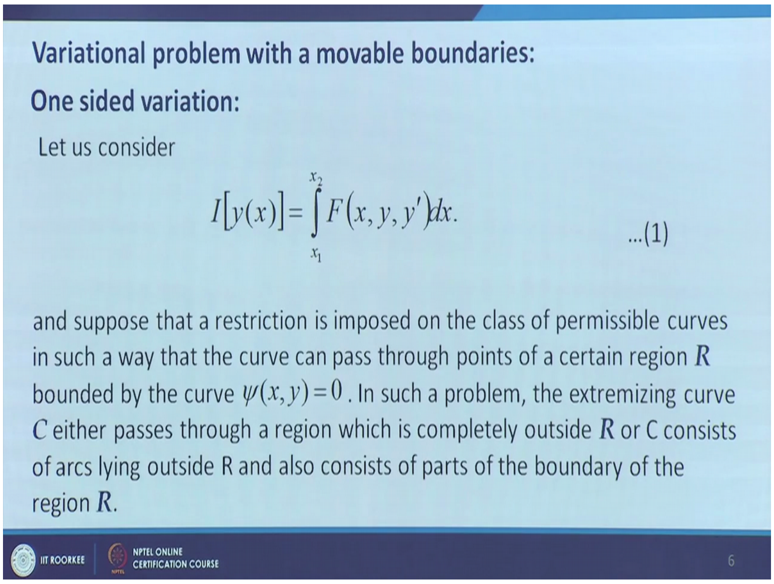

Now let us take the case of one sided variation, here what we will happen is that what will happen is we are given a functional I y x is equal to integral x 1 to x 2 F x, y, y dash dx. Let us suppose that there is a restriction which is imposed on the clause of permissible curves in such a way that the curve can pass through points of a certain region R bounded by the curve Psi x, y equal to 0 so we have a certain region R which is bounded by the curve Psi x, y equal to 0, the restriction is imposed on the clause of permissible curves that those curves can pass only through this region and in such a problem the extremizing curve C either passes through a region which is completely outside R or C consist of arcs lying outside R and also consist of parts of the boundary of the region R.

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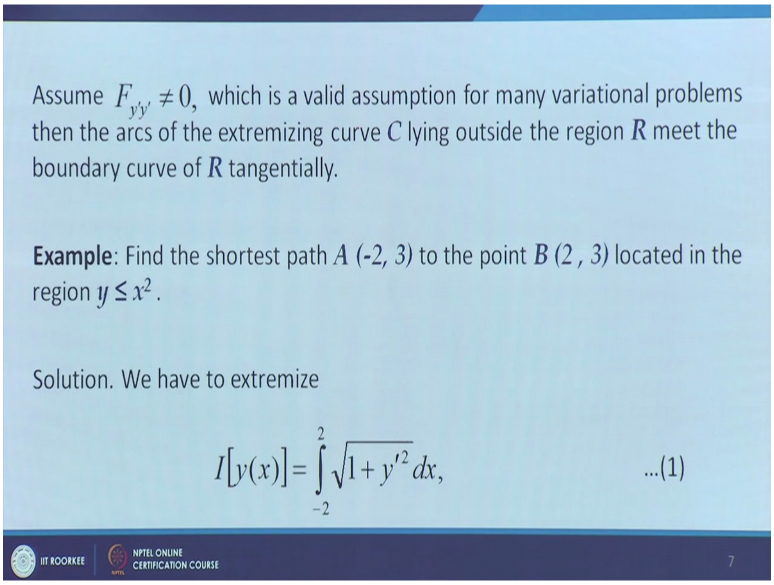

Now let us so here we assume that the derivative of F with respect to y dash twice is not equal to 0 this assumption is a valid assumption in many variational problems this means that the arcs of the extremizing curve C lying outside the region R meet the curve or the boundary curve of R tangentially. Let us study a problem on this article let us consider this problem find the shortest distance shortest path A minus 2, 3 to the point B 2, 3 located in the region y less than or equal to x square.

Let us see the curve the boundary curve of the region y less than or equal to x is given by y equal to x square.

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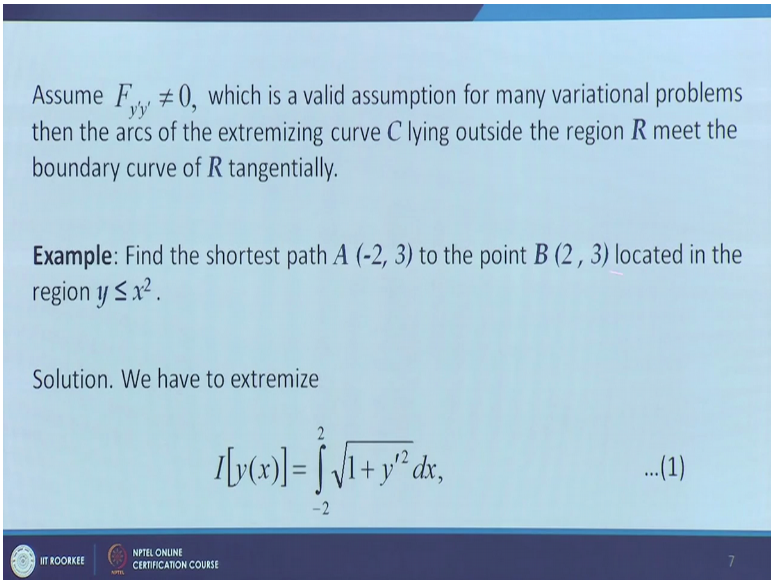

