**Integral Equations, Calculus of Variations and their Applications**

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**Lecture 46**

**Euler's equation-11**

Hello friends, welcome I you to my lecture on Euler's equations-II.

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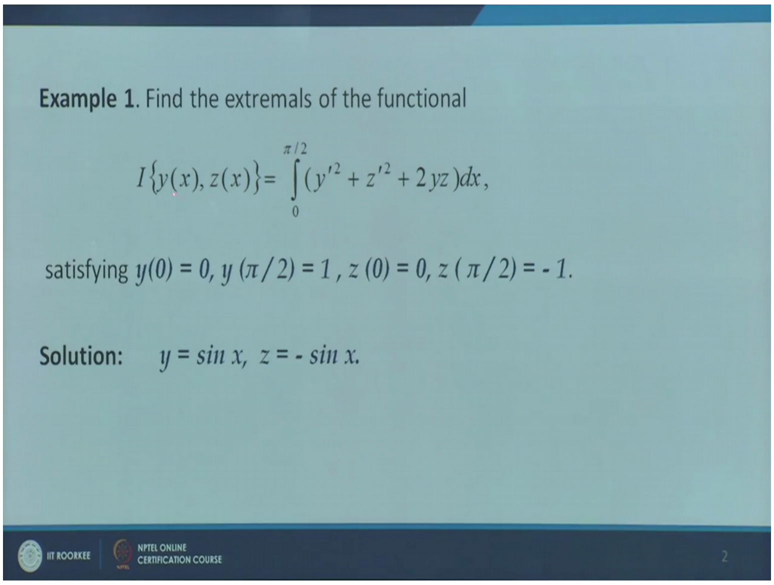


Now here, we shall be discussing the two examples, where the functional involves several dependent variables with (())(0:38) depend on a single variable like , the article which we have done in the previous lecture. So (the) all these examples are dependent on that. So find the externals of the functional I if (())(0:51) y(x), z(x) equal to integral 0 to pi by 2 y dash square plus z dash square plus 2 by z dx. So here y and z are two dependent variables that depend on a single independent variable x, y dash is dy by dx, z dash is dz by dx.

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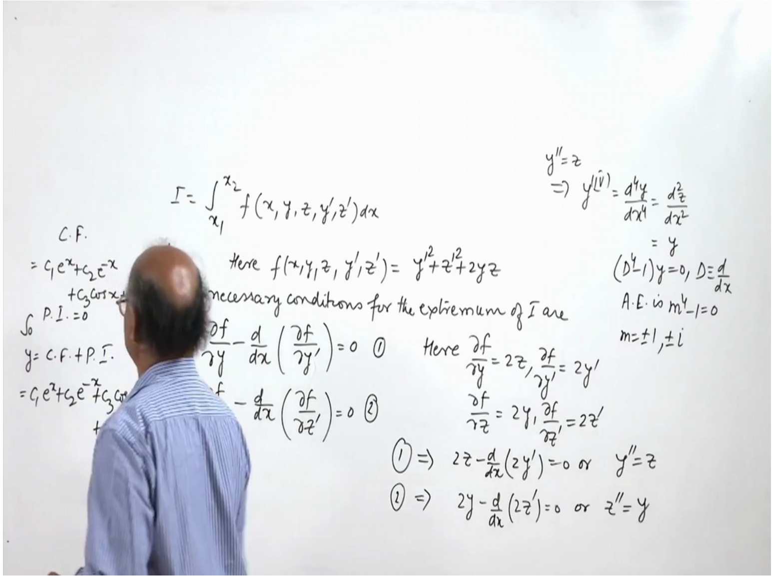


So here if you be compare it with the standard form, x1, x2 f(x, y, z, y dash, z dash) dx, we have this. So here f(x, y, z, y dash, z dash) is equal to y dash square plus z dash square plus 2 yz. Now the necessary condition conditions for the extremum of I are, so here in the article we have taken y1 and y2 as the two dependent variables that depend on a single variable x. here, instead of y1, y2 we have y and z. So we have (the) differential equations as delta f over delta y minus d over dx delta f over delta y1 dash is equal to (y1) y dash equal to 0. This is 1 equation, because y1 is equal to y here and then the second equation is delta f by delta z minus d over dx. These are the two necessary conditions for the extremum of I.

Now let us find delta f over delta y. So here delta f over delta y is equal to 2z delta f over delta y dash is equal to 2y y dash, delta f over delta z is equal to 2y delta f over delta z as z dash is equal to 2z dash. So let me call this as equation 1,this as equation 2 then 1 implies 2z minus d over dx of 2y dash is equal to 0 or dy dash over dx becomes y double dash 2 we can cancel. So y double dash is equal to z or 2 gives you 2y minus d over dx of 2z dash, derivative of z dash with respect to x is z double dash and 2 we can cancel. So we get z double dash is equal to y. So we have 2 equations y double dash is equal to z and z double dash is equal to y.

Now let us differentiate this equation, y double dash is equal to z twice. So y double dash is equal to z gives you, this is d4 y over dx 4 (d dx) this is $4^{th}$ derivative of dy. So this is d4y over dx 4 is equal to d square z by dx square and d square z by dx square is equal to y. So we get d4 minus 1y equal to 0, where D is d over dx. Now this is $4^{th}$ order linear differential equation, but the coefficients are constant. So auxiliary equation is m4 minus 1 equal to 0, the roots are the roots of this equation are m equal to plus minus 1 plus minus i and so we can write the complementary function. The complementary function will be c1e to the power x plus c2 e to the power minus x plus c3 cos x plus c4 sin x, okay. Now particular integral is 0, because the right hand side is 0. So y is equal to particular integral is 0. So y is equal to complementary function plus particular (())(6:08) integral, which is equal to c1e to the power x plus c2e to the power minus x plus c3 cos x plus c4 sin x.

