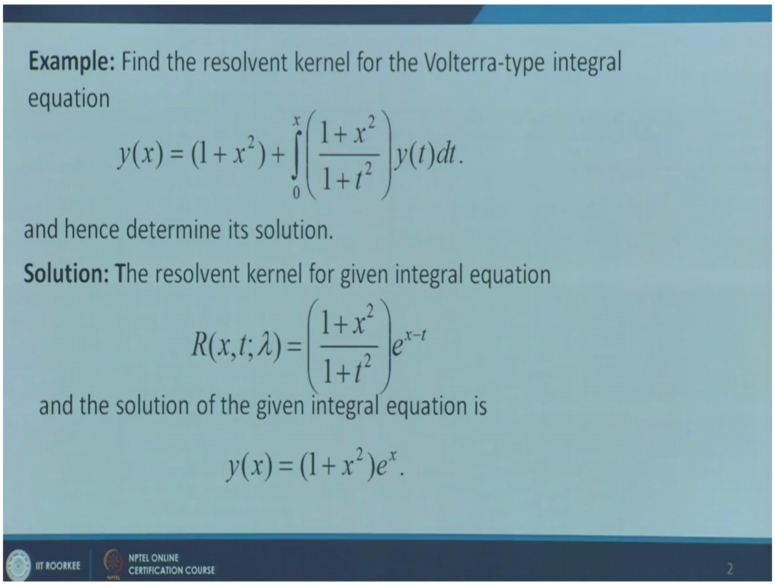
Integral Equations, Calculus of Variations and their Applications
Doctor P.N. Agrawal
Department of Mathematics
Indian Institute of Technology Roorkee
Lecture 26
Neumann Series and Resolvent Kernels II

Hello friends I welcome you to my second lecture on Neumann series and resolvent kernel. In the last lecture we discussed one problem on the solution of Volterra integral equation of second kind by finding the resolvent kernel. Here I start the one more such example to make things more clear. Let us consider the equation y x equal to 1 plus x square, integral 0 to x, 1 plus x square by 1 plus t square, y t d t.

You can see that it is a Volterra integral equation of the second kind. And so we will solve this equation by first finding the resolvent kernel and then determining the solution of this integral equation.

(Refer Slide Time: 01:07)



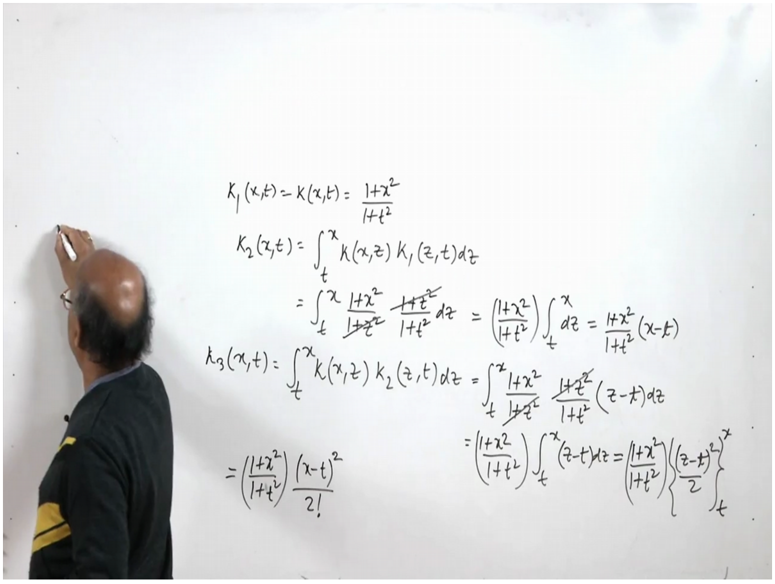
So let us write we will first find the iterated kernels. Let us recall that the iterated kernels K 1 x t is given by K x t and we are given here K x t equal to 1 plus x square by 1 plus t square. The kernel of the integral equation is K x t which is given as 1 plus x square by 1 plus t square. Then K 2 x t, we know that K 2 x t is given by integral t to x, K x z into K 1 z t d z.

So we can write it as integral t to x K x z. K x z will be 1 plus x square upon 1 plus z square into K 1 z t, K 1 z t is same as K z t and K z t will be 1 plus z square divided by 1 plus t square into d z. So 1 plus z square will cancel with 1 plus z square and 1 plus x square by 1 plus t square is independent of z, so we can write it outside. And what we get is integral t to x d z which after integration gives us. So thus we have obtained the second iterated kernel K 2 x t. Let us find K 3 x t also in a similar manner.

So K 3 x t is given by integral t to x, K x z into K 2 z t d z. This is equal to integral t to x K x z. K x z is 1 plus x square by 1 plus z square. And then K 2 z t, so K 2 x t is this when we replace x by z we get K 2 z t. So 1 plus z square divided by 1 plus t square into z minus t d z. So again we will cancel this and we will have 1 plus x square by 1 plus t square integral t to x, z minus t d z. After integration here what we have is 1 plus x square by 1 plus t square and then z minus t whole square by 2.

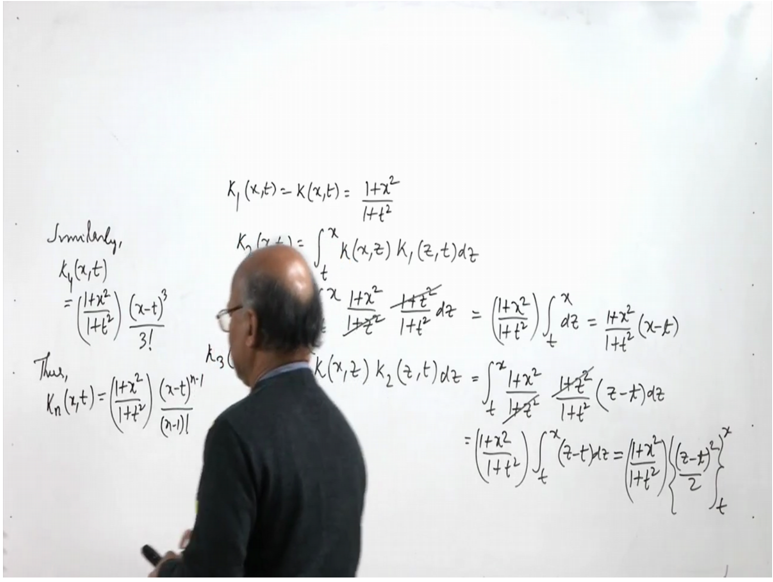
The evaluation of this will give us 1 plus x square by 1 plus t square and then x minus t whole square by 2 which we can write as whole square by 2 factorial.

(Refer Slide Time: 04:51)



You can continue this process. You can find K 4 x t similarly. You can see that it comes out to be 1 plus x square by 1 plus t square, x minus t whole to the power 3 divided by 3 factorial. So generalizing we have thus K n x t is equal to 1 plus x square by 1 plus square and we have x minus t raise to the power n minus 1 over n minus 1 factorial.

