Integral Equations, Calculus of Variations and their Applications
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Lecture 28
Equations with Convolution Type Kernels II

Hello friends, I welcome you to my second lecture on equations with convolution type kernels. Let us consider the system of algebraic equations phi 1 x equal to 1 minus 2 times integral 0 to x, e to the power 2 times x minus t, phi 1 t d t plus integral 0 to x, phi 2 t d t. And then phi 2 x equal to 4 x minus integral 0 to x, phi 1 t d t plus 4 times integral 0 to x, x minus t, phi 2 t d t. So as we discussed in the previous lecture let us take the Laplace transform of both of these equations on both sides.

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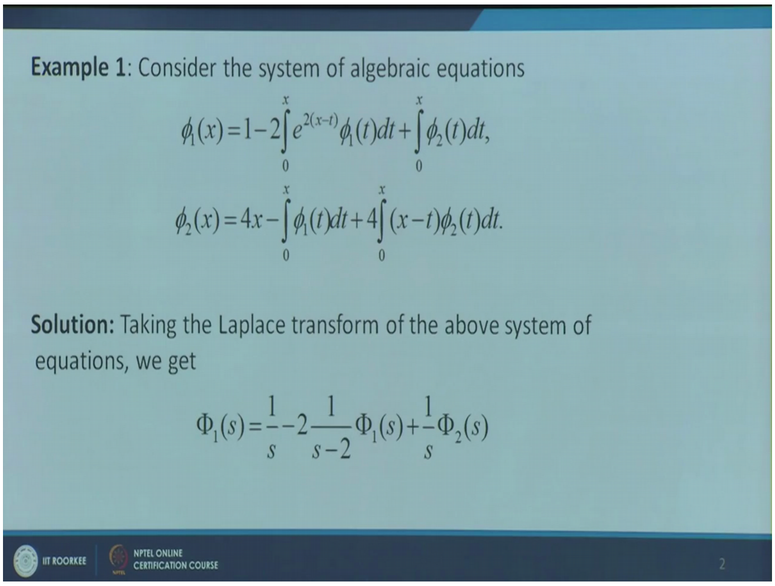

So what we get is when we take the Laplace transform of the first equation we get phi 1 s equal to Laplace transform of 1 which is 1 by s minus (Lap) 2 times Laplace transform of e to the power x into Laplace transform of phi 1 x plus. Now the third term on the right side integral 0 to x phi 2 t d t. When we take the Laplace transform of this integral it can also be regarded as the convolution of the function 1 and phi 2 t d t.

We can write it as integral 0 to x into phi 2 t d t where (con) 1 function is taken as the function 1. So Laplace transform of integral 0 to x, 1 into phi 2 t d t is Laplace transform of 1 into Laplace transform of phi 2 x. What we are doing here when we take the Laplace transform of integral 0 to x, phi 1 x minus t phi 2 t d t. This is Laplace transform of phi 1 x into Laplace transform of phi 2 x.

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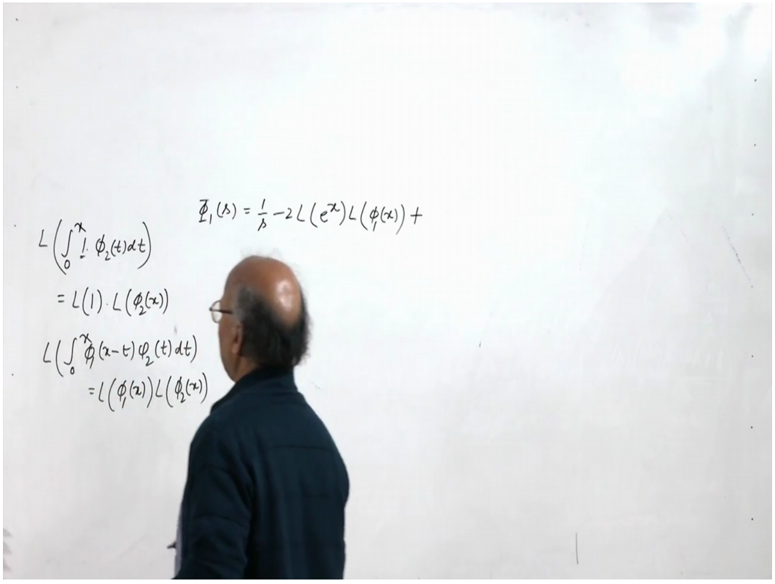

So let us take phi 1 function to be the constant function 1, okay. So let phi 1 x be equal to 1 for all x. Then phi 1 x minus t will be equal to 1, so Laplace transform of integral 0 to x, phi 2 t d t will be the product of the Laplace transform of 1 and Laplace transform of phi 2 x. So Laplace transform of 1 means 1 over s into Laplace transform of phi 2 x, so that is phi 2 s. And then when we take the Laplace transform of the second equation, Laplace transform of phi 2 is phi 2 s equal to Laplace transform x is 1 over s square.

And then minus Laplace transform of, again integral 0 to x, phi 1 t d t, its Laplace transform will be product of Laplace transform of 1 and Laplace transform of phi 1 x. And then 4 times Laplace transform of integral 0 to x, (in) x minus t into phi 2 t d t will be Laplace transform of x into Laplace transform of phi 2 t, where we have taken phi 1 x function to be x so that phi 1 x minus t is equal to x minus t. So this implies phi 1 s equal to 1 over s minus 2 over s minus 1, phi 1 s plus 1 over s, phi 2 s.

Then phi 2 s equal to 4 over s square minus 1 over s, phi 1 s plus 4 over s square, phi 2 s. Or we can write them as, let us collect the coefficient of phi 1 s. So 1 plus 2 over s minus 1 into phi 1 s minus 1 over s, phi 2 s equal to 1 over s. Second equation can be rewritten as 1 over s, phi 1 s plus 1 minus 4 over s square, phi 2 s equal to 4 over s square.

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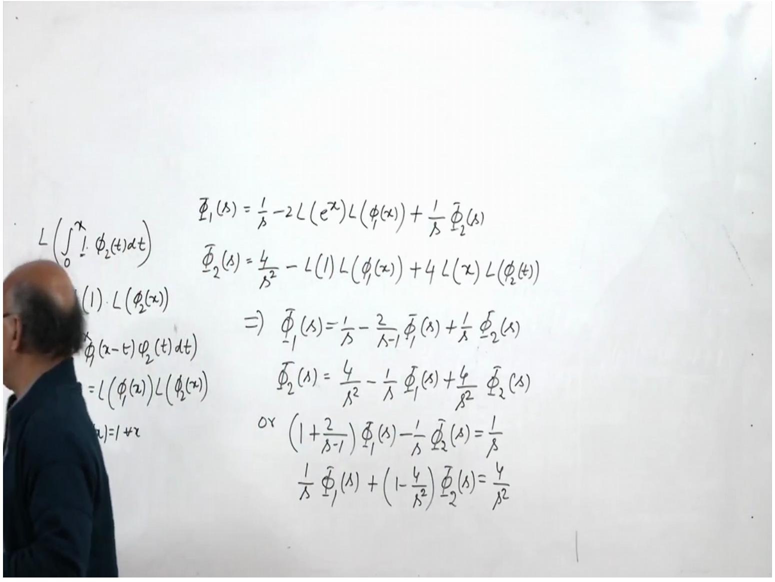

Simplifying we get, so 1 plus 2 over s minus 1 becomes s plus 1 over s minus 1 into phi 1 s minus 1 over s, phi 2 s equal to 1 over s. Then 1 over s, phi 1 s plus s square minus 4 divided by s square into phi 2 s equal to 4 over s square. Now these two are linear equations in phi 1 s and phi 2 s.