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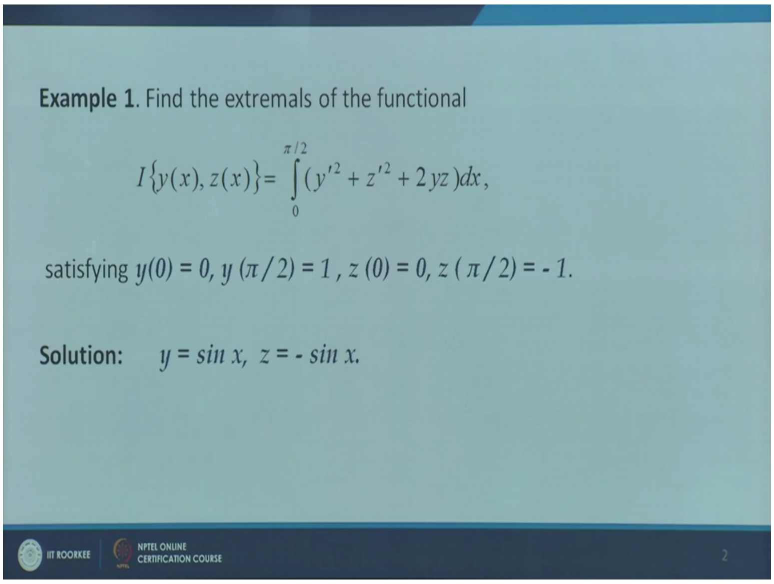


Now let us in order to find the 4 constants c1.c2, c3, c4 we use the given conditions on y and z. the two conditions are for the values of y at 0 and pi by 2, the boundary conditions and the two boundary conditions are given for z at x equal to 0, z is 0 at x equal to pi by 2, z equal to minus 1. So we need 4 conditions to determine the 4 constants c1, c2, c3, c4. So two conditions we shall get for the two boundary conditions given for y and two conditions we shall get for the boundary conditions given for z.

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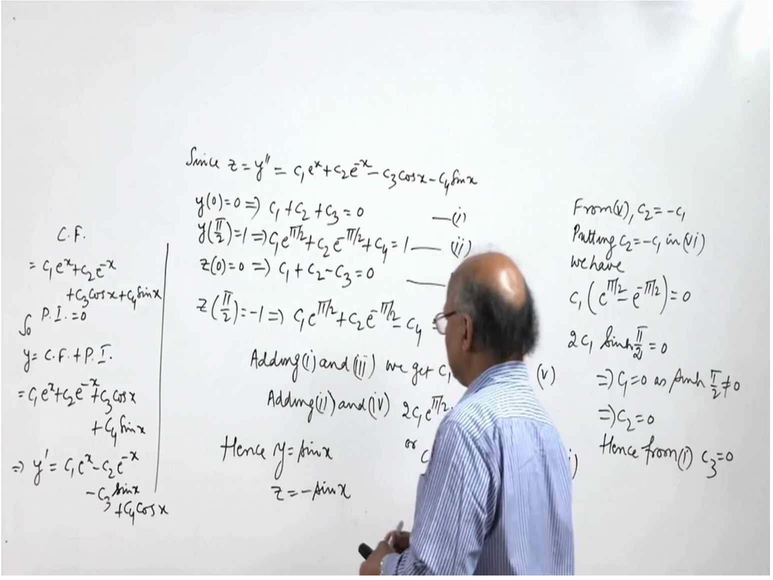


So let us write the corresponding expression for z also. So we see, here that y double dash is equal to y double dash is equal to z. So z is equal to since z is equal to y double dash we have first we find y dash. So y dash will be equal to c1 e to the power x minus c2e to the power minus x minus c3cosx and then we have c4sinx and y double dash, y double dash will be equal to c1e to the power x c2e to the power minus x into minus 1. So plus c2e to the power x we have and then we have plus c3sinx, derivative of cosx is minus sinx here and derivative of sinx is cosx here. So minus c3 cosx, it will be and then we have minus c4sinx, y is c1e to the power x plus c2e to the power minus x c3cosx plus c4sinx when we differentiate it, we get y dash equal to c1e to the power x minus c2e to the power minus x minus c3sinx plus c4cosx. We differentiate it again we get c1e to the power x plus c2e to the power minus x minus c3cosx minus c4sinx, okay.

Now y(0) equal to 0 gives y(0) equal to 0 gives you put 0 for x, you get c1e to the power 0, which is 1 plus c2 then we have c3cos0 is 1. So we get c1 plus c2 plus c3sinx is 0. When x is equal to 0. So this is equal to 0 and then y at pi by 2 is 1. So this will give you c1e to the power pi by 2 plus c2e to the power minus pi by 2, then cos pi by 2 is 0, sin pi by 2 is 1, so we get c4 equal to 1. Now we have z(0) equal to 0. So replace here x equal to 0. So we get c1 plus c2 and then minus c3 equal to 0 and the $4^{th}$ condition is z(pi by 2) is minus 1. So z let us put pi by 2 for x. So c1e to the power pi by 2 plus c2 e to the power minus pi by 2 and then we get cos pi by 2 is 0. So minus c4 is equal to minus 1, okay. Let me call it equation number 1, this equation number 2. Here, equation number 3 and this one as 4, we can solve these 4 linear equations for the 4 unknown coefficients constant c1, c2, c3, c4.

So if you add 1 and 3, adding 1 and 3 what we get is, c3 will cancel out, 2c1 plus c2 equal to 0, we get c1 plus c2 equal to 0. if we add 2nd and 4th, we shall have (c1) 2c1e to the power pi by 2 plus 2c2e to the power minus pi by 2 and c4, c4 will cancel 1 and minus 1 will cancel we will get equal to 0. So here we get c1e to the power pi by 2 plus c2e to the power minus pi by 2 equal to 0. Now let me call it equation number 5 and this as equation number 6. So from 5 we get c2 equal to minus c1. Let us put this value in 6. So putting we have c1 times e to the power pi by 2 minus e to the power minus pi by 2 equal to 0.