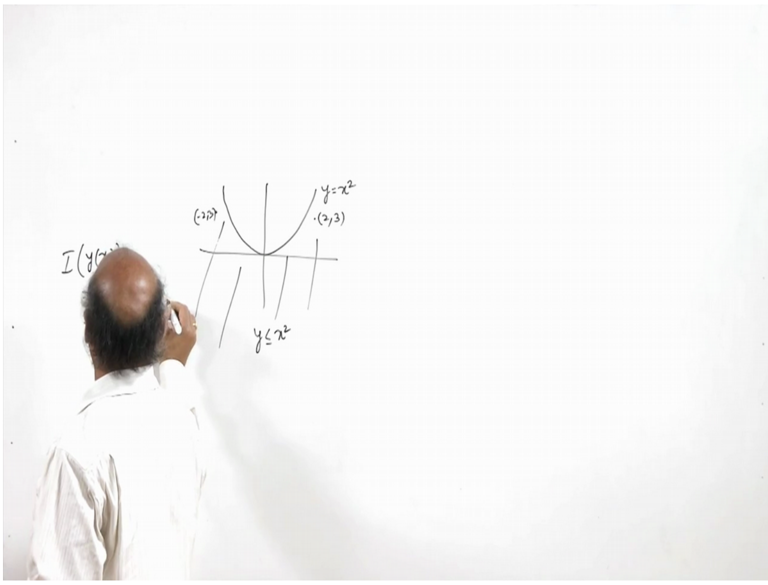

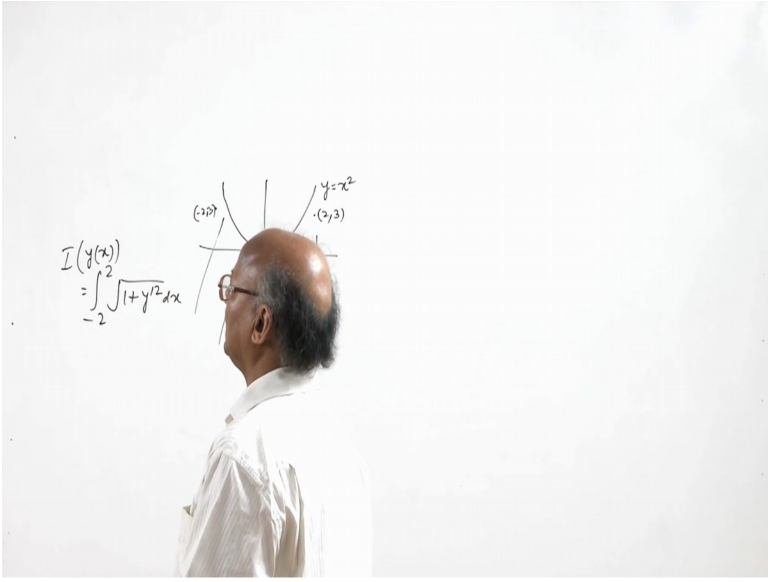

So this is the curve y equal to x square. Now we are given that y is less than or equal to x square. So this region this region is y less than or equal to x square. Now we have to find that path which is the shortest from the point minus 2, 3 to the point 2, 3. So the point is here let us say minus 2, 3 and its there is a point here 2, 3 the two points are symmetric with respect to the y axis and so we have to find the shortest path from the point minus 2, 3 to the point 2, 3 which is located in the region y less than or equal to x square.

So we have to find the path which is shortest from the point minus 2, 3 to 2, 3 and does not cross this curve y is equal to y is equal to x square, it lies the path must lie in the region y less than or equal to x square. So we have to extremize here I y x equal to integral over minus 2 to 2 x varies from minus 2 to 2 under root 1 plus y dash square dx.

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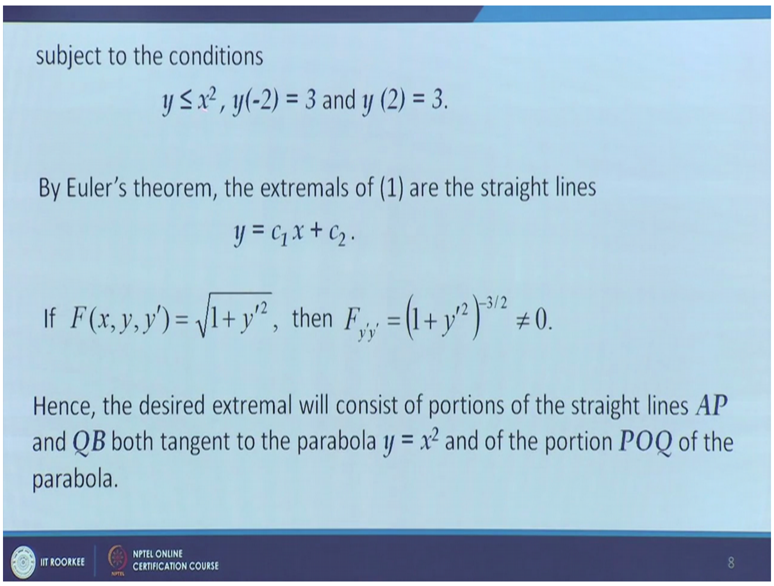

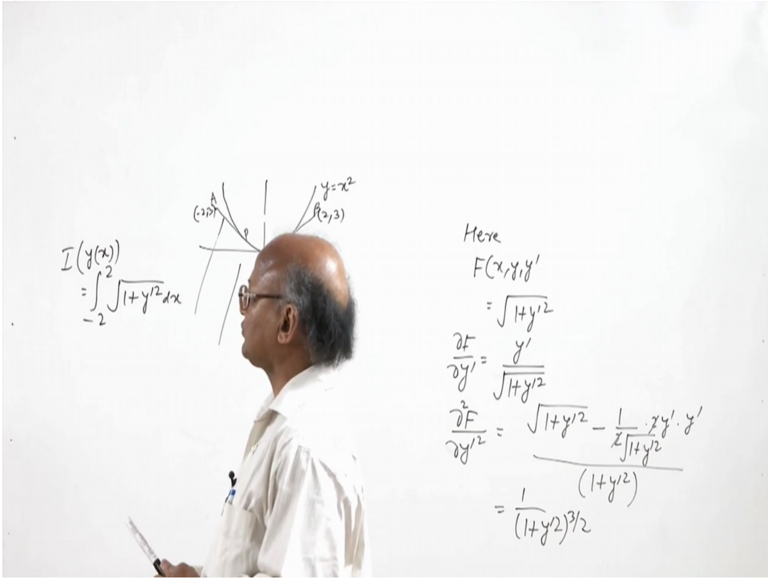

Now subject to the conditions that y is less than or equal to x square and y minus 2 is equal to 3, y 2 is equal to 3. By Eulers theorem we know that the extremals of this functional okay under root 1 plus y dash square dx the extremals are straight lines this we have seen earlier in in a lecture that y is equal to c 1 x plus c 2. Now if F x, y, y dash is under root 1 plus y dash square then we can see that here F x, y, y dash is equal to under root 1 plus y dash square.