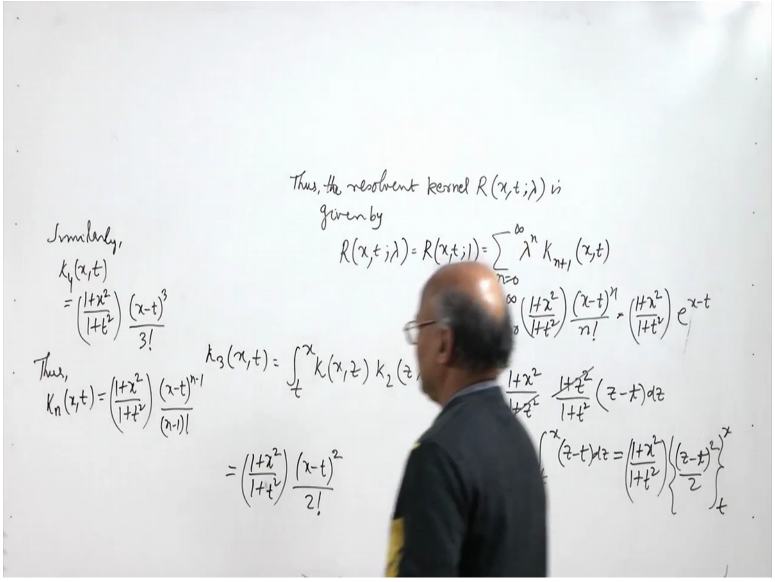
(Refer Slide Time: 05:44)



Now let us find the resolvent kernel R x t lambda. So thus the resolvent kernel is, now you can see in the given integral equation lambda is equal to 1 so we have R x t 1 equal to sigma n equal to 0 to infinity, lambda to the power n. Then K n plus 1 x t. So this is lambda is 1 here, so summation n equal to 0 to infinity and then we have 1 to the power n then K n plus 1 will be 1 plus x square over 1 plus t square, x minus t raise to the power n divided by n factorial.

And this is nothing but 1 plus x square by 1 plus t square and then e to the power x minus t. Sigma equal to 0 to infinity, x minus t raise to the power n by n factorial is e to the power x minus t.

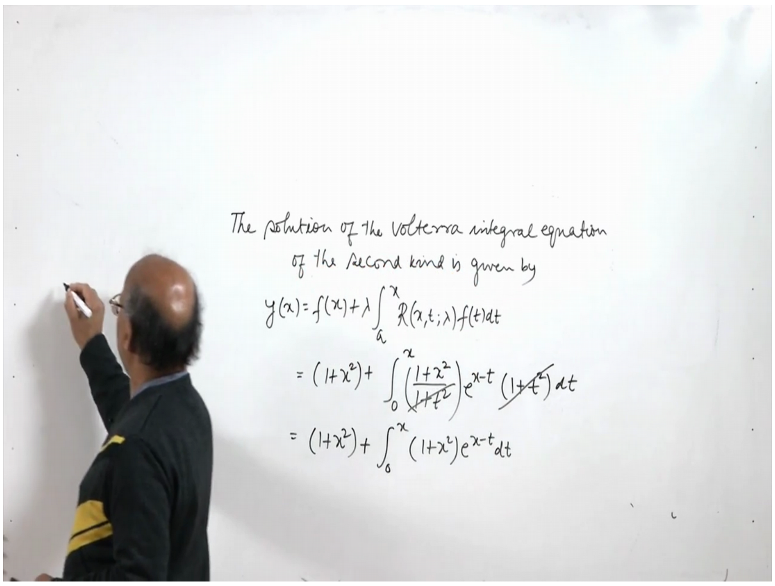
(Refer Slide Time: 07:21)



Now having found R x t lambda so now let us find the solution of the given integral equation. So let us put the value of R x t lambda. Here the solution of the integral equation is given by Volterra integral equation of the second kind y x equal to f x plus lambda integral a to x, R x t lambda, f t d t. So let us put the values here, f x is equal to 1 plus x square, lambda is equal to 1, a is given to be equal to 0, so 0 to x and R x t lambda we have found 1 plus x square by 1 plus t square into e to the power x minus t.

And then we have f t is 1 plus x square, d t. So what we get is 1 plus x square, f t is 1 plus t square. So this 1 plus t square and the 1 plus t square here will get cancelled and we will have 1 plus x square, integral 0 to x, 1 plus x square e to the power x minus t, d t.

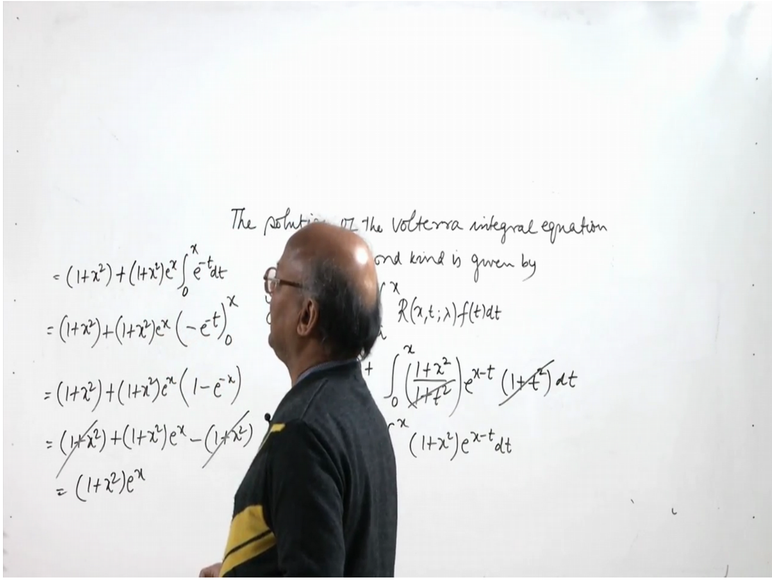
(Refer Slide Time: 09:50)



And this can be written further as 1 plus x square, 1 plus x square into e to the power x we can write outside and we have to simply integrate 0 to x, e to the power minus t, d t. So what we get is 1 plus x square plus 1 plus x square, e to the power x and minus e to the power minus t integral 0 to x. So we have 1 plus x square, 1 plus square, e to the power x and then we have 1 minus e to the power minus x.

So what we have is these two cancel and we have. So y x is equal to 1 plus x square into e to the power x is the solution of the given integral equation.

(Refer Slide Time: 11:10)



Now let us discuss how to find the solution of a Volterra integral equation of second kind when its kernel we take as K x t. Kernel K x t is of a particular form. Let us say this is the equation. Y x is equal to f x plus lambda times a to x, K x t y t d t and the kernel K x t is a polynomial of degree n minus 1 in t. So that we can always write it in the form K x t equal to a not x plus a 1 x into x minus x t and so on a n minus 1 x over n minus 1 factorial, x minus t to the power n minus 1.

You can see that this since we have assumed K x t to be a polynomial in t of degree n minus 1 we can always write it in this form where (max) it is a polynomial in t of degree n minus 1.