Now e to the power pi by 2 minus e to the power pi by 2 is equal to 2sin hyperbolic 2 cos hyperbolic pi. So we get 2 c1 cos hyperbolic pi by 2, cos hyperbolic sorry, it is sin hyperbolic pi by 2. So since sin hyperbolic pi by 2 is not 0. This implies c1 equal to 0, c1 equal to 0 makes it c2 equal to 0. So which imply further c2 equal to 0. Now c1 c2 equal to 0. So 3rd equation one gives you hence from 1, c3 equal to 0. So we get c1 c2 c3 equal to 0 and therefore, c1 equal to 0, c2 equal to 0 gives you c4 equal to 1. So and 2nd equation gives you c4 equal to 1, so c4 is equal to 1, c1, c2, c3 equal to 0. So y is equal to hence y is equal to sinx and z is equal to minus sinx, we can see that the values of y and z which we have got satisfy the given boundary conditions.

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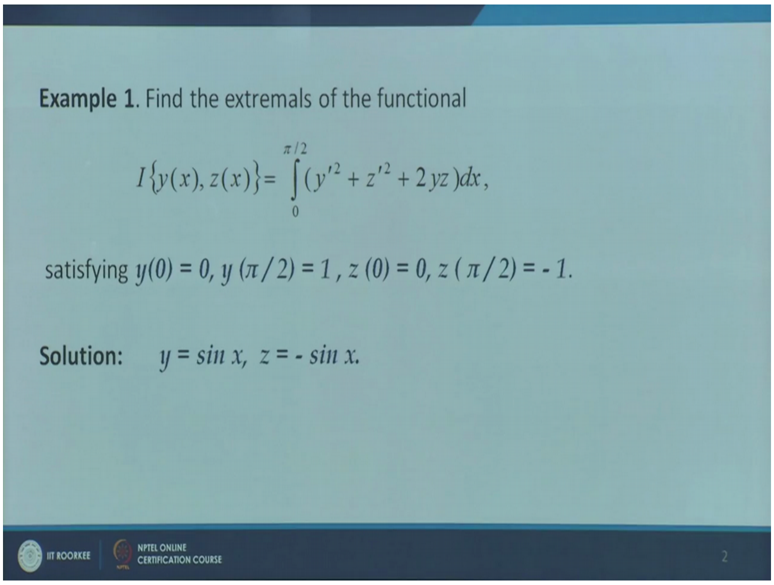


y is equal to sinx at x equal to 0, yo then y at pi by 2 y at pi by 2 is 1 and it is also 1 given z0 equal to 0, because sinx is 0 and z pi by 2 is minus sin pi by 2, which is minus 1. So there be solution, y equal to sinx, z equal to minus sinx.

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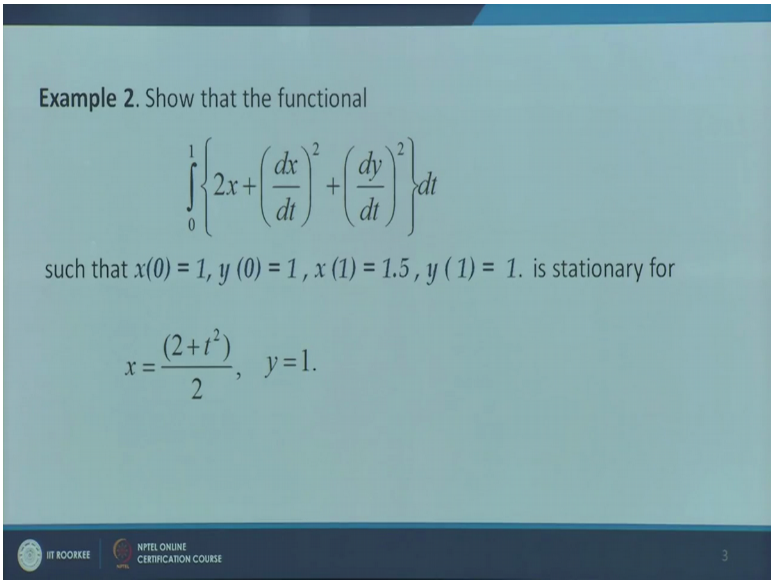


Now let us take one more example. So this time we are given the functional where x1 is 0 x2 is 0 the limits of integration are 0 and 1.

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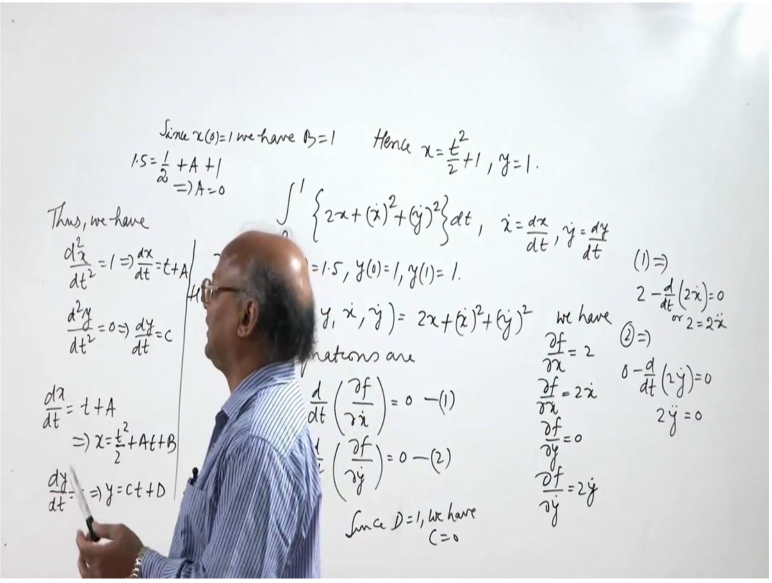


And we have 2 dependent variables x and y, which depend on a single independent variable, which is t. So we have 2x plus dx by dt whole square let me put it as x dot whole square plus y dot whole square and we are given at x(0) equal to 1, x(1) equal to 1.5, y(0) equal to 1, y(1) equal to 1. So we are given 2 boundary conditions for x and 2 for y. Now so here f (t, x, y, x dot, y dot) we have two independent variable, x and y, 1 independent variable t, so this is equal to 2x plus x dot square plus y dot square. The Euler's equations are delta f over delta y1. So instead of y1 we have x here minus d over dx will be replaced by d over dt, then delta f over delta y1 dash. So we have delta f over delta x dot equal to 0. This is first equation. Second is delta f over delta y2, which is delta f over delta y minus d over dt delta f over dash y dot.

