Integral Equations, Calculus of Variations and their Applications
Professor Doctor P N Agrawal
Department of Mathematics
Indian Institute of Technology, Roorkee
Mod 06 Lecture Number 25
Neumann series and Resolvent kernel

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Hello friends Welcome to my lecture on Neumann Series and resolvent kernel There will be two lectures on this topic This is first of the two lectures on Neumann Series and resolvent kernel Here we will discussing how to find the

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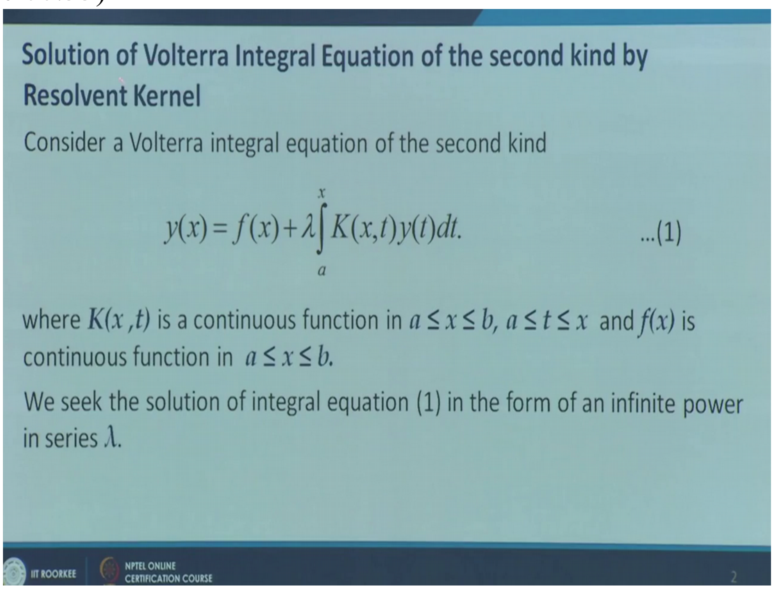

solution of the Volterra integral equation of the second kind by finding the resolvent kernel

Let us consider a the Volterra integral equation of the second kind which is given by y x equal to f x plus lambda times integral a to x K x t into y t d t We assume here K x t is a continuous function in the interval a less than equal to x less than or equal to b a less than equal to t less than equal to x So in the t x plane the region over which K x t is a continuous function is given by these inequalities f x is the continuous function on the closed interval a b We seek the solution of this equation in the form of an infinite series in powers of lambda

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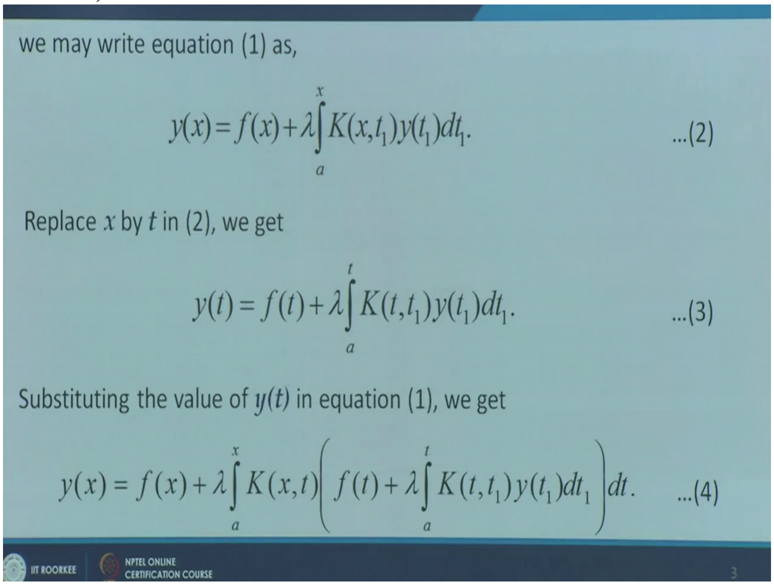

The equation integral equation Volterra integral equation of second kind

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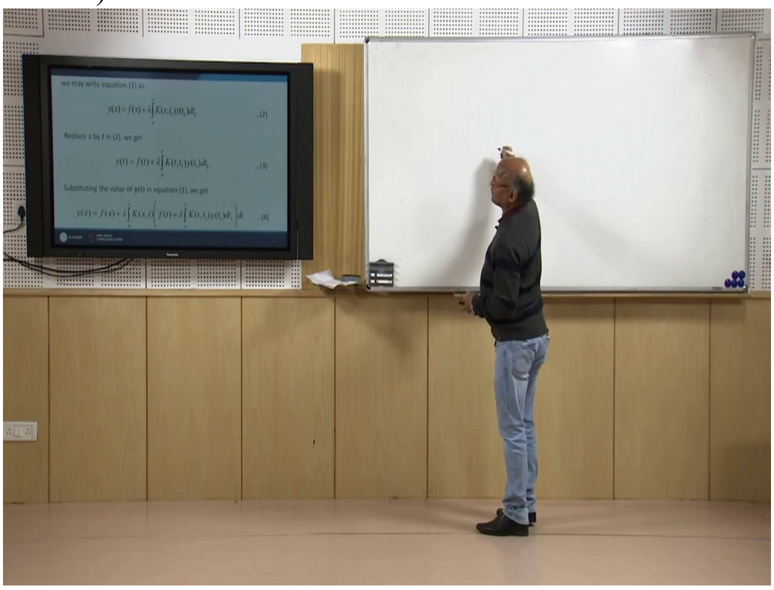

let us write as y x equal to the integral equation is equal to y x equal to f x plus lambda times integral a to x K x t y t d t This is the equation given to us