(Refer Slide Time: 12:04)

and the coefficients $a_k(x)$, $k = 0, 1, 2, \ldots, (n-1)$ are continuous in $a \leq x \leq b$. Then, the resolvent kernel $R(x, t; \lambda)$ of (1) is given by

$$R(x,t;\lambda) = \frac{1}{\lambda}\frac{d^n}{dx^n}g(x,t;\lambda)$$

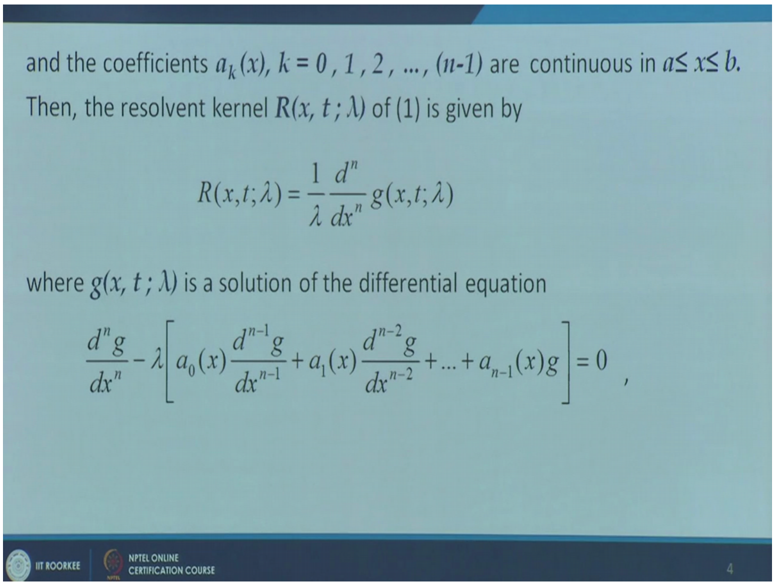
where $g(x, t; \lambda)$ is a solution of the differential equation

$$\frac{d^n g}{dx^n} - \lambda\left[a_0(x)\frac{d^{n-1}g}{dx^{n-1}} + a_1(x)\frac{d^{n-2}g}{dx^{n-2}} + \ldots + a_{n-1}(x)g\right] = 0 \quad ,$$

Now the coefficients a K x here the coefficients a K x where K varies from 0 to n minus 1, they are continuous in the close interval a b. Then in this case in terms of that the resolvent kernel R x t lambda is given by 1 over lambda, d n over d x n, g x t lambda. And g x t lambda is a solution of this differential equation of nth order.

So d n g over d x n minus lambda, a not x d n minus 1 g over d x n minus 1 plus a 1 x d n minus 2 g over d x n minus 2 and so on, a n minus 1 x into g equal to 0. This is nth order differential equation with variable coefficients.

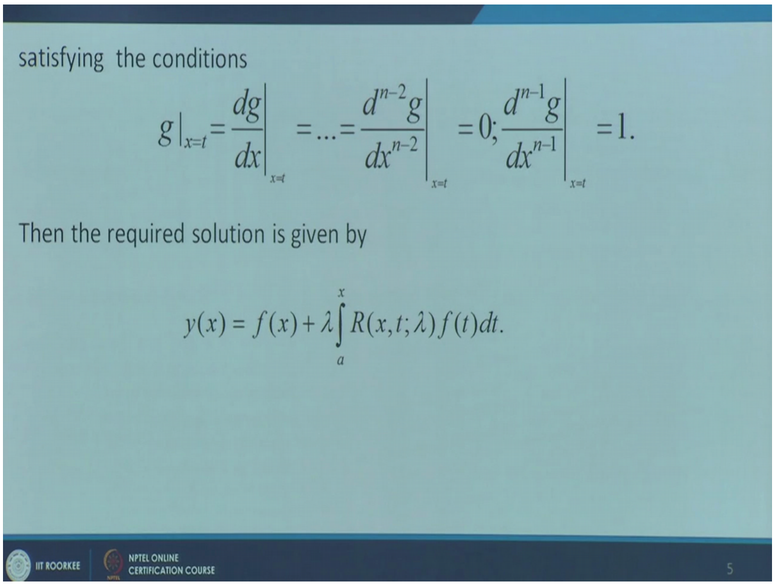
(Refer Slide Time: 12:49)



And the conditions on g are g at x equal to t is 0. The d g over d x at x equal to t is also 0 and n minus 2th derivative of g with respect to x at x equal to t is 0. While n minus 1th derivative of g with respect to x at x equal to t is 1.

And then the required solution is given by y x equal to f x plus lambda times integral a to x, R x t lambda f t d t as usual. As we have earlier seen how to find the solution once the resolvent kernel is known. So by putting the value of the resolvent kernel R x t lambda here we can then find the required solution of the given integral equation.

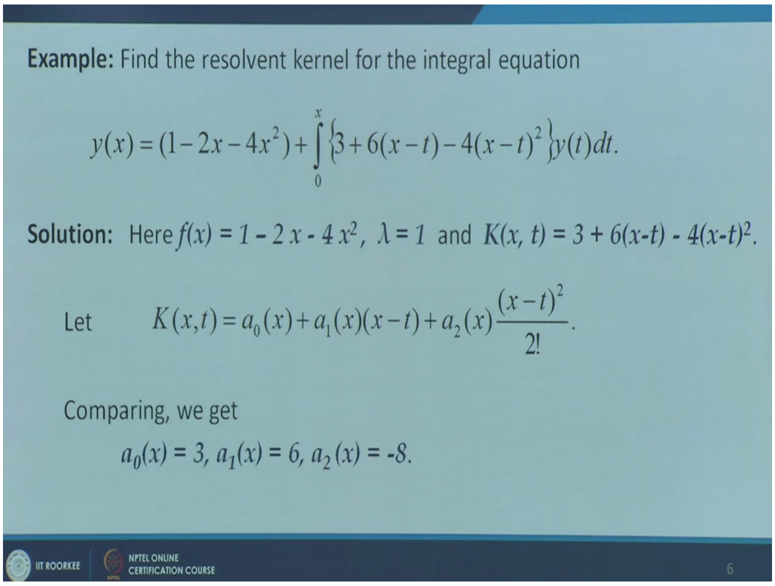
(Refer Slide Time: 13:35)



Now let us see how this method works by taking a little example. So let us say y x is equal to 1 minus 2 x minus 4 x square and we have integral 0 to x 3 plus 6 times x minus t minus 4 times x minus t whole square y t d t. So here you can see that f x function is 1 minus 2 x minus 4 x square, lambda is equal to 1, the lower limit a is equal to 0 and K x t is 3 plus 6 times x minus t minus 4 times x minus t whole square which is a polynomial in t of degree 2.

And already it is written in the form in which we want K x t to be. So we want K x t to be in the form a not x plus a 1 x, x minus t, a 2 x into x minus t whole square by 2 factorial. It is in that form already. So here f x is this, lambda is 1, K x t is 3 plus 6 times x minus t minus 4 times x minus t whole square. So let us compare this form of K x t with the general form of K x t when it is a polynomial in t of degree 2. So you can see that when we compare a not x is equal to 3, a 1 x is equal to 6 and a 2 x is equal to minus 8, okay.

(Refer Slide Time: 14:53)

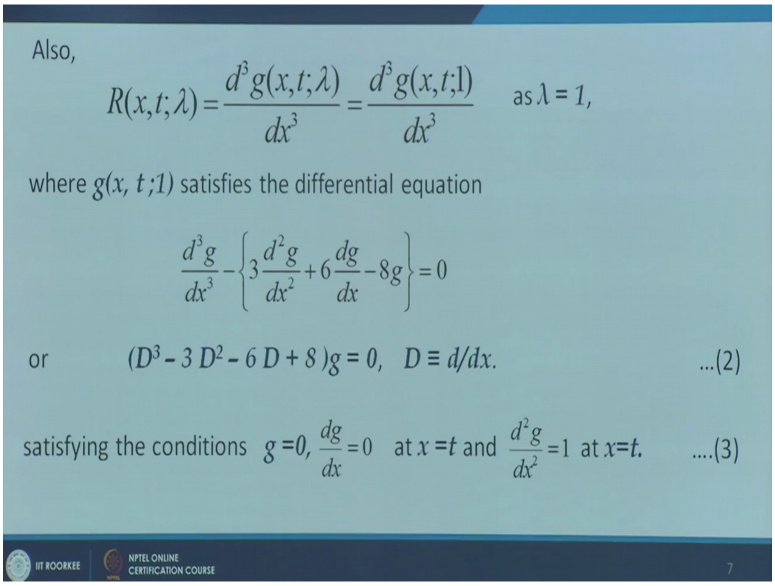


Now R x t lambda will be d cube g x t lambda over d x cube. Why? Because here K x t lambda is a polynomial in t of degree 2. So when it is a polynomial in t of degree n minus 1, the nth order differential equation in g we have. So R x t lambda is d cube g x t lambda over d x cube and lambda is 1. So we can also write it as d cube g x t 1 over d x cube.