Now for the given function f we have partial derivative of f with respect to x is 2 partial derivative of f with respect to x dot 2x dot, partial derivative of f with respect to y is 0, partial derivative of f with respect to y dot is 2y dot. Now so equation 1 and 2, 1 gives us 2 minus d over dt of 2x dot equal to 0 and 2 gives us delta f over delta y is 0, 0 minus d over dt of 2y dot. Now this is what 2 we can cancel. This is or I can say 2 equal to 2x double dot. So we have x double dot equal to 1 and here, we have (y) 2y double dot equal to 0. So thus we have d square x by dt square equal to 1 and d square y over dt square is equal to 0.

Now this will give you when we integrate dx by dt equal to some constant A and here, some it will be equal to t plus some constant A when we integrate and here when we integrate we will get dy by dt equal to some constant c. So we further integrate this equation. So dx by dt over dx by dt equal to t plus A gives us, x equal to t square by 2 plus At plus B and dy by dt equal to C gives you on integration y equal to Ct plus D.

Now let us use the given conditions x(0) is equal to 1. Since, x(0) equal to 1 we have B equal to 1, and x(1) is equal to 1.5. So 1.5 equal to 1 by 2, then A plus 1. So this implies A is equal to 0 and hence, x is equal to t square by 2 plus 1. Now for y we have y(1) y(0) equal to 1, gives you 1 equal to D and y(1) equal to 1 gives us 1 equal to C plus D. So since D is 1, we have C equal to 0. So C is equal to 0, D equal to 1, so we get y equal to 1.

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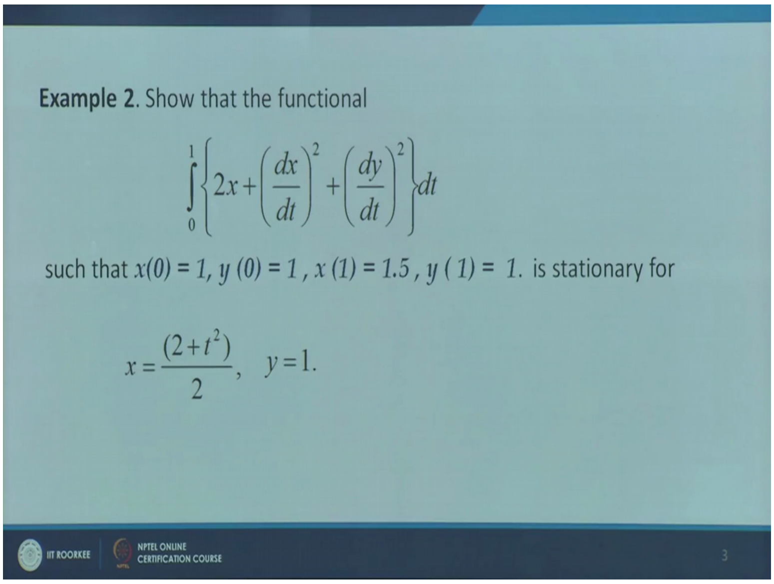


So we have get the extremal of the given functional as x equal to t square by 2 plus 1 and y equal to 1.

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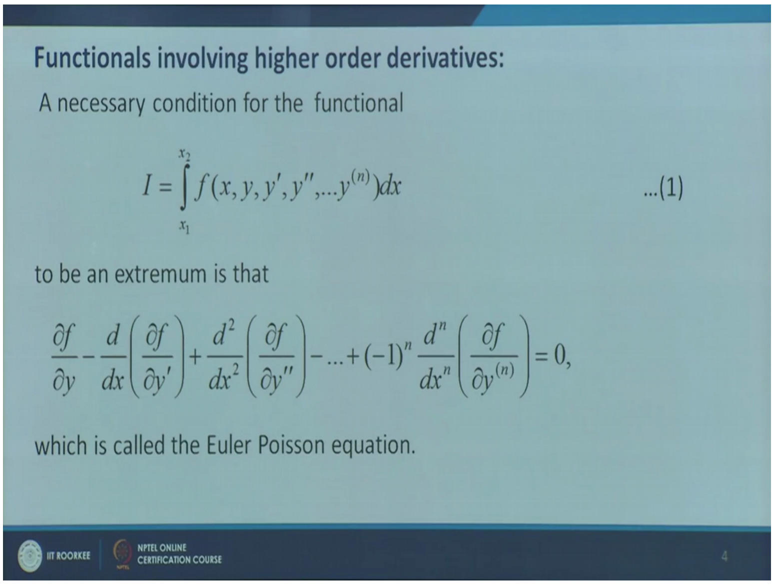


now let us discuss the case of those functional which are to be extremize and they involve higher order derivatives. So here we shall be having at independent variable x and a dependent variable y and its higher order derivative that is y double dash, y triple dash and so on and nth derivative of y. So a necessary condition for the x functional, which involves higher order derivatives of y with respect to x to be an extremum is that partial derivative of f with respect to y minus d over dx partial derivative of f with respect to y dash plus d square x by dt dx square partial derivative of f with respect to y double dash n so on. The term corresponding to the nth derivative is minus 1 to the power n dn over dxn partial derivative of f with respect to yn. Now this equation is called as the Euler's poisson equation. Now as in the case of the previous lecture, here also we shall prove this condition we shall find this condition for the case n equal to 2. The result for n equal to 2 can be easily extended to the case of n.