So when we differentiate it partially with respect to y dash we get y dash divided by under root 1 plus y dash square, let us differentiate it again so what we get derivative of this with respect to y dash is 1 then under root 1 plus y dash square minus 1 by 2 under root 1 plus y dash square into 2 y dash into y dash divided by 1 plus y dash square, so this gives you 1 plus y dash square minus y dash square so 1 upon 1 plus y dash square to the power 3 by 2.

So and this is clearly not 0, hence the desired extremal will consist of portions of the straight lines AP and QB both tangent to the parabola, so what we will happen is this will touch this is let us say A, this is B, this is AP, this is BQ. So the extremal will consist of the portions of the straight lines AP, QB and both tangent to the parabola y equal to x square and of the portion POQ, so AP and QB will be tangent to the parabola and so this is the extremal minus 2, 3 to P that is the straight line AP which is tangent to the curve y equal to x square then the arc of the curve POQ and then the straight line QB which is tangent to the parabola at the point Q.

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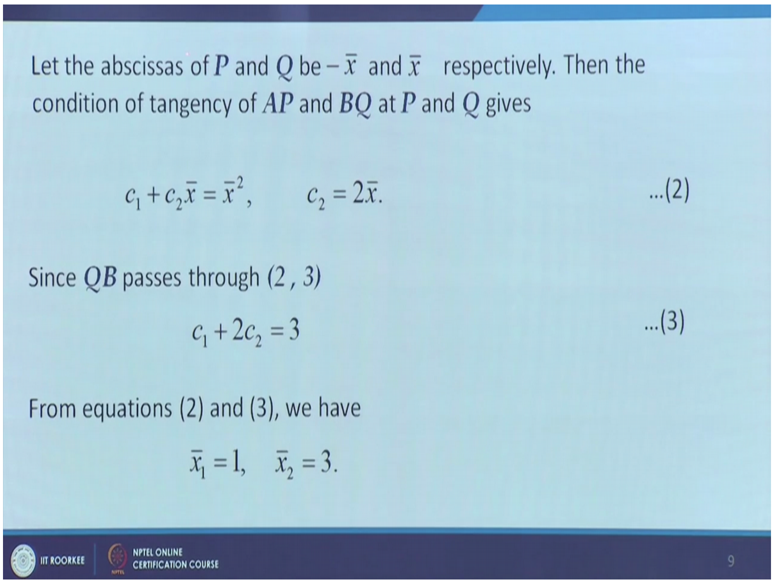

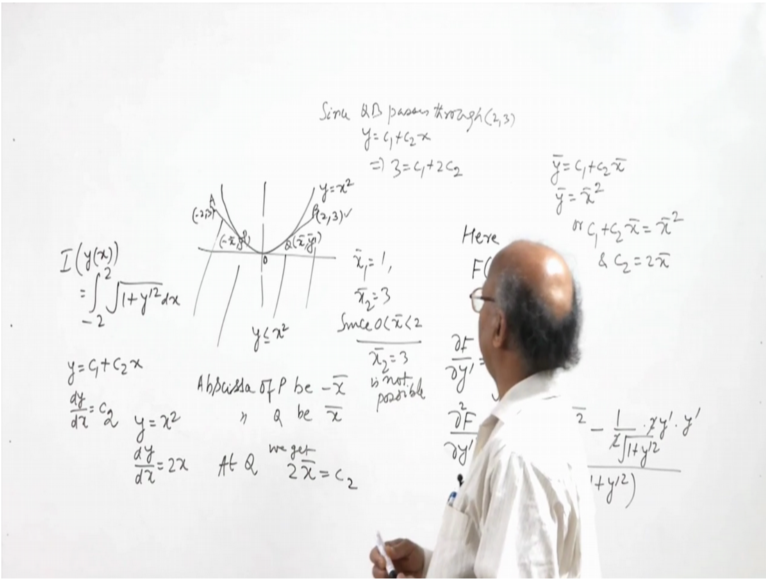

So let us let us say that the abscissas of P and Q be minus x bar and x bar respectively. So let abscissa of P be minus x bar because they are symmetric so and abscissa of Q be x bar then the condition of tangency of AP and BQ at P and Q gives. Now dy by dx y is equal to c 1 x plus c 2, so dy by dx is equal to c 1, okay at the point y y is equal to x square, so at the point of contact the the slopes must be same so dy by dx will be equal to 2x and therefore at the point of contact x bar at Q we will have 2x bar is equal to c 1, 2x bar equal to I have taken here c 1 plus c 2x, c 1 plus c 2x, so I will get here c 2 and therefore this will be 2x bar is equal to c 2 further more x bar, y bar lies on the parabola so let us say this is this is x bar, y bar and this is minus x bar, y bar the point P.

So x bar, y bar lies on the parabola as well as on the line c 1 x plus c 2 x, so y bar is equal to c 1 plus c 2 x bar and y bar is also equal to x bar square. So we can say that c 1 plus c 2 x bar is equal to x bar square and we have c 2 equal to 2x bar from the condition of tangency. So c 2 is equal to 2x x bar is equal to c 2 by 2 what we get is c 2 equal to 2x bar, so QB passes through now more over this line QB QB passes through 2, 3. So what we get y equal to c 1 plus c 2 x gives you 3 equal to c 1 plus 2 c 2, since QB passes through this 2, 3 point so we get 3 equal to c 1 plus 2 c 2.