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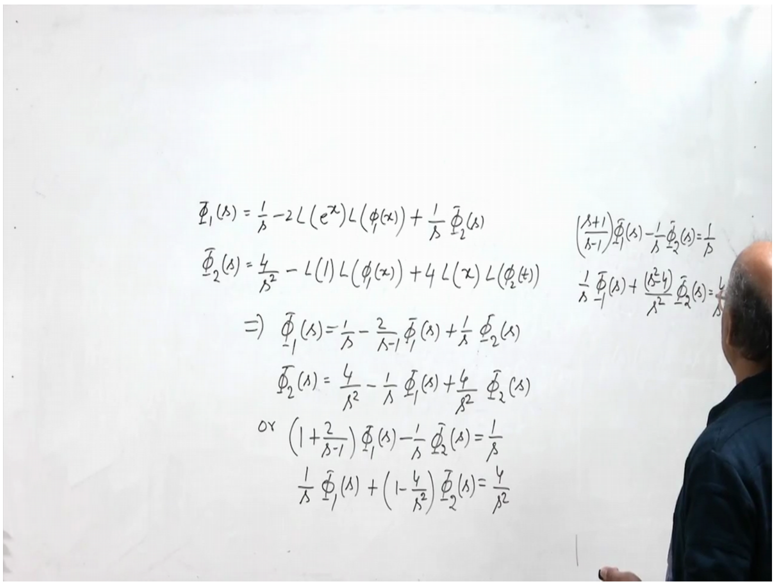

We can solve them for the values of phi 1 s and phi 2 s. So let us apply Cramer's (ru) rule. So applying Cramer's rule phi 1 s will be equal to determinant 1 over s, 4 over s square and then minus 1 over s, s square minus 4 divided by s square over determinant of the coefficient matrix. So s plus 1 over s minus 1 minus 1 over s and we have 1 over s, s square minus 4 over s square. So this gives you the value of phi 1 s.

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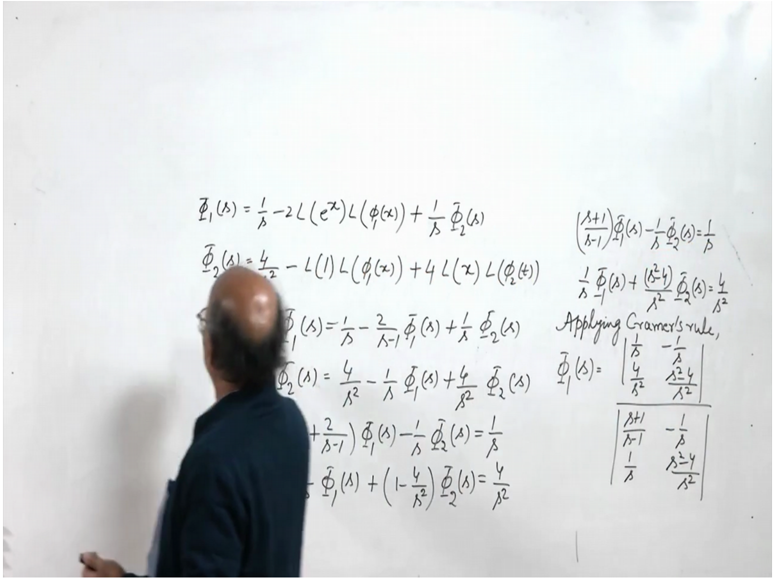

Similarly phi 2 s we can write. Phi 2 s will be equal to determinant s plus 1 over s minus 1, 1 over s and then 1 over s, second column will be 1 over s, 4 by s square divided by determinant of s plus 1 over s minus 1, 1 over s minus 1 over s, s square minus 4 divided by s square.

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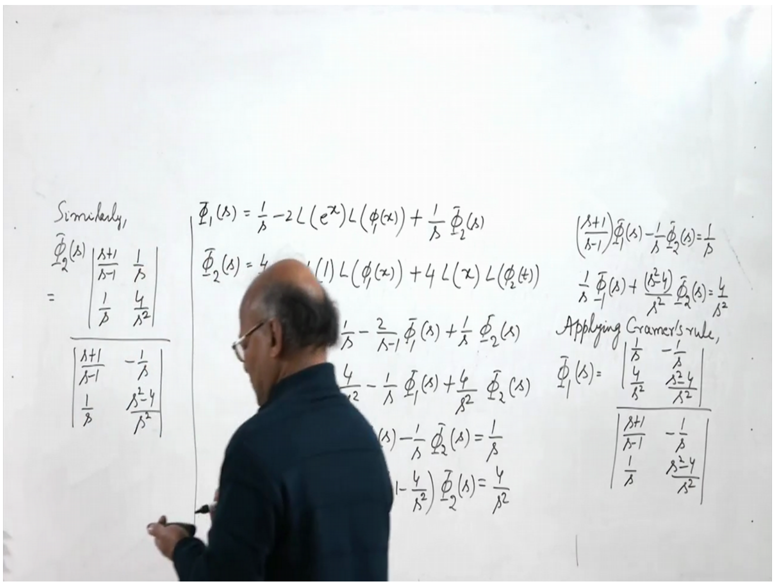

Now we can evaluate the values of these determinants and then we will see that after we simplify the expressions of phi 1 s and phi 2 s and then take the inverse Laplace transform you will see that phi 1 x comes out to be e to the power minus x minus x e to the power minus x.

Phi 2 x comes out to be 8 by 9 e to the power 2 x plus 1 by 3 x e to the power minus x minus 8 by 9 e to the power minus x. after simplifying (ph) phi 1 s and phi 2 s, we will have to break them into partial fractions and then we will have to take the inverse Laplace transform of phi 1 s and phi 2 s to arrive at the solution of the given system of Volterra integral equations of second kind.