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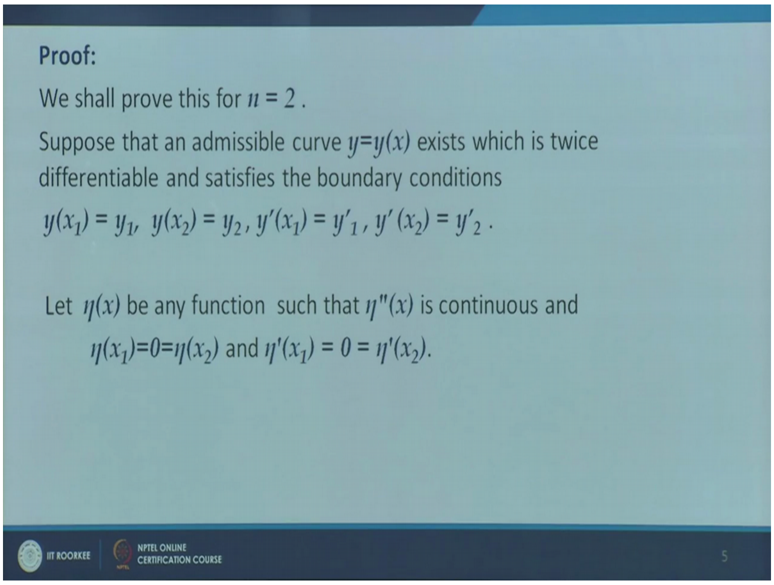


Now so let us say that we found we have a curve y equal to yx, which is twice differentiable and satisfies the boundary conditions. So y is equal to y1 at x equal to x1. Y assumes value y2 at x equal to x2 and its derivatives at x1 and x2 are denoted by y1 dash and y2 dash. let eta(x) be a function which is twice differentiable and it is the second derivative is continuous. It is (())(23:21) differentiable twice and its values at x1 and x2 are 0s. Its derivative also vanishes at x1 and x2.

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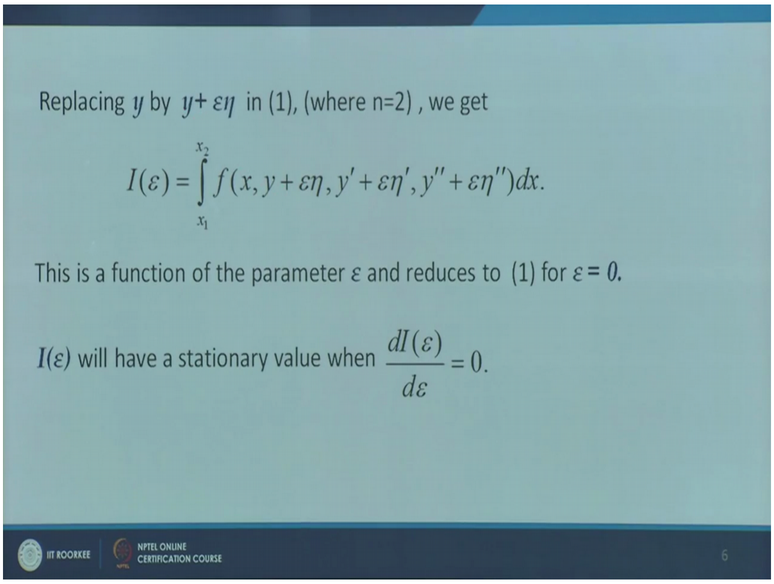


What we do is in the functional I, uh for n equal to 2, we replace y by y plus epsilon eta.

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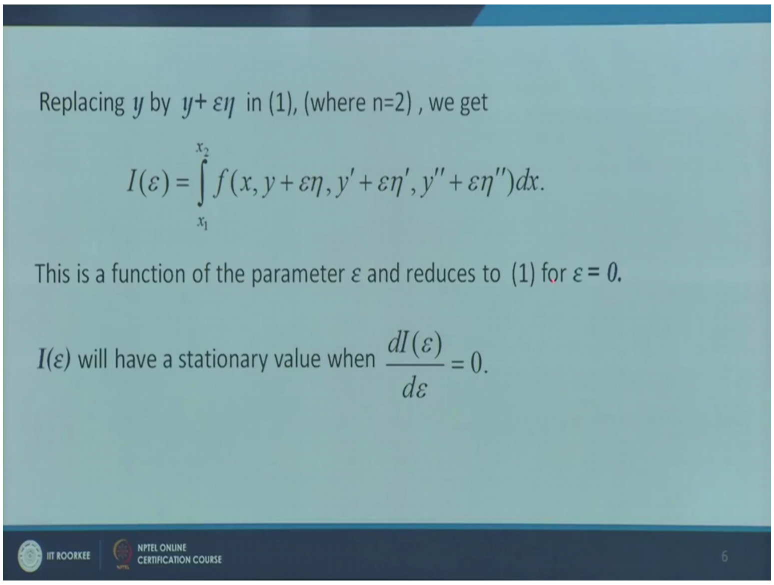

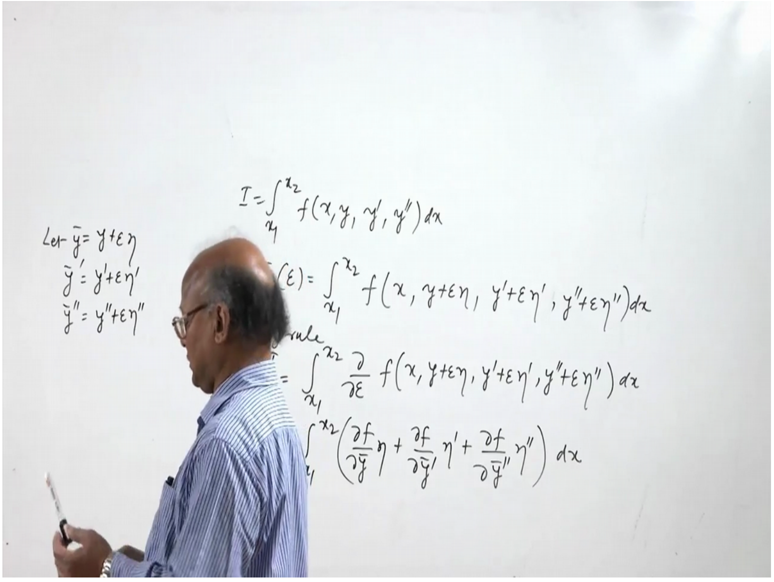


So then we shall have for the case n equal to 2, I is equal to for the case n equal to 2 we have the functional integral x1 to x2 f(x, y, y dash, y double dash). So let us replace y by y plus epsilon eta and y dash by y dash plus epsilon eta dash, y double dash by y double dash plus epsilon eta dash, double dash. So then I epsilon is equal to I will be depending on parameter epsilon and we will have I depending on epsilon equal to this. Now this is a function of the parameter epsilon and you can see here when epsilon is equal to 0. This I becomes the given I. So I epsilon reduces to the given I for epsilon equal to 0. Now this I epsilon will have a stationary value provided the I over the epsilon is equal to 0.