Now from these equations this equation and this equation c 1 plus 2 c 2 equal to 3, we can put here x bar equal to c 2 by 2 or we can say and then simplify, okay we will get x 1 bar we will get two values x 1 bar is equal to 1 and (x 1 bar) x 2 bar is equal to 3 we get two values of x bar one value is we have taken as x 1 bar, x 1 bar is 1 or x 2 bar is 3. So there are two values of x bar one is x 1 bar which is 1 and another value of x bar we have taken as x 2 bar x 2 bar is 3.

Now what happens is (x bar) x 1 bar is equal to 1 and x 2 bar is equal to 3, these are two values of x bar. So x 2 bar equal to 3 is not possible because x 2 bar x 2 bar because x bar lies between 0 and 2 since 0 is less than x bar less than 2, so x 2 bar is equal to 3 is not possible.

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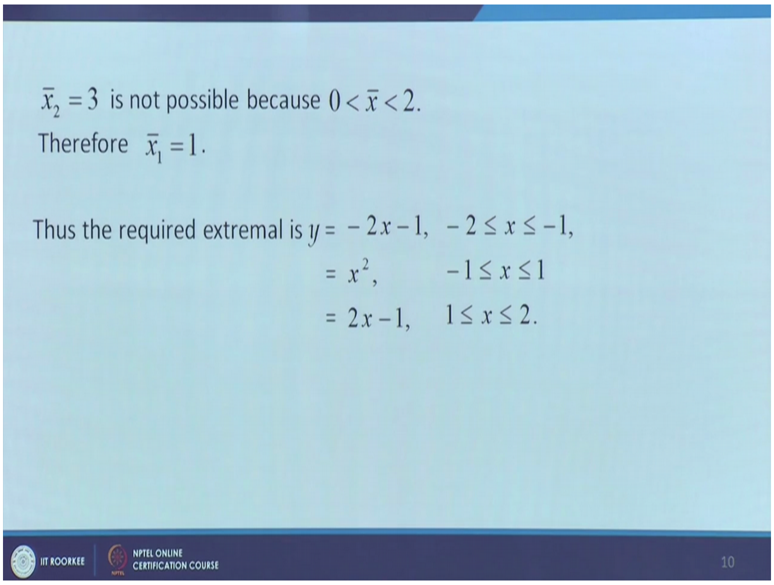

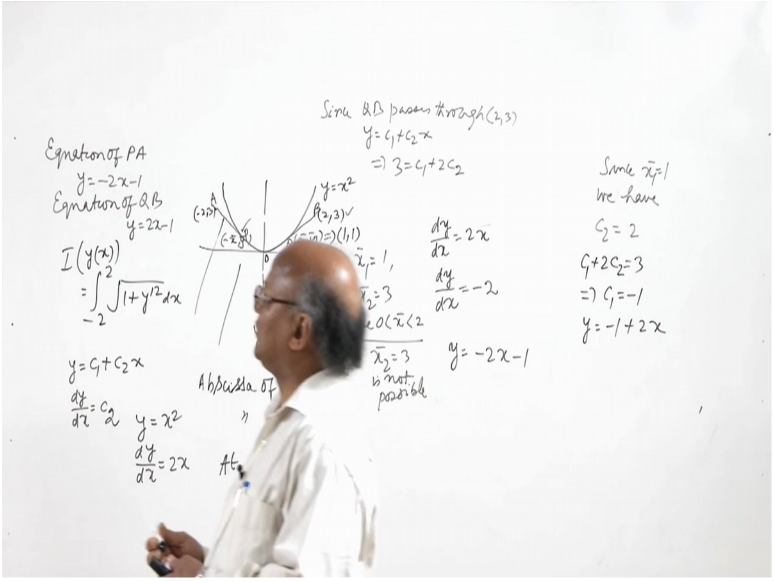

So therefore we take x 1 bar is equal to 1 and x 1 bar is equal to 1 so we will get the following extremal we had c 2 equal to 2x bar, so since x 1 bar is equal to 1 we have c 2 equal to 2, x bar is equal to 1 is admissible so we get c 2 equal to 2 and we have c 2 equal to 2 so we get c 1 equal to c 1 plus c 2 c1 plus 2 c 2 equal to 3 gives us c 1 equal to minus 1. Now what we have what happens so y is equal to c 1 plus c 2 x, so y is equal to minus 1 plus 2 x, okay. So y equal to 2 x minus 1 is the equation of the line QB and for the line AP we shall have minus x bar, y bar are the coordinates of this y equal to c 1 x plus c 2 x.

So the slope of this line is dy over dx is equal to what we have dy over dx equal to 2x and so 2x means x 1 bar is equal to 1, so minus x 1 bar so dy by dx will be minus 2 at this point, dy by dx will be minus 2, okay. So y will be equal to minus 2x minus 1 when we take x is equal to minus 1 it will be 2 minus 1, c 1 will be equal to c 1 is equal to minus 1, so we get x 1 ya it passes through minus x bar, y bar. So minus x bar means minus 1 and y bar is equal to when we take y is equal to c 1 x plus c 2 in this case y comes out to be 1.

So this will be x is equal to 1, so y is equal to 1, so this is 1, 1 point. So this is minus 1, 1 point, so here ya so the equation will be minus 2x minus 1 this is the equation of PA equation of PA will be y equal to minus 2x minus 1 equation of QB will be y equal to 2x minus 1. So the extremal will heave the equations, y equal to minus 2x minus 1 from minus 2 to minus 1 the extremal will coincide with the tangent with the line straight line AP which is tangent to the parabola at the point P, the equation of AP is y equal to minus 2x minus 1 so y equal to minus 2x minus 1 when x lies from minus 2 to minus 1 and then the arc of the parabola POQ, so from minus 1 to 1 y is equal to x square.

And then from the point 1, 1 to the point to the point 2, 3 that is less than or equal to x less than or equal to 2 we have y is equal to 2x minus 1. So this is the extremal extremal consist of the straight lines AP or POQ and the straight line segment QB and there equations are given by this.