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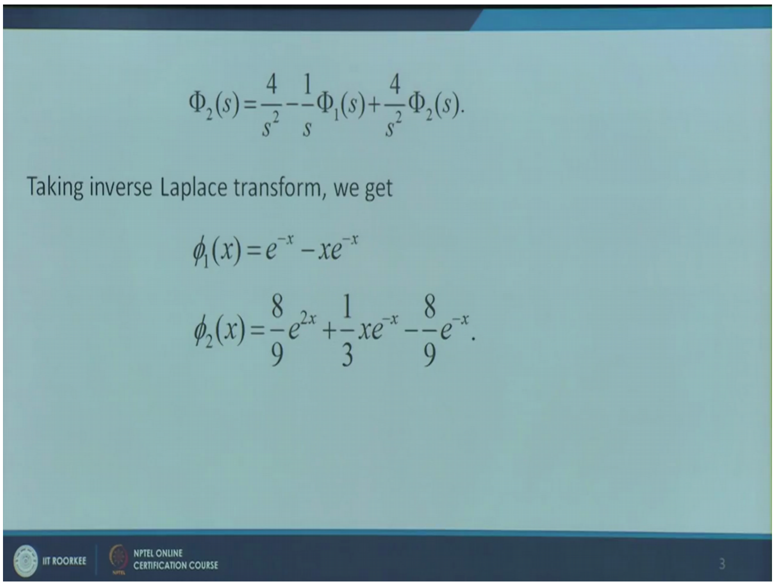

So how we do this? Let us take another example. So here we are given another system of algebraic equations, phi 1 x equal to e to the power s minus integral 0 to x, phi 1 t d t plus 4 times integral 0 to x, e to the power x minus t into phi 2 t d t and then phi 2 x equal to 1 minus integral 0 to x, e to the power t minus x, phi 1 t d t plus integral 0 to x, phi 2 t d t.

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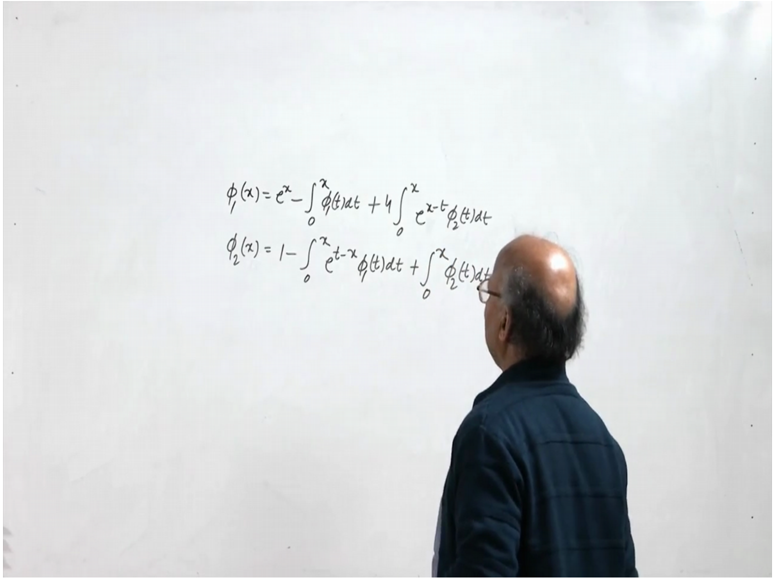

Again let us take the Laplace transform of both of these equations on both sides. So (fa) we get the Laplace transform of the first equation gives you phi 1 s equal to Laplace transform of e to the power x which is 1 over s minus 1 minus Laplace transform of integral 0 to x, phi 1 t d t. So this can be regarded as 1 into phi 1 t d t and therefore Laplace transform of 1 which is 1 by s into Laplace transform of phi 1 x which is phi 1 s plus 4 times Laplace transform of e to the power x function.

So this is 1 over s minus 1 and then Laplace transform of phi 2 x, so that is phi 2 s. And then second equation gives Laplace transform of phi 2 x is phi 2 s, Laplace transform of 1 is 1 over s minus. Now here we have e to the power t minus x. So we will have to write it as e to the power t minus x we have to write as minus x minus t. So we will have Laplace transform of e to the power minus x into Laplace transform of phi 1 x which is phi 1 s.

And then we have Laplace transform of 1 which is 1 over s into phi 2 s. Now what we will do is, or phi 2 s can be written as 1 over s minus 1 over s plus 1, phi 1 s plus 1 over s, phi 2 s. Let us now write it as an algebraic equation in phi 1 s and phi 2 s, linear system.

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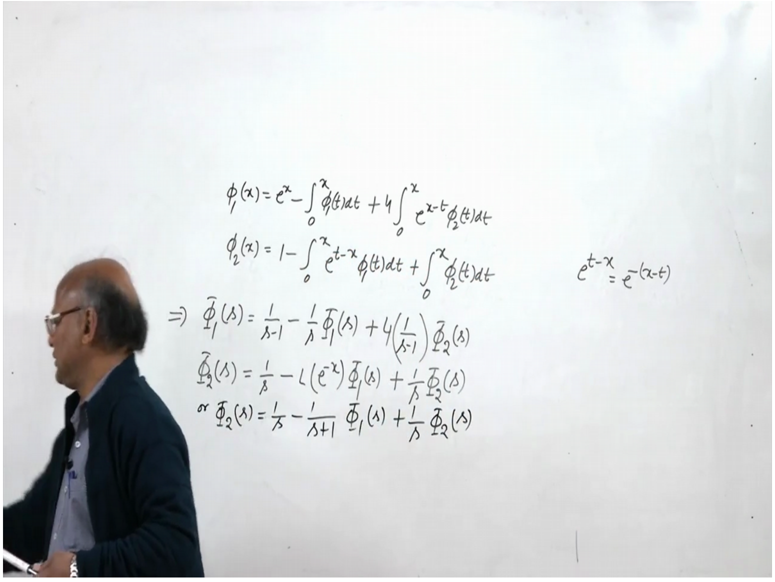

So we will have let us say pick up the first equation so, 1 plus 1 by s into phi 1 s minus 4 upon s minus 1 into phi 2 s equal to 1 over s minus 1. The second equation gives you 1 over s plus 1 into phi 1 s plus 1 minus 1 upon 1 by s, 1 minus 1 upon s into phi 2 s equal to 1 upon s. Again let us simplify or let us solve this system of linear algebraic equations in phi 1 s and phi 2 s by applying Cramer's rule.

So phi 1 s, let us write first the expression for phi 1 s. Phi 1 s will be 1 over s minus 1, 1 over s, then minus 4 upon s minus 1 and we have 1 minus 1 upon s. So that is s minus 1 upon s divided by s plus 1 over s. We have 1 over s plus 1, then we have minus 4 upon s minus 1 and then we have s minus 1 upon s. So if you simplify it further what we get?

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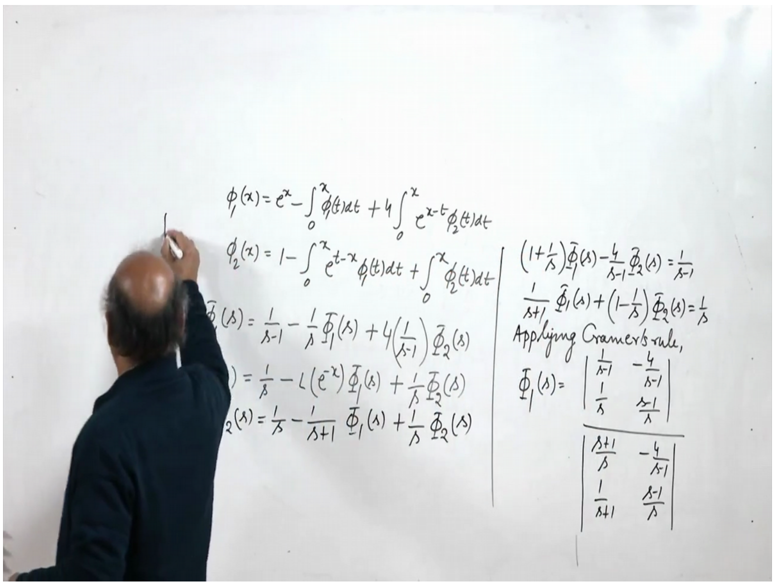

So let us see the value of the determinant in the numerator. So 1 over s minus 1 into s minus 1 over s. So we get 1 over s minus, minus 4 upon s minus 1 into 1 by s. So we get plus 4 upon s into s minus 1. Then we have in the denominator, s square minus 1 by s square and we have plus 4 upon s square minus 1. So how much is this? In the numerator we get s into s minus 1, then we have s minus 1 plus 4. So we get s plus 3 divided by s square into s square minus 1.