So let us differentiate it with respect to epsilon by using the Leibniz rule. So we can write like this, because x1 and x2 are constants. Now let us say, let y bar be equal to y plus epsilon eta, y bar dash be equal to y dash plus epsilon eta dash, y bar double dash be equal to y double dash plus epsilon eta double dash. So then when we differentiate this with respect to epsilon we shall have delta f over delta y bar into eta plus delta f over delta y bar dash eta dash delta f over delta y bar double dash eta double dash dx.

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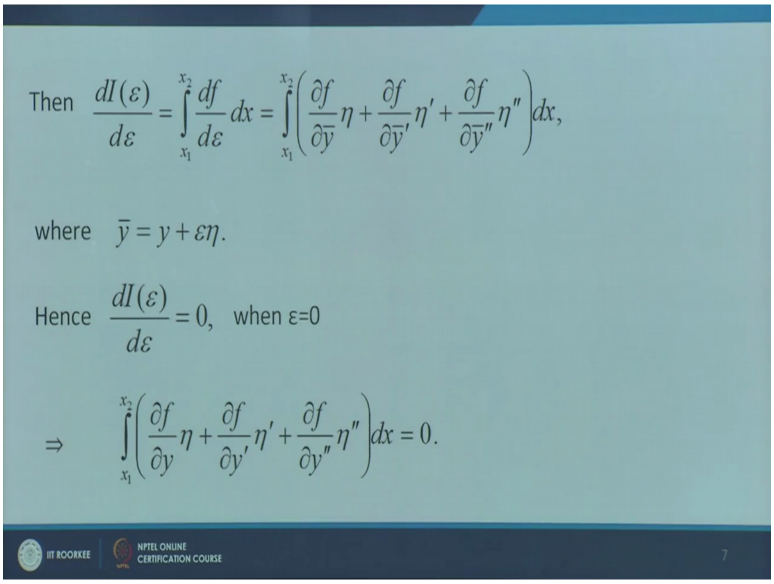


Now this is to be this is equal to 0. When I is equal to 0, so we shall have when dI epsilon equal to 0 dI by epsilon equal to 0 gives you integral x1 to x2.

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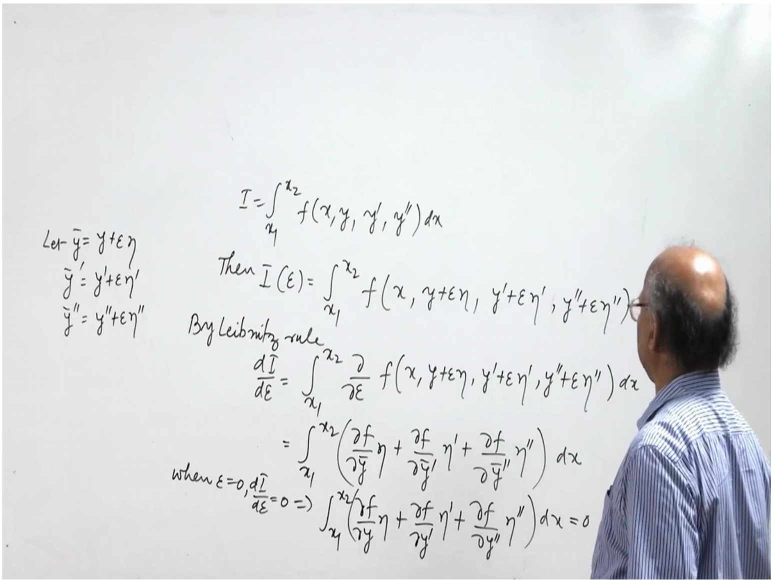


Now when epsilon is equal to 0, y bar becomes y. So delta f by delta y eta plus delta f y bar dash becomes y dash eta dash then this becomes y double dash eta double dash dx equal to 0. Now let us see we will now integrate the second term here and the 3rd term here by parts. So this for second terms will be integrated once with respect y (())(27:36) by integration by parts and this term will be integrated, 3rd term will be integrated twice by parts.

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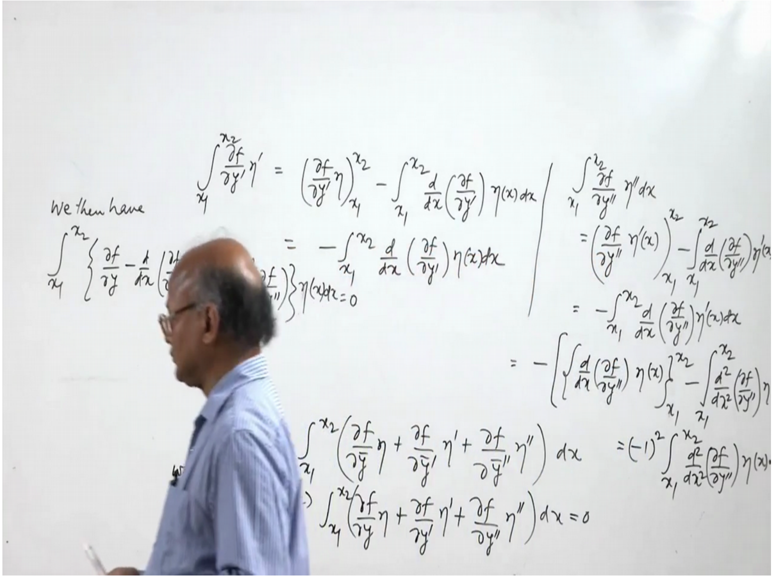


So let us see what we get when integrate by parts. So integrating delta f over delta y dash eta dash by parts we get delta f over delta y into eta, we get this. Now we have assume that eta(x) (())(28:32) that x1 and x2 since eta(x1) is equal to eta(x2) is equal to 0. This term will be zero, so is equal to we will get minus x1 to x2 d over dx this. Now let us integrate the 3rd term twice with respect to Leibniz rule with respect y integration by parts. So x1 to x2, let us integrated by parts, so we get delta f over delta y double dash eta dash (x) we get this.