Now this g x t 1 is the solution of this differential equation. So this is the third order differential equation because n is equal to 3 here and we have put the values of a not, a 1 and a 2. So d cube g over d x cube we seen that a not x is equal to 3, a 1 x is 6, a 2 x is minus 8. So substituting these values we get this third order differential equation with constant coefficients and we know how to solve nth order differential equation with constant coefficients.

So let us write it in the notation of D where D is the derivative with respect to x. So D cube minus 3 D square minus 6 D plus 8, this differential operator operates on g equal to 0. So it is a homogeneous differential equation of third order with constant coefficients. Now the conditions on g are g is equal to 0 and d g over d x is equal to 0 at x is equal to t. And d square g over d x square is 1 at x equal to t, okay.

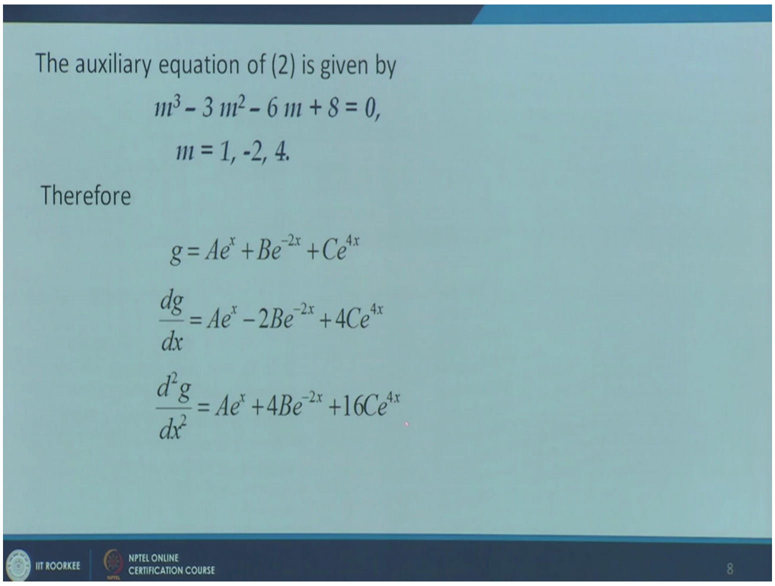
(Refer Slide Time: 16:26)



Now we can write the auxiliary equation corresponding to this homogeneous equation. So auxiliary equation is m cube minus 3 m square minus 6 m plus 8 is equal to 0 and you can check that m is equal to 1, minus 2, 4. They are the values of m which satisfies this equation. Now since these three values are real and distinct the solution of the this third order differential equation in g is written as g equal to A e to the power x, B e to the power minus 2 x, C e to the power 4 x.

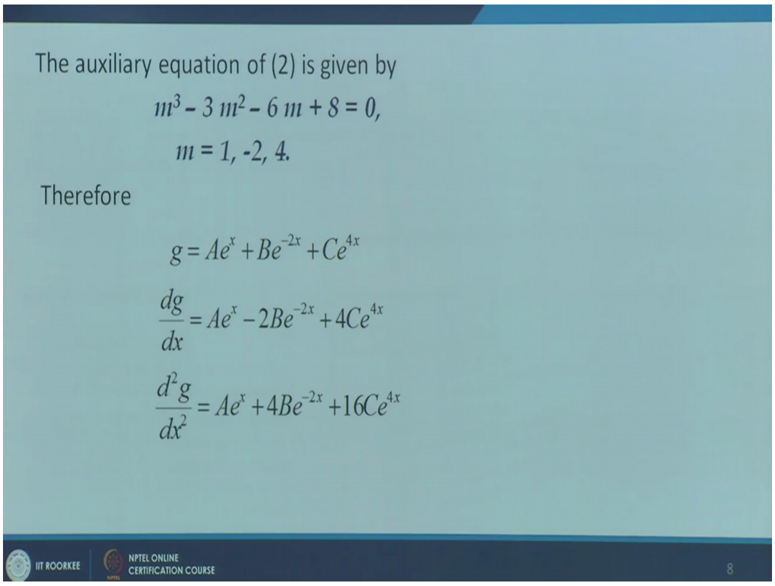
Now in order to use the conditions on g let us differentiate it with respect to x. So we get d g by d x is equal to A e to the power x minus 2 B e to the power minus 2 x and we get 4 C e to the power 4 x. And when we differentiate it once more we get d square g by d x square which is A e to the power x plus 4 B e to the power minus 2 x plus 16 C e to the power 4 x.

(Refer Slide Time: 17:25)



Now let us use the conditions on g. When you will use the conditions on g, the conditions on g are at x equal to t, g is 0, d g by d x equal to 0 and at x equal to t second derivative is 1. If you use these three conditions on this 3 equations, okay, we shall arrive at the three equations in A, B, C. Solving those three linear equations in A, B, C we get the values of A, B, C.

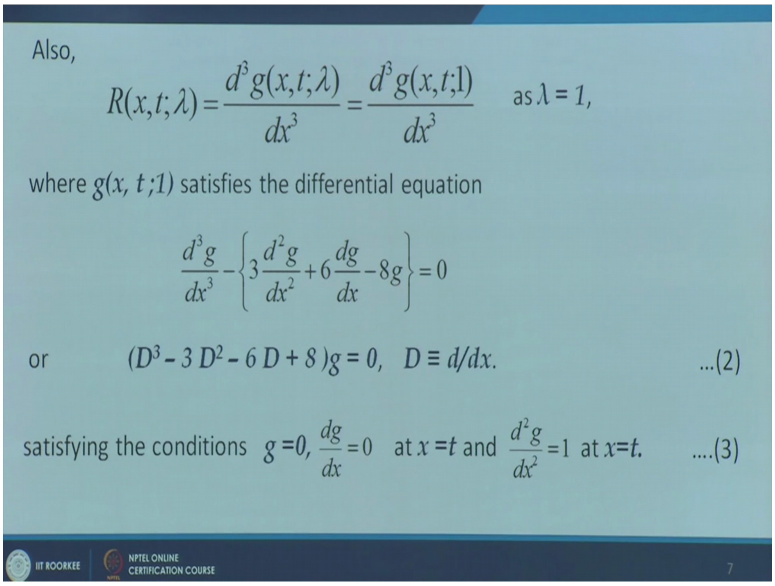
(Refer Slide Time: 17:57)



And the values of A, B, C when we put in the expression of g x t we get here the following expression of g x t lambda. G x t 1 is equal to minus 1 by 9 e to the power x minus t, 1 by 18 e to the power minus 2 x plus 2 t, 1 by 18 e to the power minus 4 x minus 4 t. Now we differentiate it once, okay.