So what we have is s square minus 1 whole square plus 4 s square. So this is s plus 3 upon s into s minus 1 and then we get s square, s square minus 1 divided by, s square minus 1 whole square plus 4 s square is s square plus 1 whole square, so we get this. We can write it further as s into s plus 3, this is s minus 1 into s plus 1, so s plus 1 divided by s square plus 1 whole square.

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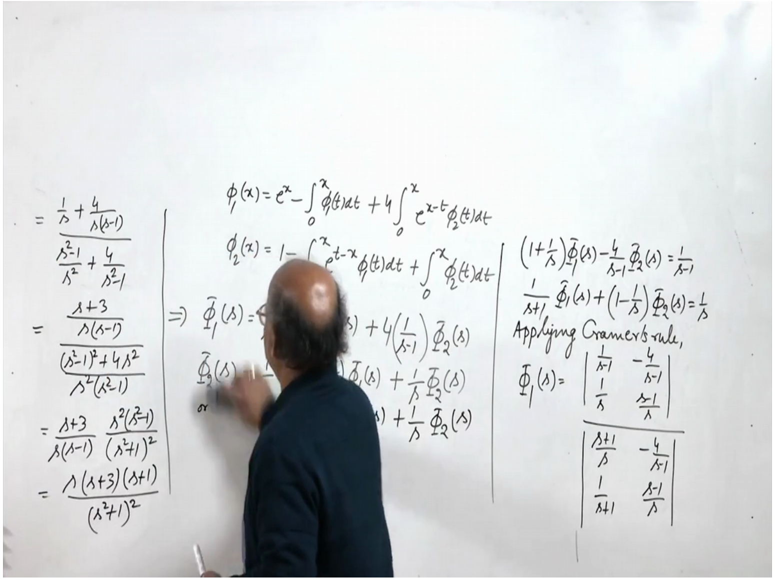

Now what we do is I am doing this problem because it is a bit too complicated. So we have this further equal to A s plus B upon s square plus 1 plus C s plus D upon s square plus 1 whole square. Let us break this into partial fractions. So then we have the identity s into s plus 3 into s plus 1 equal to A s plus B into s square plus 1 plus C s plus D. Let us multiply, so we get here A s cube plus B s square plus A s plus B plus C s plus D.

So A plus C into s plus B plus D. In the left hand side what we will have? This is s square plus 3 s plus s square plus 4 s. So s cube plus 4 s square plus 3. Now this is an identity, so let us equate the coefficients of like powers of s. So this gives you A equal to 1, equating the coefficients of s cube. B equal to 4, equating the coefficients of s square. Now equating the coefficients of this is 3 s. So equating the coefficients of s we have A plus C equal to 3 and we have B plus D equal to 0. So we have four equations.

We will have the values of all the four A, B, C, D. So A equal to 1 gives you C equal to 2 and B equal to 4 gives you D equal to minus 4.

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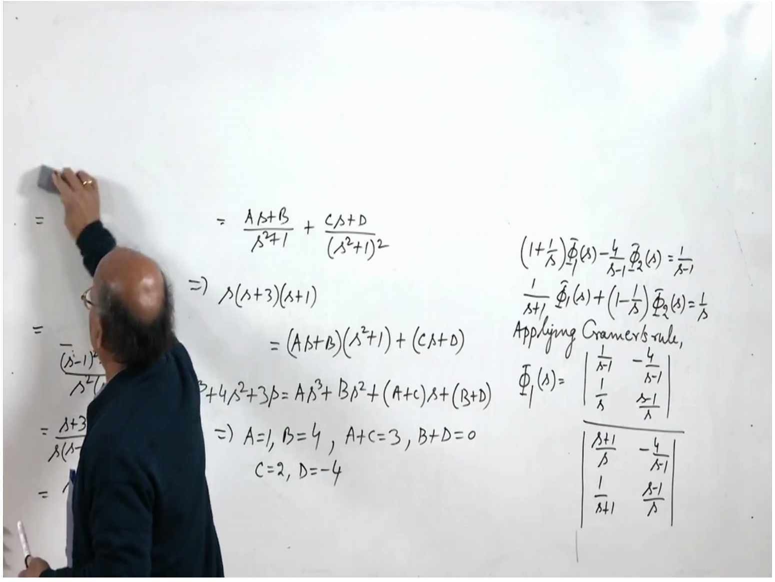

So thus we have phi 1 s equal to A s plus B upon s square plus 1. A means 1 and B is 4. So s plus 4 divided by s square plus 1. C is equal to 2 so 2 s minus 4 divided by s square plus 1 whole square, this is phi 1 s.

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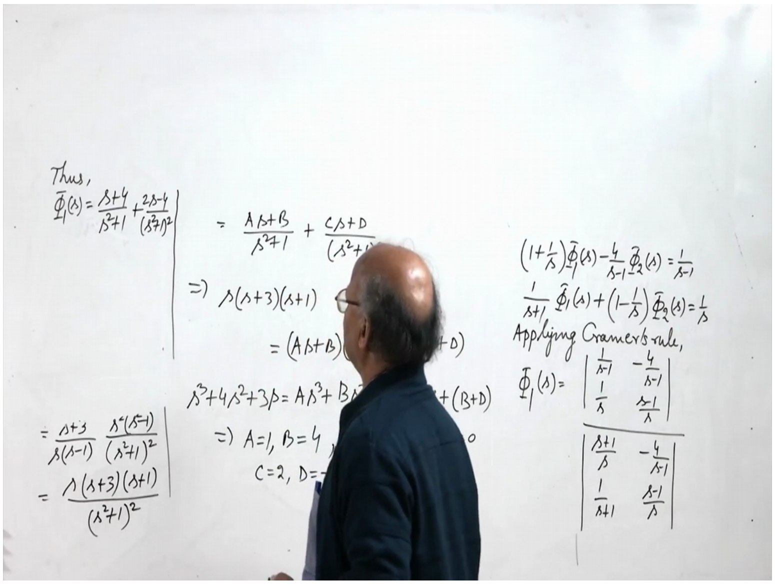

Now let us take inverse Laplace transform of this. So inverse Laplace transform of phi 1 s gives you inverse Laplace transform of, now let us break it into two parts. S over s square plus 1 plus inverse Laplace transform of 4 times, 4 we can take outside, 1 over s square plus 1. Then inverse Laplace transform of 2 s upon s square plus 1 whole square, then inverse

Laplace transform of minus 4 times inverse Laplace transform of 1 over s square plus 1 whole square.