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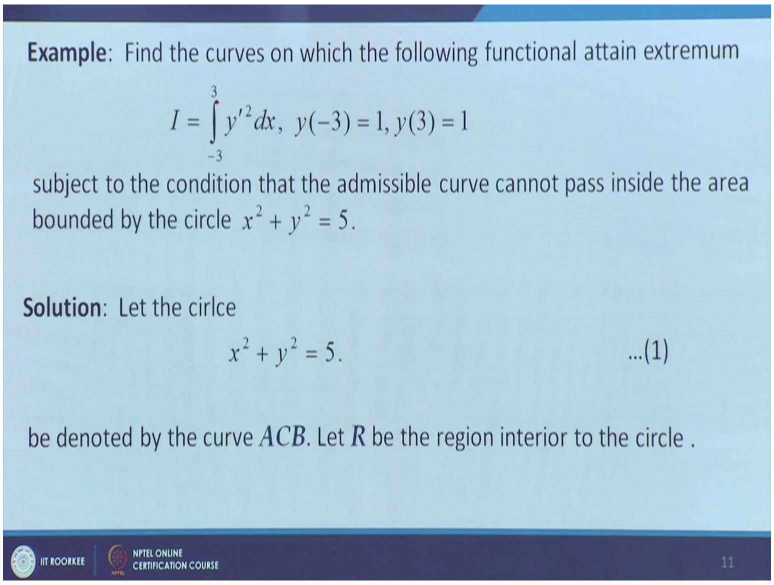

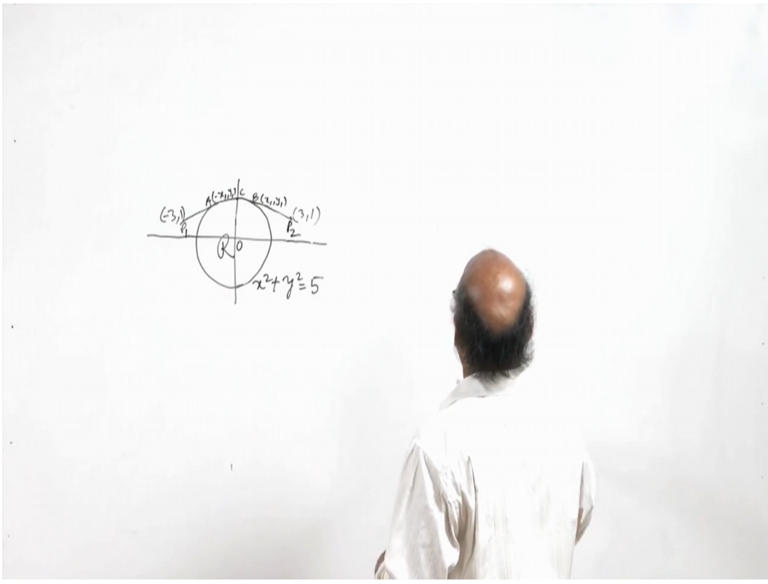

Now let us take another problem here we have to extremize the functional I equal to integral over minus 3 to 3 y dash square dx, where y assumes value 1 at x equal to minus 3 and it assumes value 3 when x is equal to it assumes the value 1 when x is equal to 3 and it is subjected to the condition that the admissible curve cannot pass inside the area bounded by the circle x square plus y square equal to 5.

Now here we have the circle x square plus y square is equal to 5 at x equal to minus 3, y is equal to 1, so this is let us say the point minus 3, 1 and this is the point 3, 1. The condition is given that the admissible curve cannot pass inside the area bounded by the circle x square plus y square equal to 5 and the circle is this. Now the curve okay these straight line segments straight lines will be tangent to the curve like this.

So we have the circle x square plus y square equal to 5 it is denoted by the curve ACB, A, C, B, ACB and let us say this is P and Q you can see that the curve does not pass through the area this one. Now or let us say R be the region interior to the circle R be the region interior to the circle.

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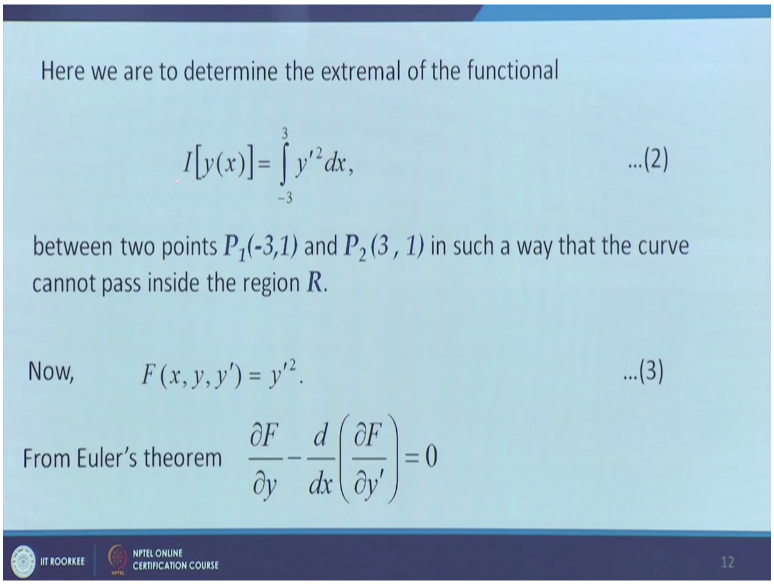