Now eta dash(x) we have assumed to be equal to 0 at x1 and x2, eta dash(x1) is equal to eta dash(x2) is equal to 0. So this will make this term 0 and we will get minus integral x1 to x2 d over dx this. Now let us integrate it once more with respect to x by parts. So when we integrated once more, we get minus okay we will get this. Now eta(x2) is 0 eta(x1) is 0. So this becomes 0 and we get minus 1 square this. So let us substitute the value of the second term and the value of the 3rd term here and we shall have be then that integral x1 to x2 delta f over delta y and then the value of this is minus d over dx delta f over delta y dash and there we have minus 1 square into eta(x) dx equal to 0 is what we get.

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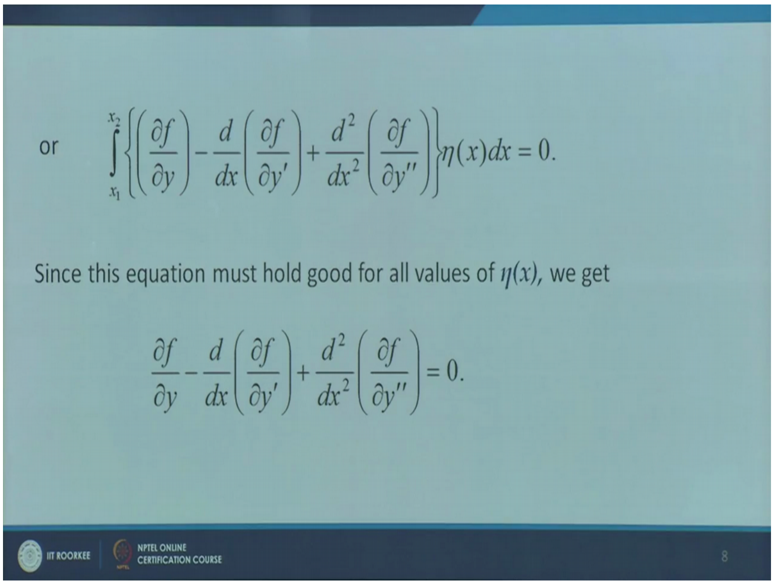


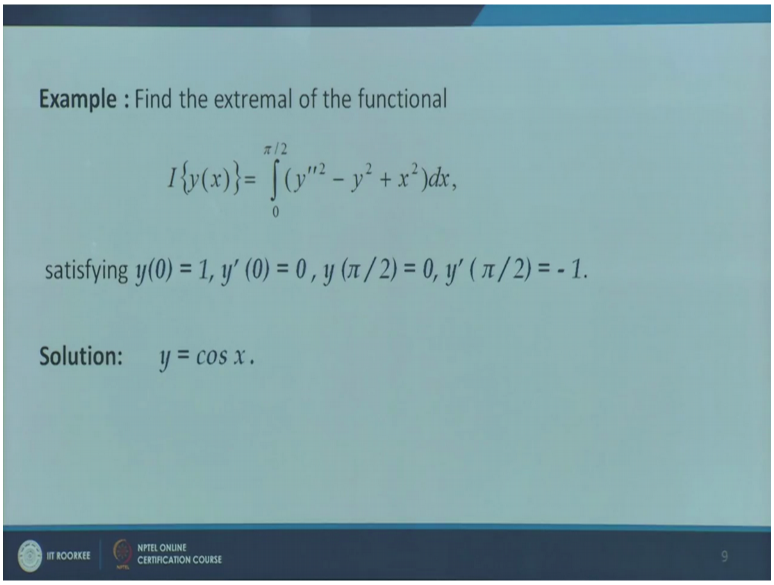


now so we get we arrive at this equation. Now since this equation holds good for all values of eta(x), we must have this equation, delta f over delta y minus d over dx delta f over delta y dash plus d square over dx square delta f over delta y double dash equal to 0. So this this is a what we have get in the case of n equal to 2 and we can easily generalize it to the case of n derivatives.

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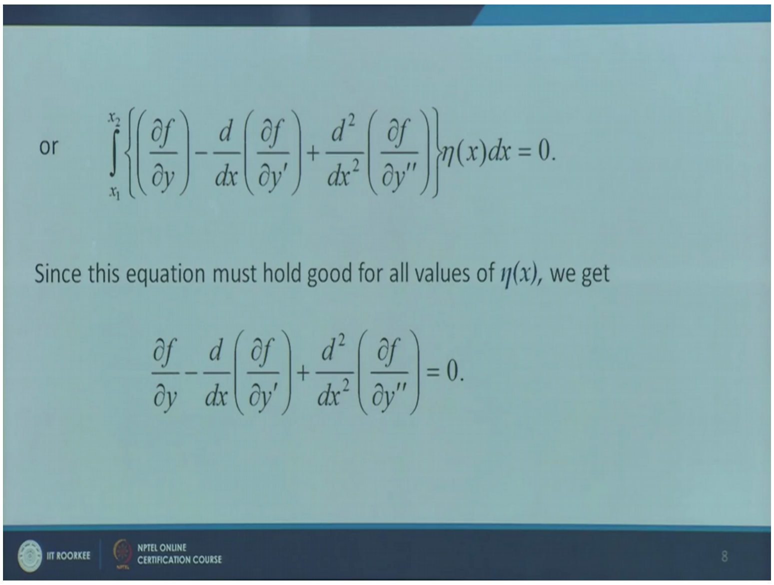