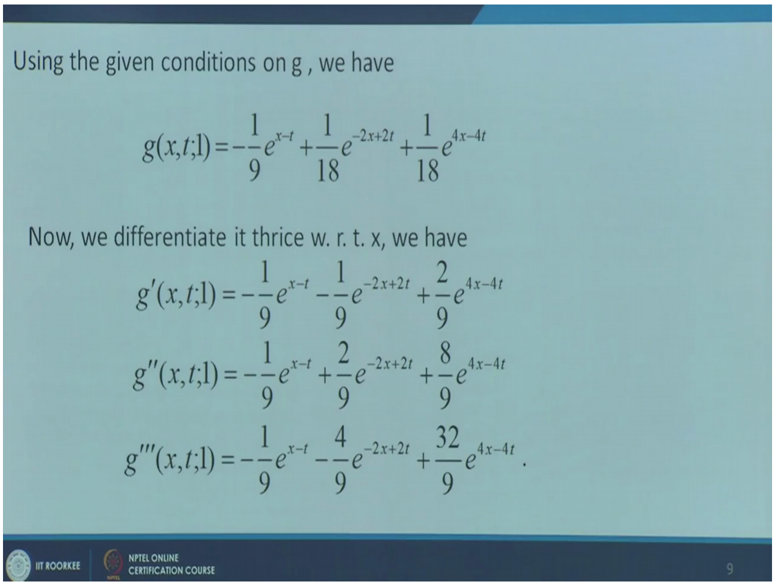
So we get (deja) g dash, in order to find why we are differentiating now because we want to know what is the R x t lambda? We want to know R x t lambda and for R x t lambda we need to know what is g x t 1 over d x cube? So we have already found g x t 1, so we will differentiate it thrice and get R x t lambda.

(Refer Slide Time: 18:42)



So g x t 1 is this that is differentiated once. We get this expression. We differentiate it again, we get this expression and when we differentiate it again we get this expression. So this is third derivative of g x t 1 and what we get is this.

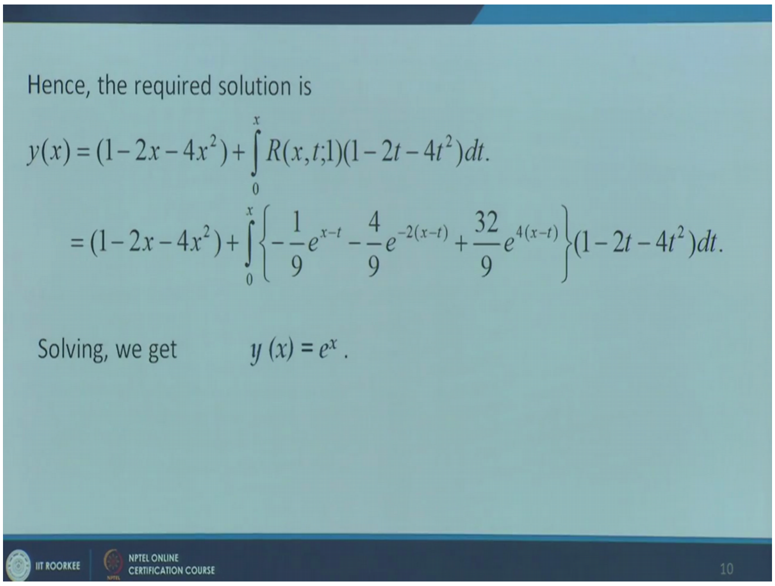
(Refer Slide Time: 19:02)



Now so this is nothing but we get R x t 1. R x t 1 is third derivative of g x t 1. So with respect to x, so y x is equal to f x plus integral 0 to x R x t 1 into f t d t. F t is 1 minus 2 t minus 4 t square. And we put the value of R x t 1 here. Now you can do the integration by parts here by simply taking e to the power x here outside in the first case. In the second you can take e to the power minus 2 x out. In the third integral you can take e to the power 4 x outside and then you integrate by parts.

Say for example in the first case you will have to integrate e to the power minus t into 1 minus 2 t minus 4 t square with respect to t and put the limits 0 x. In the second case we will be integrating e to the power 2 t into 1 minus 2 t minus 4 t square with respect to t and then (pu) we put the limits. And here we will be integrating e to the power minus 4 t into 1 minus 2 t minus 4 t square and put the limits. So after you integrate by parts these three terms and substitute the limits and simplify you will get y x equal to e to the power x.

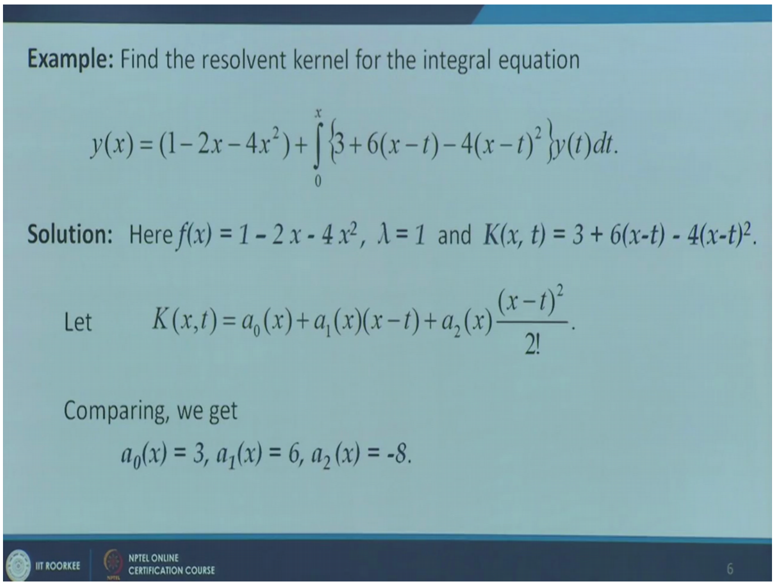
(Refer Slide Time: 20:22)



Now let me tell you that this method you can see is very cumbersome. You have to first find the differential equation in g, then you have to solve the differential equation in g. The differential equation in g has variable coefficients. Their functions depending on x and when we have variable coefficient it is not easy to solve the differential equation. Here what happened that the coefficients of the differential equation in g turned out to be constant so we could easily solve this differential equation.

And then we use the conditions over g to determine g x t lambda. And once we determined g x t lambda we have to differentiate say for example in this case three times to arrive at R x t lambda and then we (diff) integrate R x t lambda. And then we put the value of R x t lambda in this equation and integrate which is again a lengthy procedure. So if you solve this by using the Laplace transform technique, let us say for example this one.

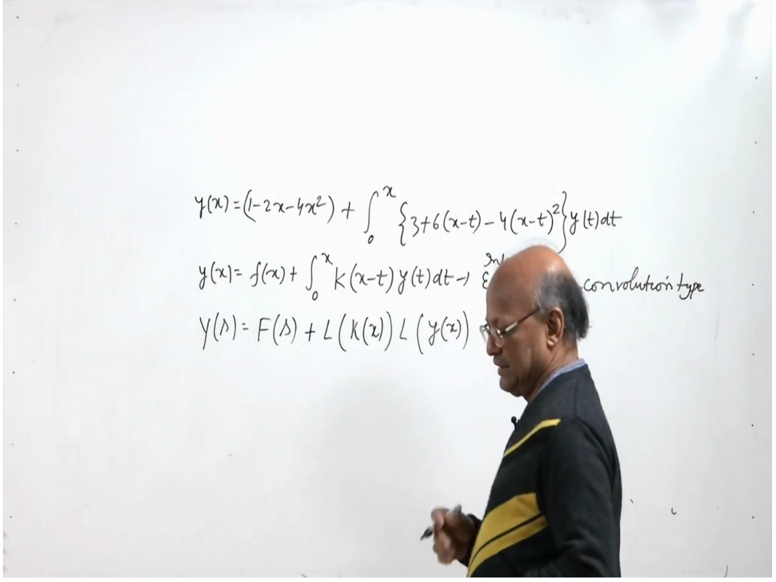
(Refer Slide Time: 21:32)



This integral equation let us solve by using the Laplace transform technique. What we will do? We have y x equal to 1 minus 2 x minus 4 x square integral 0 to x and then we have 3 plus 6 times x minus t minus 4 times x minus t whole square, y t d t. So you can compare this equation with y x equal to f x plus integral 0 to x, K x minus t, y t d t. This equation is of this form. And such equations are called as (conver) integral equations of (convo) convolution type.