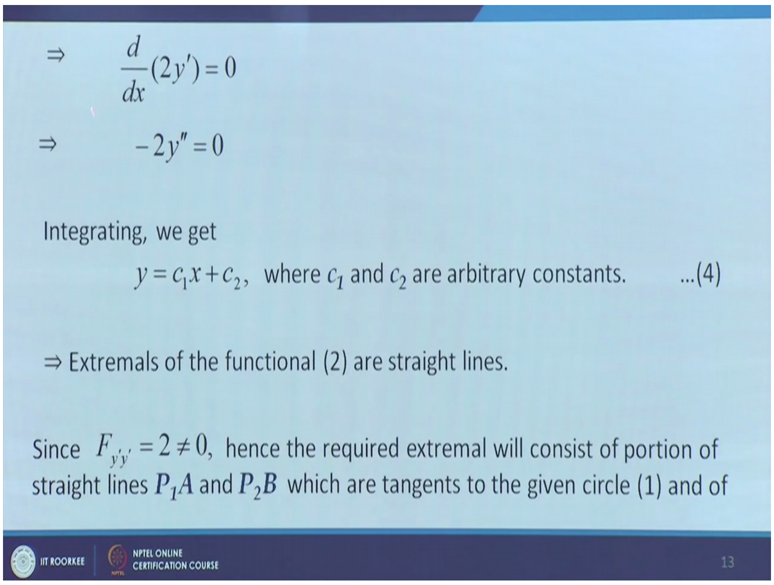

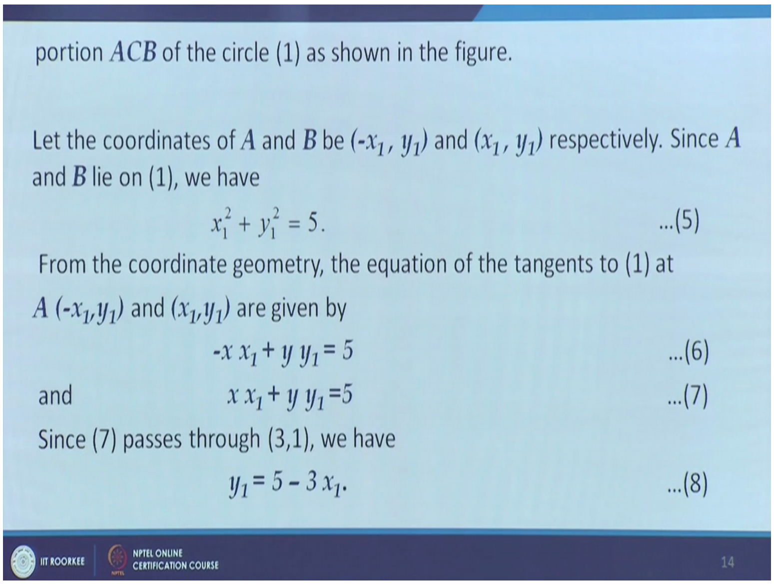

So we have to determine the extremal of the function that is we have to find the path between the points minus 3, 1 and 3, 1 in such a way that the curve cannot pass inside the region R and this is to be extremize. Now F x, y, y dash is equal to y dash square here. So from the Eulers theorem F y minus d over dx F y dash is equal to 0 and this gives you d over dx of 2y dash equal to 0 because F is independent of y, so delta F by delta y is equal to 0, delta y delta y dash is 2y dash, so this gives us 2y double dash is equal to 0. And 2y double dash is equal to 0 gives y equal to c 1 x plus c 2 where c 1, c 2 are arbitrary constants. So extremal of the functional to 2 extremal of this functional are straight lines.

Now since F y y dash y dash F y dash F y dash y dash here you can see F y dash is 2 y dash and F y dash y dash will be 2, so F y dash y dash is not equal to 0 and therefore the required extremal will consist of portion of the straight lines P 1 A let me call it as P 1 A and this as P 2 B, so P 1 A, P 2 B which are tangent to the circle 1 and of the portion ACB of the circle as shown here ACB of the circle as shown here.

Let the coordinates of A and B be minus x 1, y 1 and x 1, y 1 respectively. So this is minus x 1, y 1 and the coordinates of this are x 1, y 1. Since A and B lie on the curve x square plus y square equal to 5 the parabola the circle, so x 1 square plus y 1 square is equal to 5.

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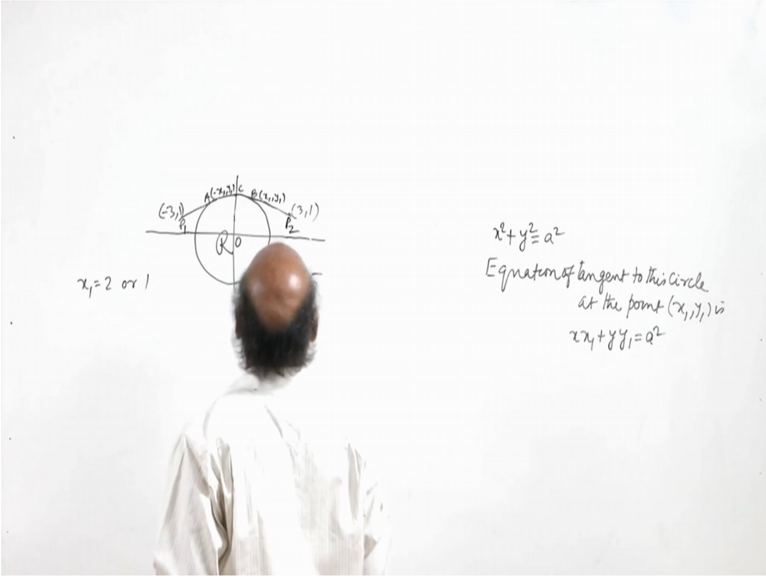

portion ACB of the circle (1) as shown in the figure.

Let the coordinates of A and B be $(-x_1, y_1)$ and (x_1, y_1) respectively. Since A and B lie on (1), we have

$$x_1^2 + y_1^2 = 5. \quad ...(5)$$

From the coordinate geometry, the equation of the tangents to (1) at A $(-x_1, y_1)$ and (x_1, y_1) are given by

$$-x\,x_1 + y\,y_1 = 5 \quad ...(6)$$

and

$$x\,x_1 + y\,y_1 = 5 \quad ...(7)$$

Since (7) passes through (3,1), we have

$$y_1 = 5 - 3\,x_1. \quad ...(8)$$

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Substituting the value of y_1 in (5), we get

$$x_1 = 2 \text{ or } 1$$

$$\Rightarrow \quad y_1 = -1 \text{ or } 2.$$

Hence coordinate of B are $(2, -1)$ or $(1, 2)$. But we see that coordinate $(2,-1)$ are not admissible . So we take $(1, 2)$ as coordinates of B. Then the corresponding coordinates of A are $(-1, 2)$.