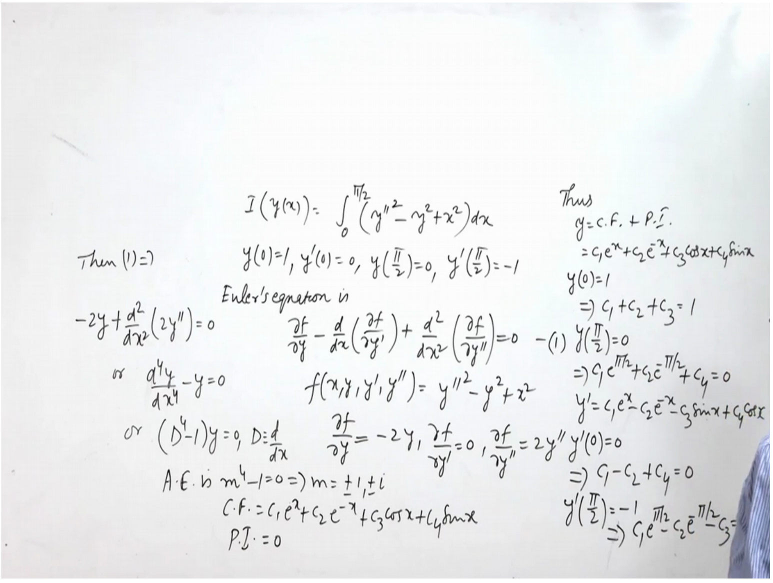


Let us take an example on this. So we have I(y(x)) equal to 0 to pi by 2 y double dash square minus y square plus x square dx. So this is an functional, which involves derivatives of 2nd order and we are given y(0) equal to 1, y dash(0) equal to 0, y(pi by2) equal to 0 and y dash(pi by 2) equal to minus 1. Now so here we have the case n equal to 2. So we shall apply this result. So Euler's equation is delta f over delta y minus d over dx okay and we are given f(x, y, y dash, y double dash) equal to y double dash square minus y square plus x square.

Let us find its derivatives, so derivative with respect to y, it is minus 2y derivative with respect y dash is 0, derivative with respect to y double dash is equal to 2y double dash. Let us put in the equation 1. So then 1 equation 1 gives minus 2y, this term is 0, d square over dx square 2y double dash or we can say d4y over dx4 minus y equal to

0 or we can say D4 minus 1y equal to 0, where D is the differential operator d over dx.

Now it is $4^{th}$ order linear differential equation with constant coefficients. So auxiliary equation is m4 minus 1equal to 0 the roots are m equal to plus minus 1 plus minus i. So complementary function is c1e to the power x, c2e to the power minus x c3cosx c4sinx. since right hand side is 0, so particular integral is 0 and thus we have y equal to.

Now let us find the 4 constants c1, c2, c3, c4, y(0) is equal to 1 gives you c1 plus c2 plus c3 equal to 1. y (pi by 2) equal to 0 y(pi by 2) is c1e to the power pi by 2 plus c2e to the power minus pi by 2 and then c3 cos(pi by 2) is 0, so c4 sin(pi by 2) is 1. So this is equal to 0, y dash let us find y dash here. Y dash is equal to c1e to the power x minus c2e to the power minus x minus c3sinx plus c4cosx or we are given that y dash 0 equal to 0. So what we have c1 minus c2 plus c4 is equal to 0 and then y dash (pi by 2) is equal to minus 1. So what we get when we put pi by 2, c1e to the power pi by 2 minus c2e to the power minus pi by 2 and minus c3 equal to 0. It is equal to minus 1. So thus we have these 4 equations. We can solve them for the values of c1, c2, c3, c4.

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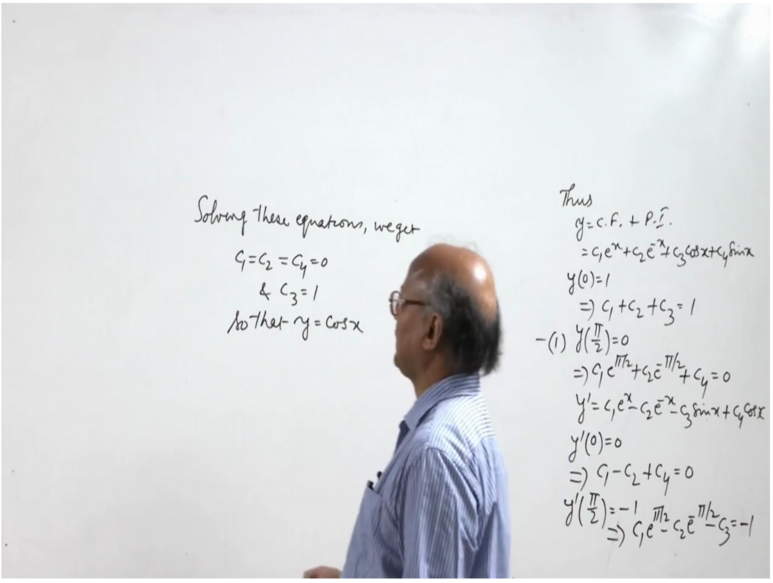


So we have 4 equations c1 plus c2 plus c3 equal to 1, c1e to the power pi by 2 plus e to the power minus pi by 2 plus c4 equal to 0, c1 minus c2 plus c4 equal to 0, c1 e to the power pi by 2 minus e to the power minus pi by 2 minus c3 equal to minus 1. So from these 4 equations, it is not difficult to find the values of c1,c2,c3,c4 and you will see that solving these equations, we get c1 equal to c2 equal to c4 equal to 0 and c3 equal to 1. So that y is equal to cosx and we can verify that the solution y equal to y equal to cosx satisfies all the 4 given boundary conditions y(0) equal to 1, y dash(0) is y dash is minus sinx, so y dash(0) is 0, y(pi by2) is 0, cos pi by 2 is 0, so y(pi by 2) is 0 and y dash is minus sinx. So y dash (pi by 2) is minus 1. So this is the required solution of the given problem that is the functional extremal of the given functional with that I would like to conclude my lecture thank you very much for your attention.