We shall be later on discussing how to find the solution of integral equation of convolution type. What we do is we take the Laplace transform of both sides. Suppose we assume that Laplace transform y x is y s and then the Laplace transform of f x is f s, then we get y s equal to f s and then Laplace transform of this. Laplace transform of this integral 0 to x, K x minus t, y t d t can be found by using the convolution theorem and we get Laplace transform of say K x into Laplace transform of y x.

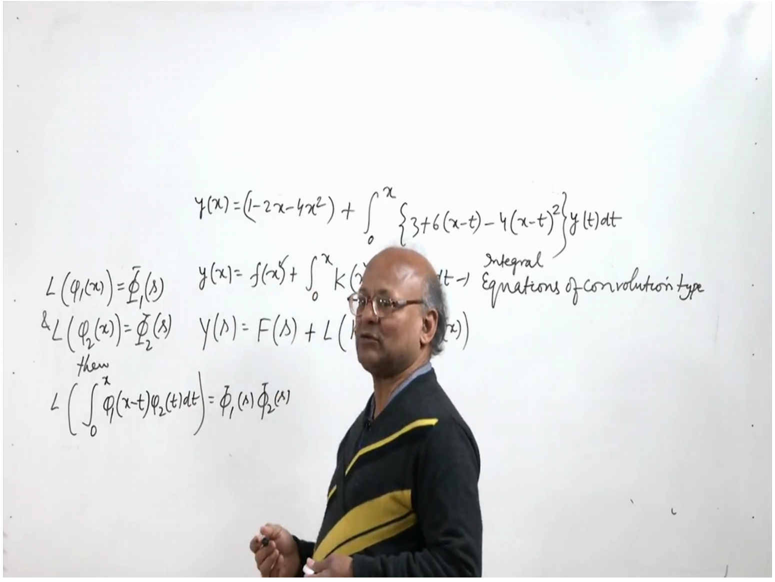
(Refer Slide Time: 23:40)



Because in the convolution theorem we say that if the Laplace transform of two functions phi 1 x and phi 2 x exist, Laplace transform of phi 1 x is say phi 1 s and Laplace transform of phi 2 x is phi 2 s then the Laplace transform of their convolution is defined as say 0 to x. We can say phi 1 x minus t into phi 2 t d t is equal to phi 1 s into phi 2 s. So here we see that we have the convolution of two functions K x and y x here.

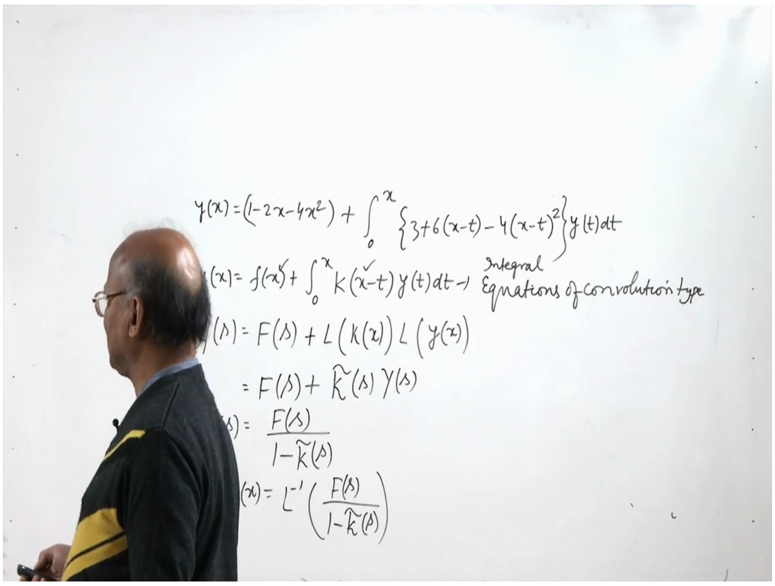
And so while taking the Laplace transform of this we have written it as the product of Laplace transform of K x into Laplace transform of y x. But here we need to assume that the functions K x and f x, the functions f x and K x are such that their Laplace transform exist. That means they are of exponential order. So once we assume that then it turns out that the function y x is also of exponential order and so we can write it as Laplace transform.

(Refer Slide Time: 25:03)



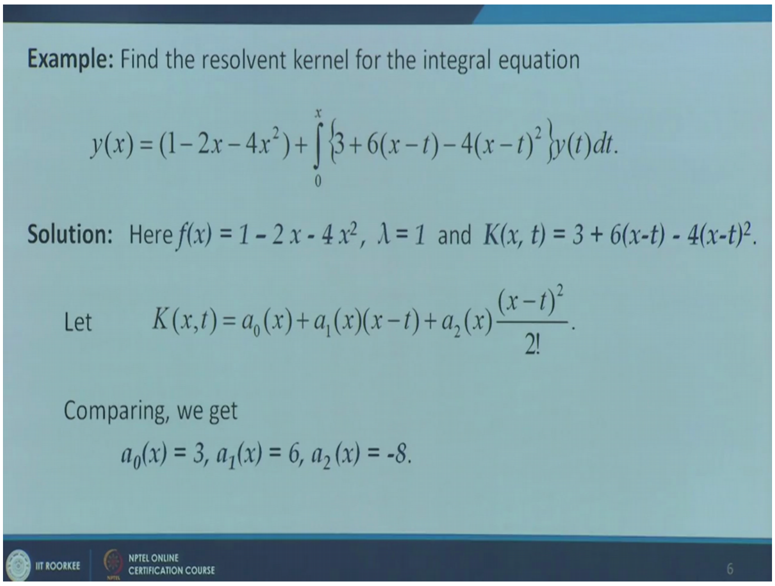
So once we have this I can write this further as and then we can solve it for y s. Y s will be equal to F s divided by 1 minus this. And after we have found y s we can take the inverse Laplace transform. So then y x, the solution of the given integral equation is then L inverse of F s over 1 minus this. And you will see that we can easily find the solution of this differential equation which was given by y x equal to e to the power x if we apply this technique. And it can be obtained in just 5-6 steps.

(Refer Slide Time: 26:19)



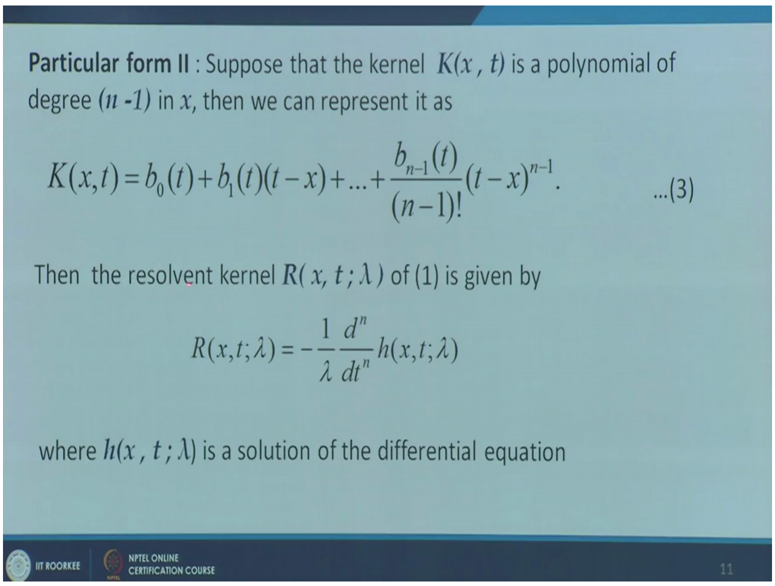
Here this is a very cumbersome method to find the solution of the integral equation. But it is a method which is purely of academic interest. But here we can assume this to be a polynomial of any degree and that can be managed here also. So one can always use to find the solution of such an integral equation by applying the technique of integral equations of convolution type.