Putting $x_1 = 1$ and $y_1 = 2$ in (6) and (7), we see that the equations of tangents A and B are $-x + 2y = 5$ and $x + 2y = 5$ respectively.

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Now from the coordinate geometry we know that if if we try tangent to the curve x square plus y square equal to a square at the point x 1, y 1 the equation of tangent to the circle at the point x 1, y 1 is x, x 1 plus y, y 1 is equal to a square. So making use of this we have the equations of the tangent lines P 1 A and P 2 B as minus x x 1 plus y y 1 equal to 5 and x x 1 plus y y 1 equal to 5.

Now this x x 1 plus y y 1 equal to 5 also passes through the point P 2 which is 3, 1. So let us put the coordinates here x equal to 3, y equal to 1. So we get y equal to 5 minus 3 x 1. Now y 1 equal to 5 minus 3 x 1 we put in this equation x 1 square plus y 1 square is equal to 5 and then we get the values of two values of x 1 they are 2 or 1 and x 1 is 2 or 1.

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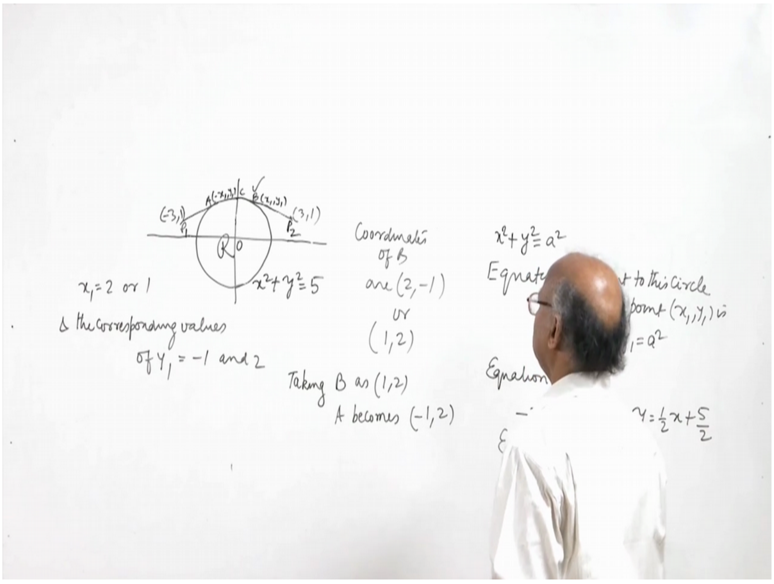

Now let us see x 1 is equal to and corresponding values of y 1 are and the corresponding values of y 1 y 1 are minus 1 and 2, okay. So the coordinates of B will be either 2 minus 1, okay coordinates of this point coordinates of B are 2 minus 1, okay 2 minus 1 or 1, 2. Now clearly the coordinates of B cannot be 2 minus 1, okay so because B lies in the first quadrant so the coordinates of B we take as 1, 2 and the corresponding coordinates of A are therefore minus 1, 2, okay. So taking B as 1, 2 A becomes minus 1, 2.

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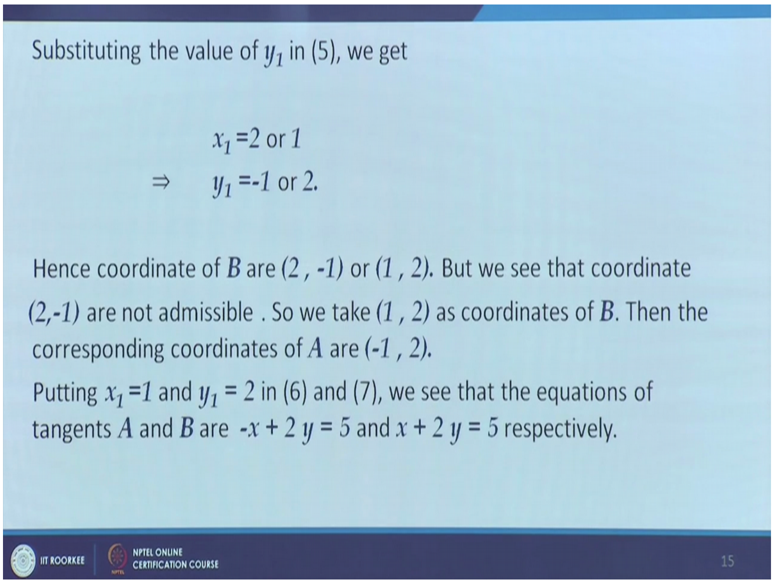

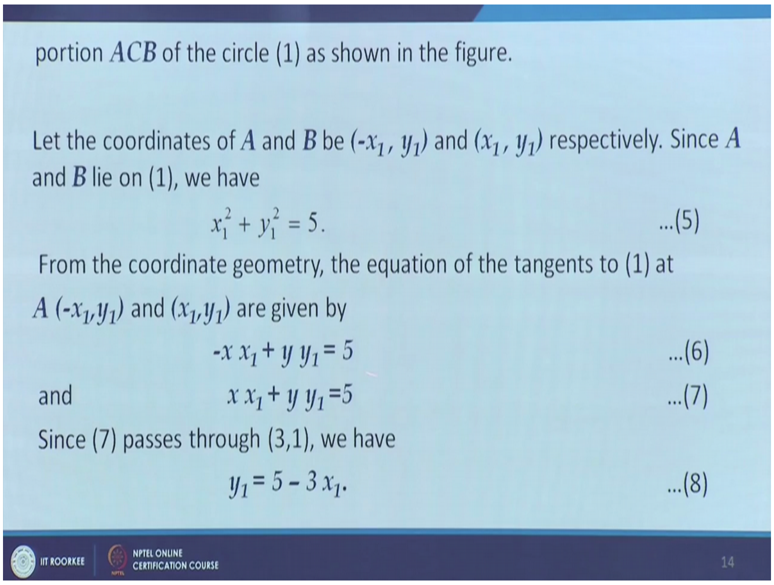

Now putting x 1 equal to 1 we have x 1 equal to 1 here, y 1 equal to 2. In these equations 6 and 7 we get the equation of the tangent lines P 1 A and P 2 B and they are the equation of the tangent at A is minus x plus 2 y equal to 5 and the equation of the tangent at the point B is x plus 2 y equal to 5.