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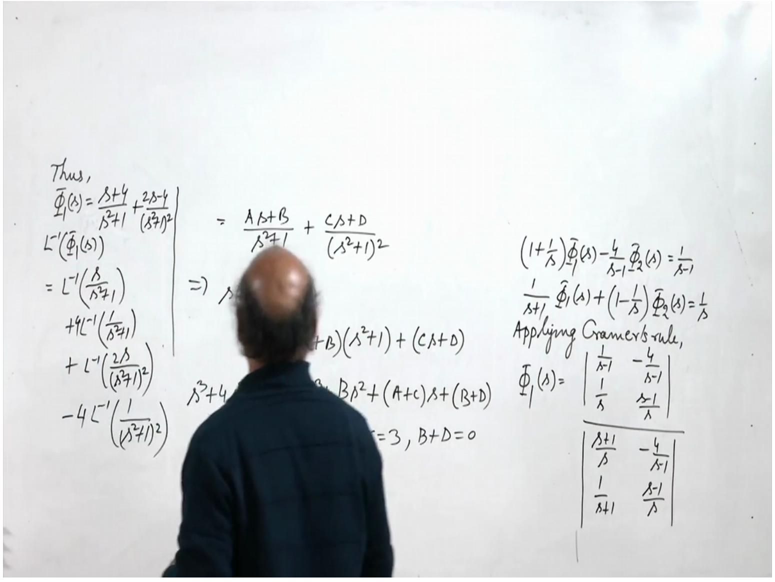

Let us find the inverse Laplace transform one by one. So L inverse of phi 1 s is equal to phi 1 x, inverse Laplace transform of s over s square plus 1 is cos x, inverse Laplace transform of 1 over s square plus 1 is sin x, then inverse Laplace transform of 2 s upon s square plus 1 whole square let us get, okay. Now we know that Laplace transform of sin x is 1 over s square plus 1. So Laplace transform of x into sin x when you multiply by x, Laplace transform of x sin x is minus d over d s of 1 upon s square plus 1.

Now when you differentiate 1 over s square plus 1 what you get is, minus 1 upon s square plus 1 whole square into 2 s. So we get minus 2 s upon s square plus 1 whole square. Minus-minus becomes plus so Laplace transform of x sin x is 2 s upon s square plus 1 whole square.

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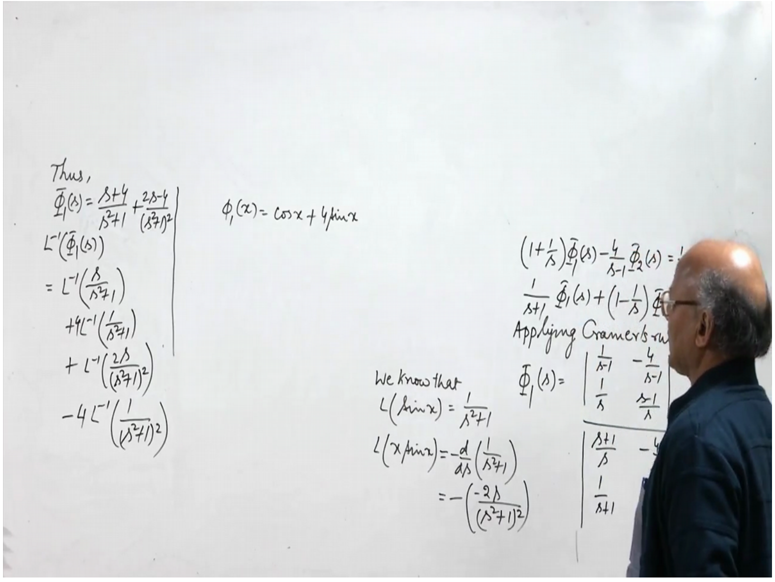

So L inverse of 2 s upon s square plus 1 whole square is 2 x sin x. Now let us find the inverse Laplace transform of 1 over s square plus 1 whole square. So we have just now seen that Laplace transform of x into sin x is equal to 2 s upon s square plus 1 whole square. Now we want inverse Laplace transform of 1 over s square plus 1 whole square.

So we have to find formula which divides the Laplace transform of a function by s. So if we know that if L f t is equal to F s then Laplace transform of integral 0 to t, f u d u is equal to F s by s.

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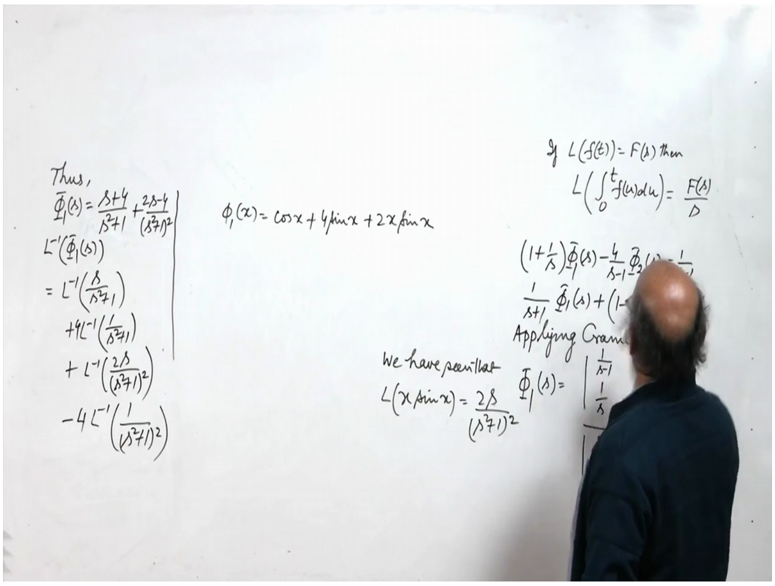

So if you want to divide this function of s by s you need to integrate this function x sin x over the interval 0 to t. So what we will have? Integral 0 to t let us consider u sin u d u. by our formula Laplace transform of this is nothing but F s by s that is 2 upon s square plus 1 whole square. So let us see what is this integral 0 to t, u sin u d u? So u times integrate by parts so minus cos u 0 to, I think we should take integral here because we are writing in terms of x so let us write here limit 0 to x.

So 0 to x minus 0 to x, derivative of u is 1, then we have minus cos u d u. So this will be equal to minus x cos x and when you put 0 it is 0. Here we have plus, integral of cos u is sin u. So what we get is minus x cos x plus sin x. So this means what? Laplace transform of minus x cos x plus sin x, this is equal to 2 upon s square plus 1 whole square.