(Refer Slide Time: 26:51)



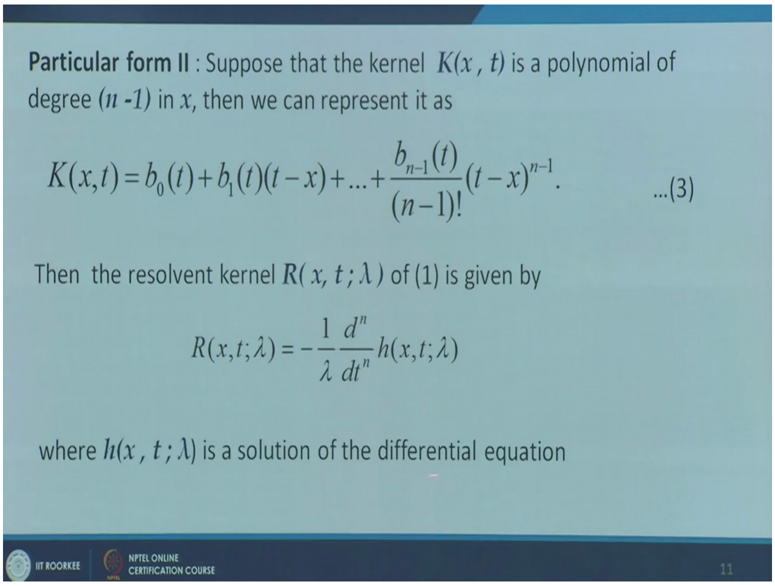
Now let us see there is another case you here. Suppose the kernel K x t is a polynomial of degree n minus 1 in x then we can (rep) always represent it as K x t equal to b not t plus b 1 t, t minus x and so on, b n minus 1 t over n minus 1 factorial into t minus x to the power n minus 1. You can see it is a polynomial in x of degree n minus 1.

(Refer Slide Time: 27:21)



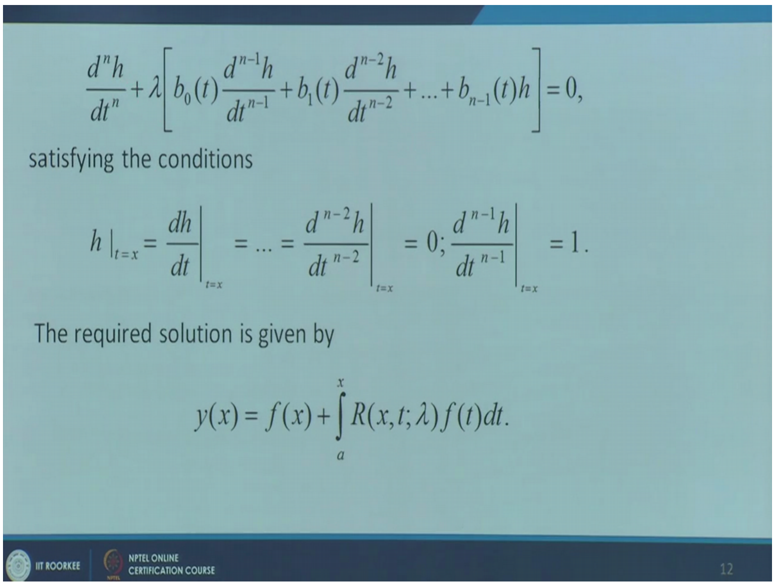
Now then the resolvent kernel R x t lambda of the Volterra integral equation 1 is given by R x t lambda is equal to minus 1 over lambda, d n over d t n of h x t lambda where h x t lambda is a solution of the differential equation of nth order with variable coefficients.

(Refer Slide Time: 27:49)



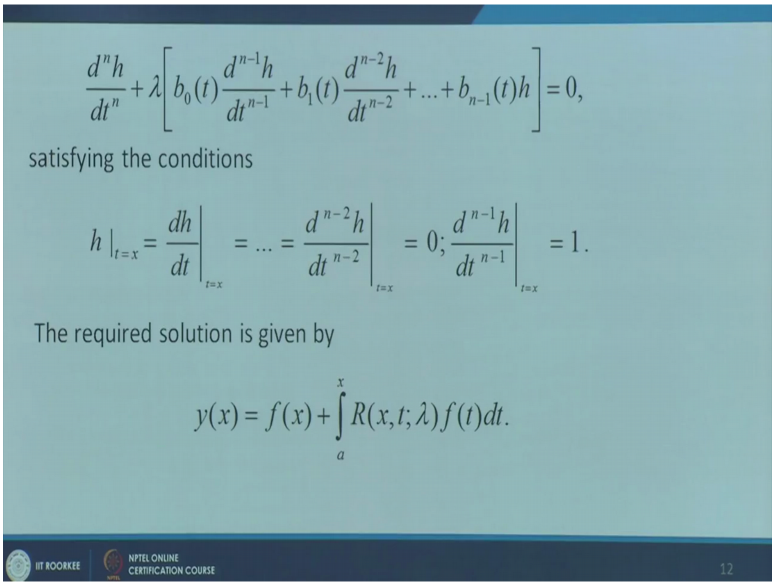
Because b not, b 1, b n minus 1 they depend on t. So this nth order differential equation homogeneous and it satisfies these conditions h at t equal to x, d h by d t at t equal to x, d n minus 2 h over d t n minus 2 at t equal to x is 0. And n minus 1 is a derivative of h with respect to t at t equals to x is 1. And then once we have these we can find the solution of this differential equation h x t lambda.

(Refer Slide Time: 28:27)



And once h x t lambda is known we can differentiate it n times and multiply by minus 1 over lambda to get R x t lambda. R x t lambda then is used here in this equation to determine the solution y x of the given integral equation.

(Refer Slide Time: 28:43)

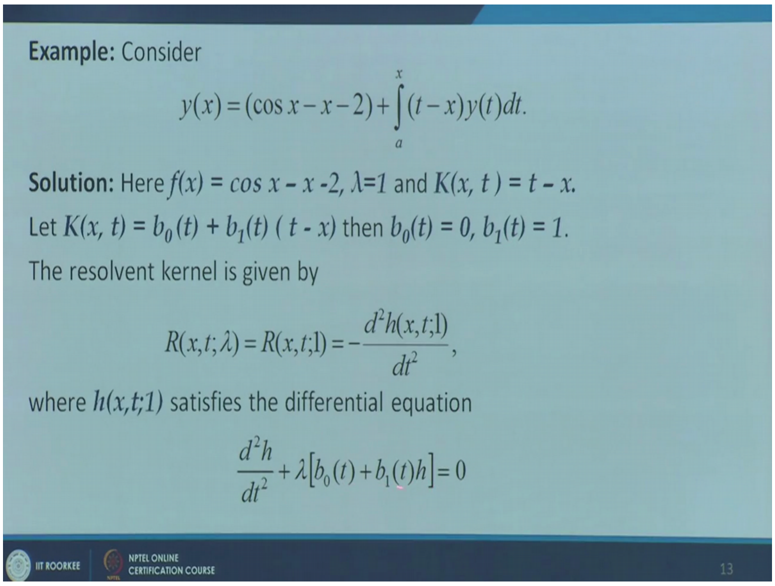


Now let us take an example on this. Suppose y x is equal to cos x minus x minus 2. So this is over function f x. You can see f x is equal to cos x minus x minus 2. Lambda is 1 here and K x t is a polynomial in x of degree 1. So it can be treated by the second particular form. You can also regard K x t here as a polynomial in t of degree 1. so it is of the form 1 as well as of the form 2.