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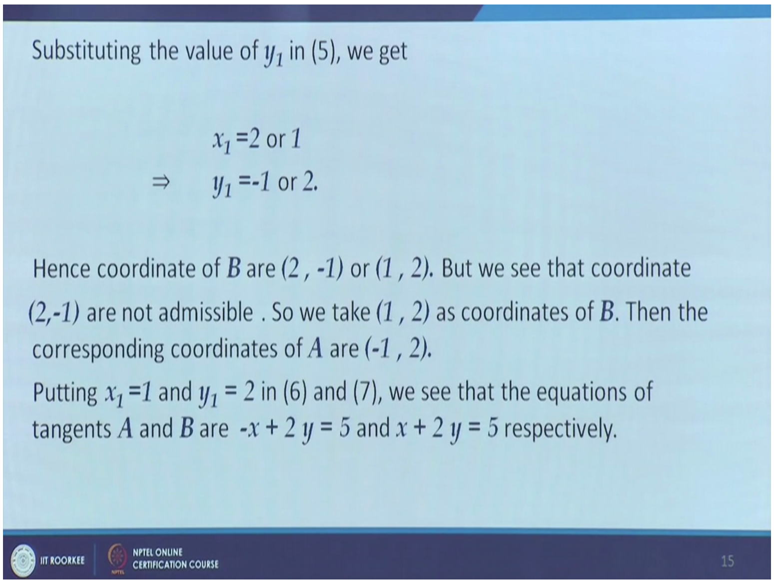

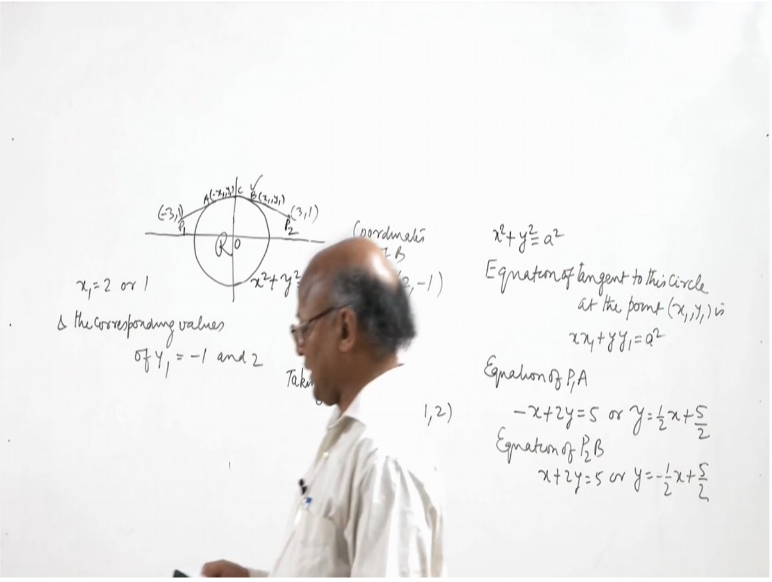

So the required extremal therefore let us see we have equation of P 1 A equation of P 1 A is minus x plus 2 y equal to 5 or y is equal to 1 by 2 x plus 5 by 2 and equation of P 2 B it is x plus 2 y equal to 5 or y is equal to 1 by 2 minus y is equal to minus 1 by 2 x plus 5 by 2. So we have the required extremal as y equal to x by 2 plus 5 by 2 y equal to x by 2 plus 5 by 2 that is tangent P 1 A and it is from minus 3 to 1 minus 3 to minus 1 from P 1 to A, A is having coordinates minus 1, 1 minus 1, 2. So for the tangent P 1 A x varies from minus 3 to minus 1 and then the arc of the circle from A to B that is ACB.

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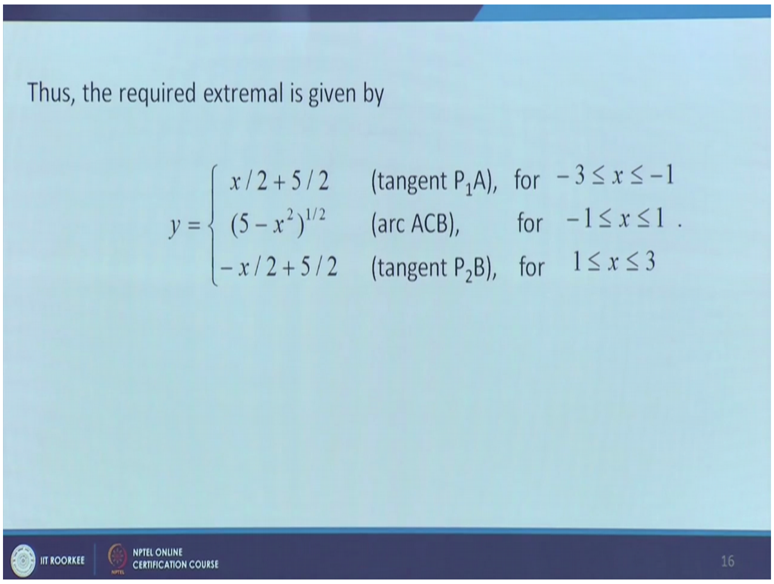

And the equation of that is 5 minus x square square root we are taking y to be positive because A, B lies above the x axis, so arc ACB so from minus 1 less than or equal to x less than or equal to 1 and then the tangent P 2 B tangent P 2 B is y equal to minus half x plus 5 by 2 from the point x 1, y 1 to the point 3, 1, x 1, y 1 is 1, 2 so x varies from 1 to 3 so this is how we find the extremal for this problem. So with this I would like to conclude my lecture, thank you very much for your attention.