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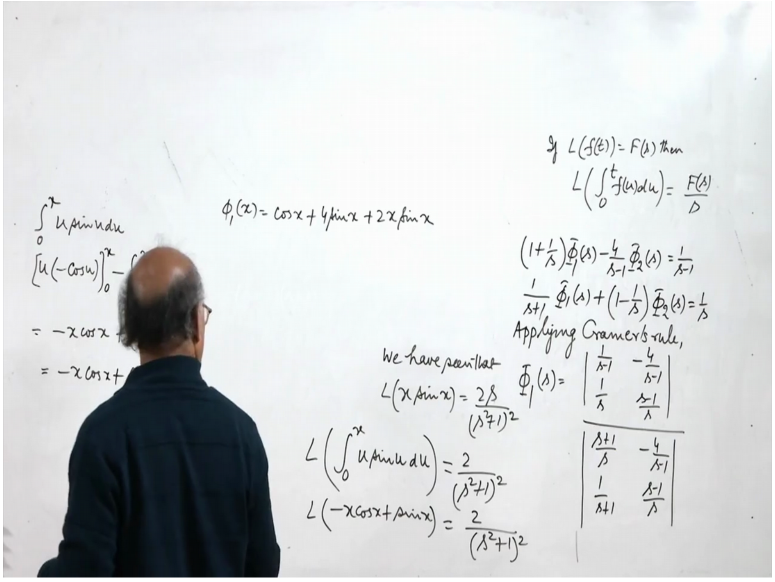

Now what we have, I have omitted that. the expression I have omitted. I think it was minus 4. We had minus 4 L inverse of 1 upon s square plus 1 whole square. So what we have? L inverse of 1 over s square plus 1 whole square is minus x cos x plus sin x by 2. So minus 4 minus x cos x plus sin x divided by 2, okay. So what is this? This is equal to right.

So cos x plus 4 sin x and then we have 2 x sin x and here we have minus-minus plus 2 x cos x and we have minus 2 sin x. So there is a mistake here. This is cos x plus 4 sin x plus x sin x minus this.

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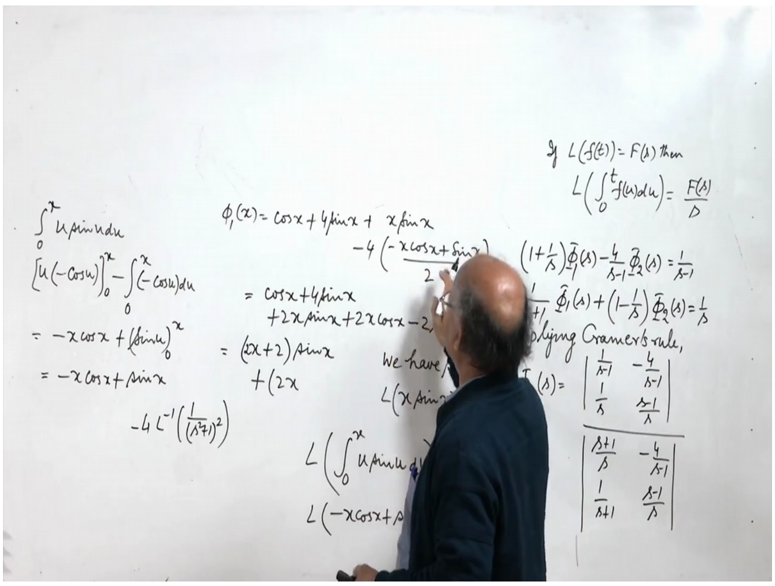

Now we have cos x plus 4 sin x plus x sin x and then we have 2 x cos x minus 2 sin x. So we will have here x plus 2 sin x and then 2 x plus 1 cos x which is your function phi 1 x. Similarly we can find phi 2 x. Phi 2 x will be given by phi 2 s is equal to determinant s plus 1 by s and then we have 1 over s plus 1, we have 1 over s minus 1 and then we have 1 over s here divided by s plus 1 by s minus 4 by s minus 1, 1 over s plus 1, s minus 1 by s.

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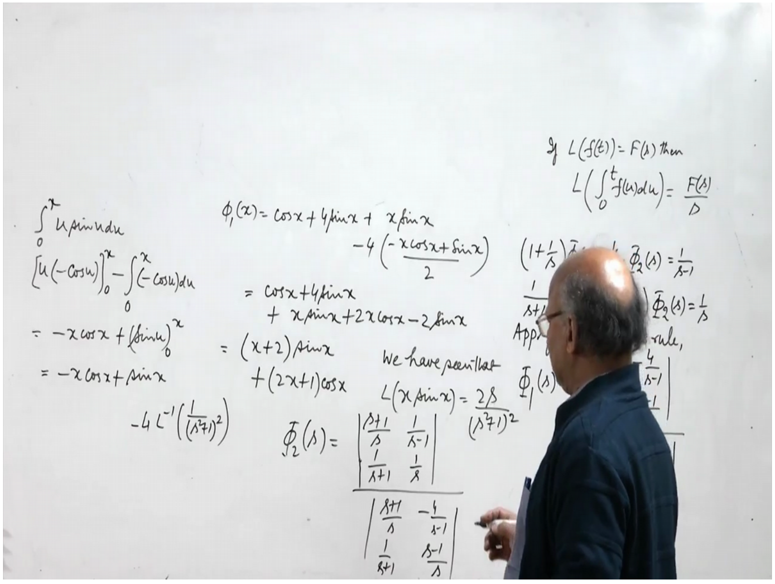

So we can solve these determinants, obtain the expressions for phi 2 s and then take the inverse Laplace transform. We have to make the partial fractions on the right hand side. So just like I did for the function phi 1 x, similarly we can do it for the function phi 2 x and we will have the value of phi 2 x as 2 plus x by 2 cos x minus 2 x plus 1 by 2 sin x. So this is how we solve the system of Volterra type integral equations of second kind.

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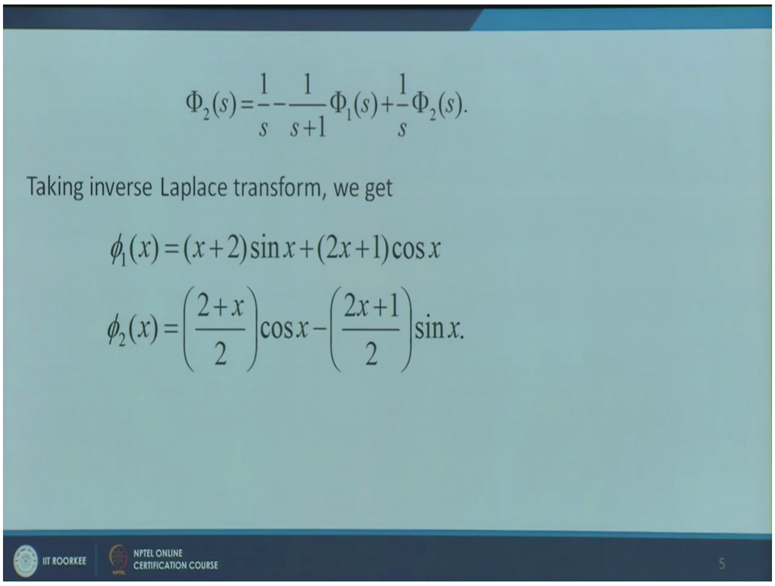

Now let us go to Volterra integral equation of the first kind of the convolution type. So here let us consider an integral equation of the first kind, integral 0 to x, K x minus t y t d t equal to f x where y t is the unknown function, f x and the kernel K x minus t is known.

So where kernel K x t is dependent solely on the difference x minus t of the arguments it is called an integral equation of the first kind of convolution type. Now if you recall when we did the first lecture on linear integral equations there we had discussed how the linear integral equations arise in the study. So there the first equation that we had discussed was the Abel’s integral equation and that Abel’s integral equation which arises by a problem in mechanics is an integral equation of this type.