So K x t is t minus x here. We assume K x t to be of the form b not t plus b 1 t into t minus x because K x t we are assuming as a polynomial in x of degree 1 in x, so we can write it like this. And then we compare this given K x t with this standard form and then we see that b not t is equal to 0 and b 1 t is equal to 1. Resolvent kernel R x t lambda will become R x t 1 because lambda is 1 here and it will given by minus d square h x t 1 over d t square because K x t is a polynomial in x of degree 1.

So we have second derivative. Now where h x t 1 satisfies the second order differential equation d square h over d t square plus lambda, b not t plus b 1 t into h equal to 0. Lambda is equal to 1 here, b not t is equal to 0 here, b 1 t is equal to 1.

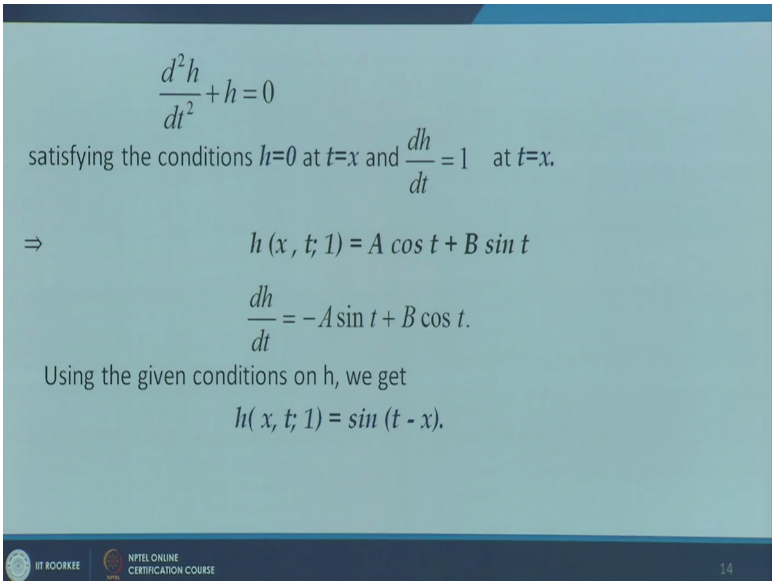
(Refer Slide Time: 30:11)



So this equation reduces to d square h by d t square plus h equal to 0. So this is the second order differential equation with constant coefficients. And the conditions imposed on h are h is equal to 0 at t equal to x and d h by d t equal to 1 at t equal to x. Now we know how to determine the solution of this differential equation with constant coefficients. So here we have the auxiliary equation m square plus 1 equal to 0 which has got two roots, plus minus i. So we have the corresponding solution here as h x t 1 equal to A cos t plus B sin t.

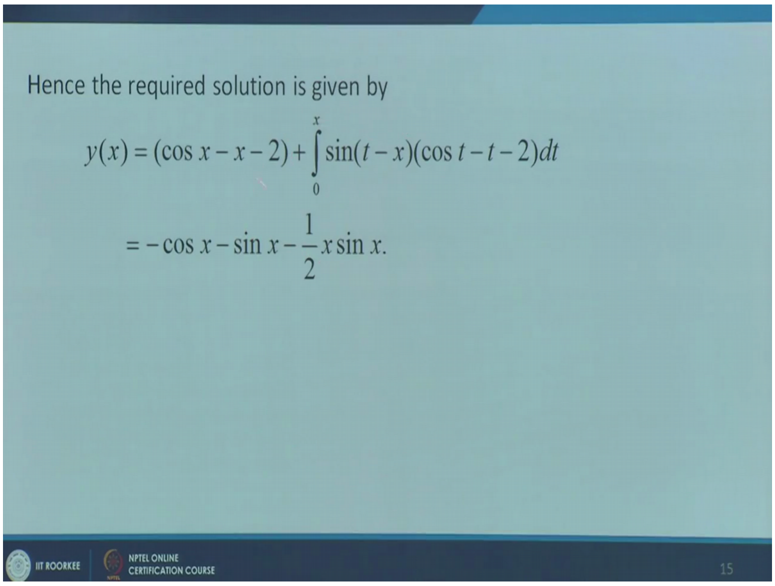
Now we will determine d h by d t here. So minus A sin t plus B cos t, substitute t equal to x and put h is equal to 0 in this equation. And then put here t equal to x and put d h by d t equal to 1. Solve these two equations for the values of A and B. Substitute and then we will see that h x t 1 comes out to be sin t minus x, okay.

(Refer Slide Time: 31:15)



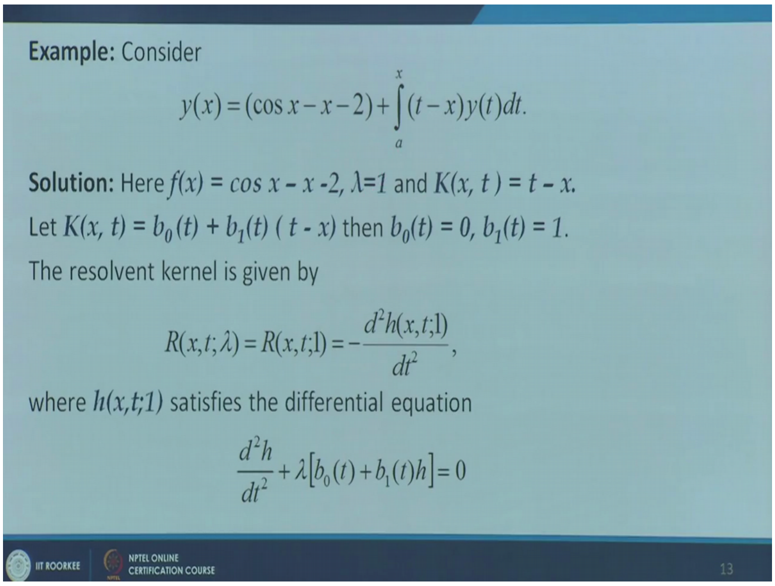
So then once we know h x t 1 we can then differentiate twice h x t 1 with respect to t multiplied by minus 1, we get R x t 1. So R x t 1 will be equal to sin t minus x and then y x is equal to cos x minus x minus 2, 0 to x sin t minus x cos t minus t minus 2 d t. So you can integrate here. We can open this sin t minus x by the formula sin A minus B. You can open this and then you can integrate by parts. You can see that the answer is y x is equal to minus cos x minus sin x minus half x sin x.

(Refer Slide Time: 32:06)



So what I would suggest that this is again this method is of purely academic interest. Otherwise we can always treat it as an integral equation of convolution type and by writing t minus x as minus times x minus t. So it is an integral equation of convolution type. And then we can solve it by taking the Laplace transform of both sides they way we solve a integral equation of convolution type.

(Refer Slide Time: 32:37)



So with this I would like to conclude my lecture. Thank you very much for your attention.