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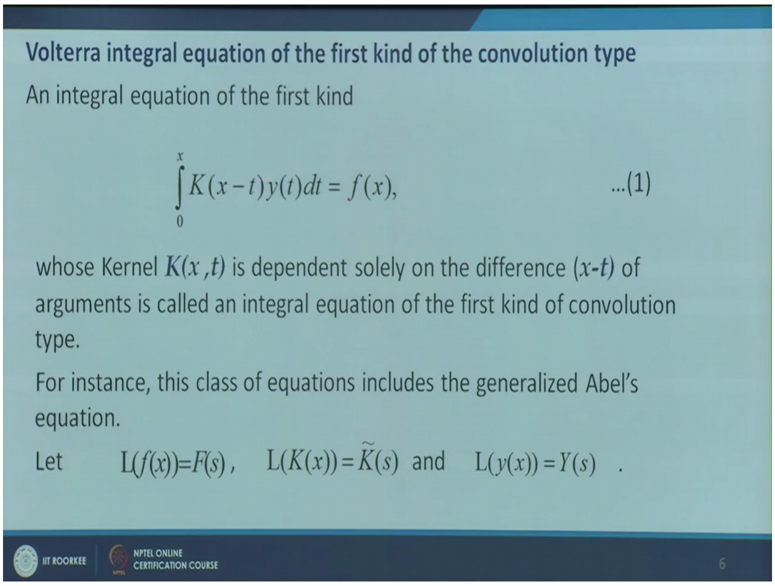

So that is an example of integral equation of the first kind of convolution type. Now so here (la) Laplace transform of f x is F s, K x is K s and L y x is equal to Y s. Then we will have by taking the Laplace transform on both sides of equation 1 and utilising again the convolution theorem, Laplace transform of the left hand side by convolution theorem will be K tilde s into Y s equal to F s.

And then assuming K tilde s is not equal to 0 we can determine the value of Y s. Y s is equal to F s by K tilde s where K tilde s is not 0. Then the solution of the linear integral equation is obtained by taking the inverse Laplace transform of this equation. So L inverse of Y s gives us y x and L inverse of F s over K s gives us the solution of the given integral equation.

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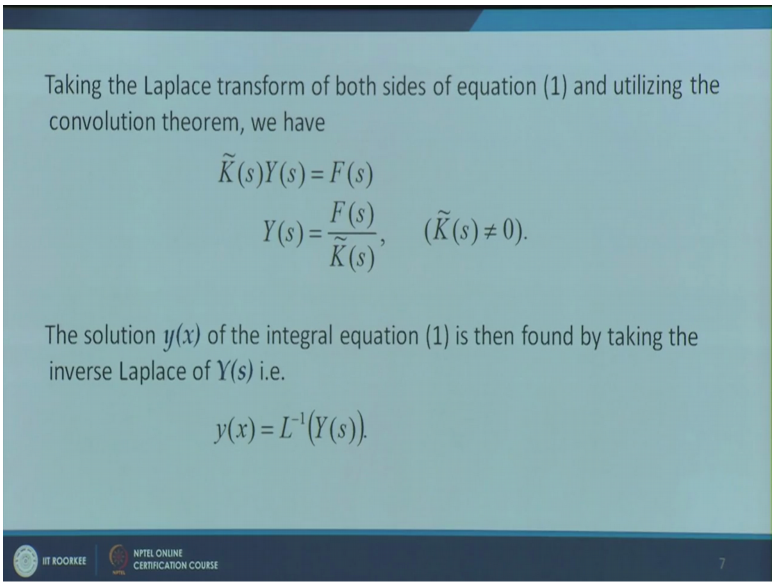

Now let us take an example. Suppose we have the integral equation 0 to x, e to the power x minus t, y t d t equal to x. Let us take the Laplace transform of both sides. So Laplace transform of this by convolution theorem will be Laplace transform of e to the power x into Laplace transform of y x. So Laplace transform of e to the power x into Laplace transform of y x equal to Laplace transform of x. So this gives you 1 over s minus 1 into Y s equal to 1 over s square.

Or we can say Y s equal to s minus 1 by s square. We can break it into parts. So 1 over s minus 1 over s square. So L inverse of Y s is L inverse of 1 over s minus L inverse of 1 over s square. And what we get is y x equal to 1 minus x.

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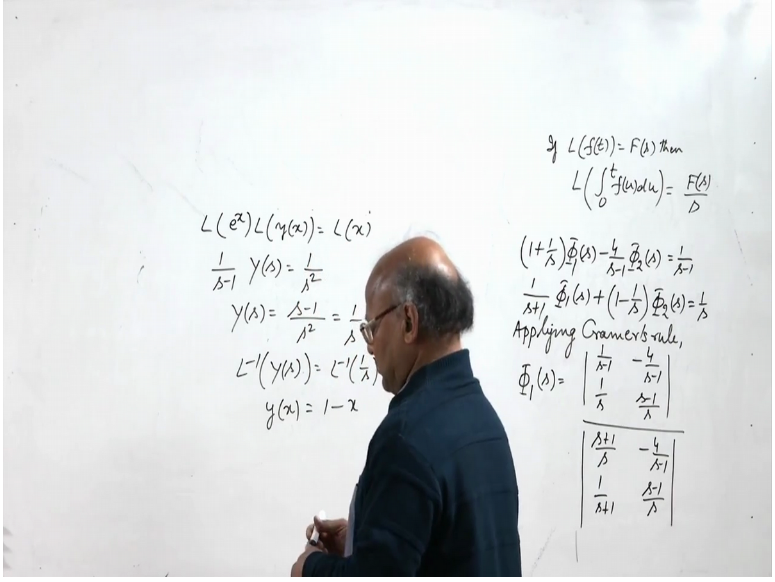

So in taking inverse Laplace transform we get the solution of the given convolution type integral equation of first kind. Now let us go to one more problem. The integral 0 to x, sin hyperbolic x minus t, phi t d t equal to x cube into e to the power minus x.

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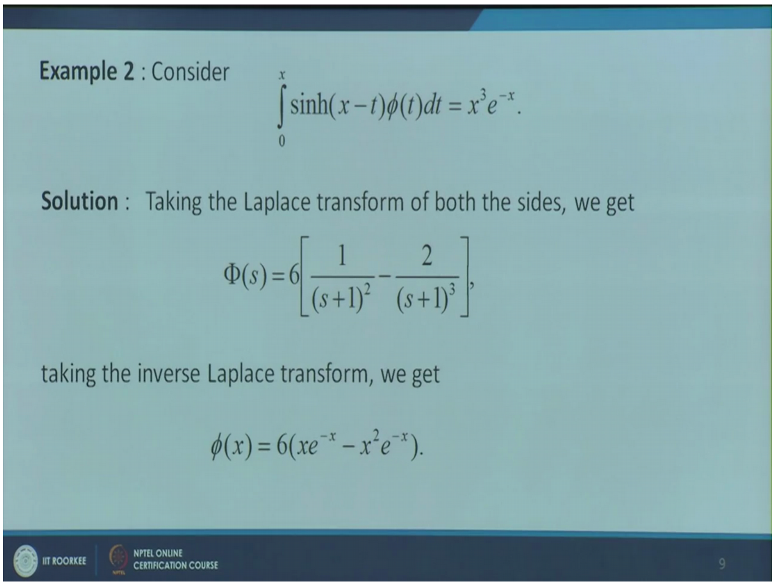

So here again let us take the Laplace transform on both sides. So Laplace transform of 0 to x, sin hyperbolic x minus t, phi t d t is Laplace transform of sin hyperbolic x into Laplace transform of phi x equal to Laplace transform of x cube e to the power minus x. Laplace transform of sin hyperbolic x is equal to e to the power x minus e to the power minus x by 2 and Laplace transform sin hyperbolic x is 1 by 2, Laplace transform of e to the power x minus 1 by 2, Laplace transform of e to the power minus x which is 1 over 2 times, 1 over s minus 1 minus 1 over s plus 1.

So we get 1 by 2 s plus 1 minus s plus 1. So 2 upon s square minus 1. So we get 1 over s square minus 1. So this is 1 over s square minus 1 into phi s equal to, now Laplace transform of x cube is 3 factorial divided by s to the power 4. When you multiply by e to the power minus x, let us apply shifting theorem, so this is 6 over, s is replaced by s plus 1. So s plus 1 to the power 4, so we will have 6 upon s plus 1 to the power 4.

So we have 5 s equal to 6 times s square minus 1 divided by s plus 1 to the power 4. Now this is 6 times s minus 1 into s plus 1 divided by s plus 1 to the power 4. So I can write it as 6 times s minus 1 upon s plus 1 to the power 4. So s minus 1 upon s plus 1 to the power 4.

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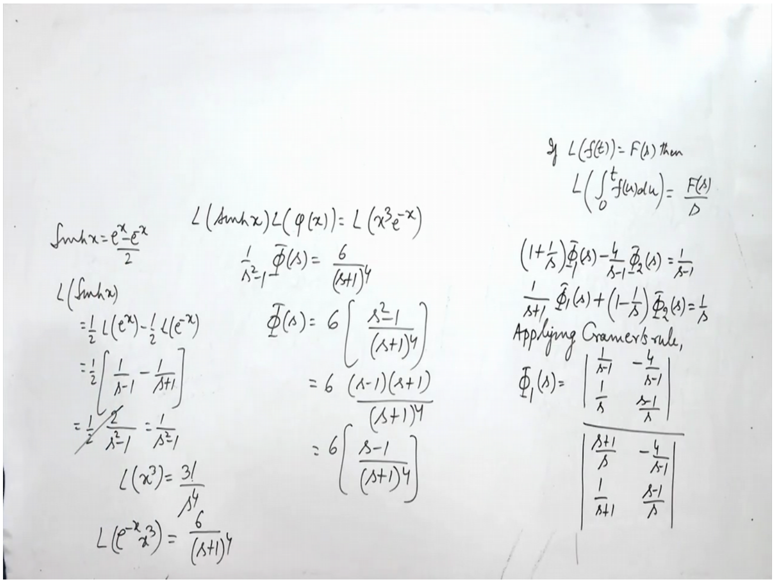

And so taking inverse Laplace transform we have 6 times L inverse of, we have s minus 1 over s plus 1 to the power 4. We can write it further as 6 times s plus 1 minus 2 divided by s plus 1 to the power 4. This will be s plus 1 to the power 3, so we have 6 times 1 over s plus 1 whole square minus 2 upon s plus 1 whole cube. So 6 times L inverse of 1 over s plus 1 whole square minus 12 1 over s plus 1 whole cube.

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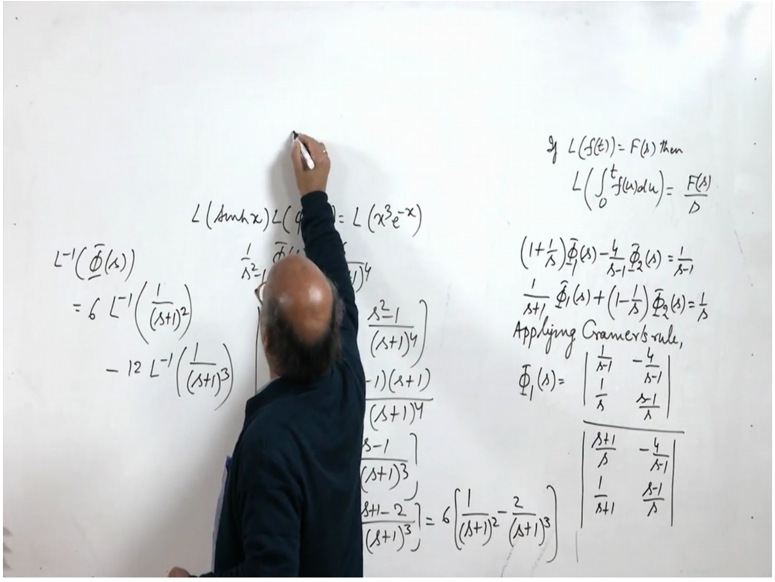

Now Laplace transform of x is 1 over s square and Laplace transform of x e to the power minus x is 1 over s plus 1 whole square. So this is 6 times x into e to the power minus x. Laplace transform of x square is 2 factorial by s cube and Laplace transform of x square e to the power minus x is 2 upon s plus 1 whole cube. So this is minus 12 times, L inverse of 1 over s plus 1 whole cube will be x square e to the power minus x divided by 2.

So we get 6 times x e to the power minus x minus x square e to the power minus x. So this is how we get the solution of the integral equation of first kind of convolution type.

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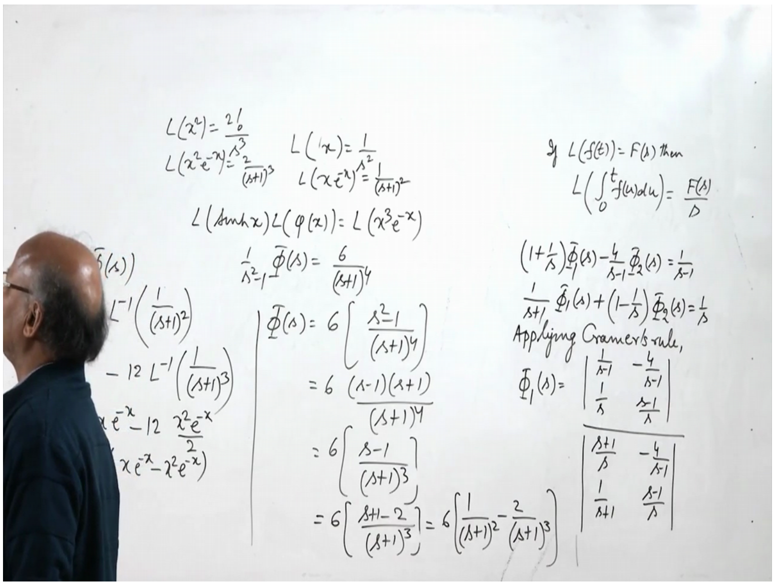

With that I would like to conclude my lecture. Thank you very much for your attention.