We can also write it as

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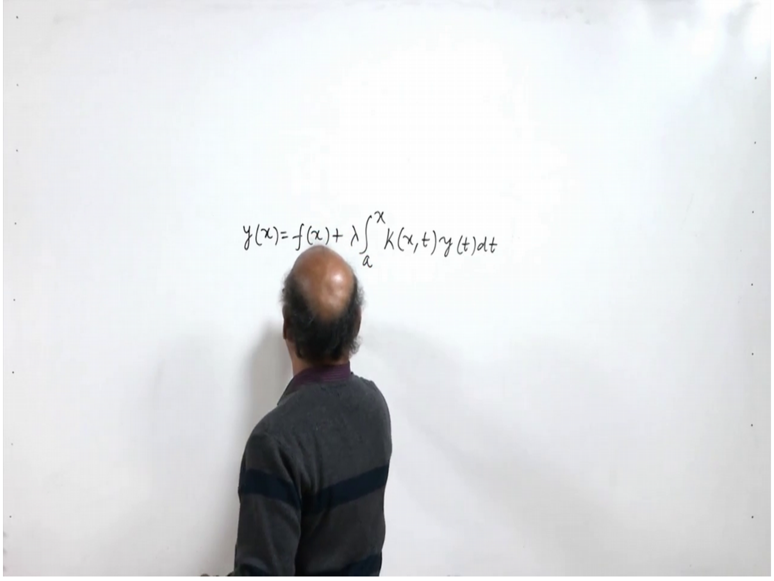

y x equal to f x plus lambda integral a to x K x t 1 y t 1 d t 1 Now in this equation in the first equation you replace x by t then what do we have So this equation in 1 let us write t for x then let us write t for x in equation 1 here y t equal to f t plus lambda integral a to x K t t 1 y t 1 d t 1 So we get y t equal to f t plus lambda

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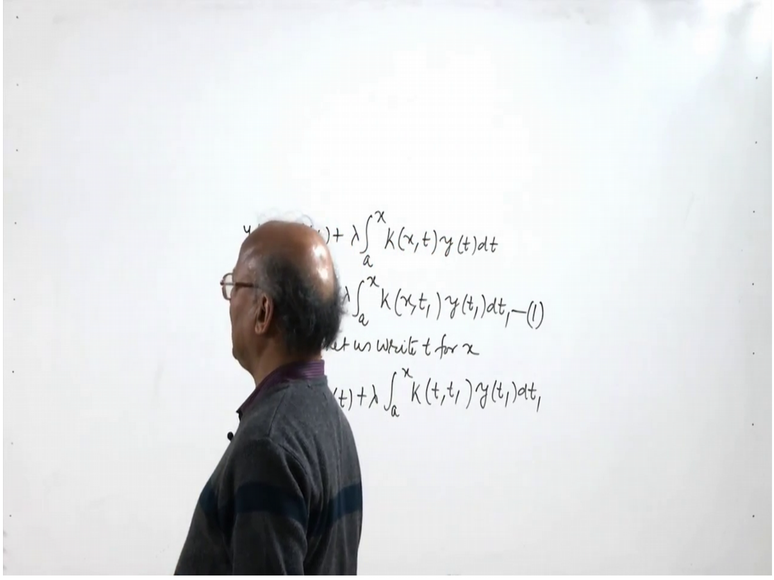

integral a to t

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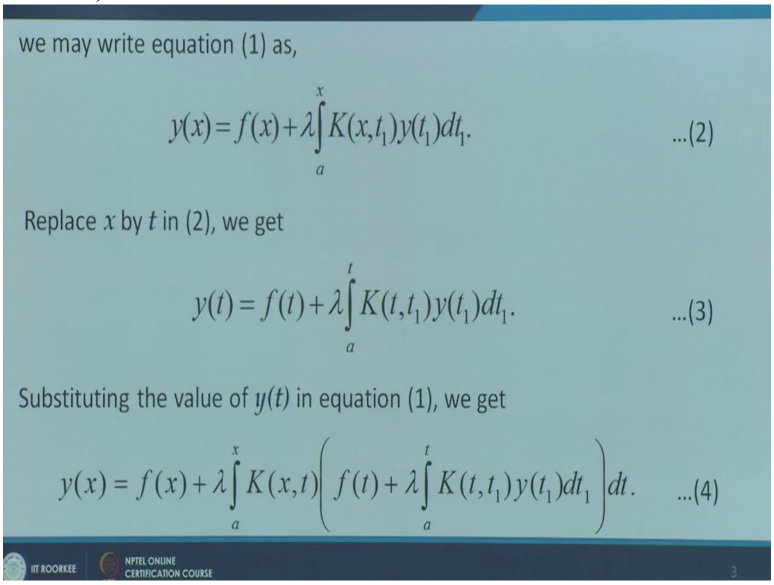

here we will have t K t t 1 by d t 1 Now let us substitute this value of y t in equation 1

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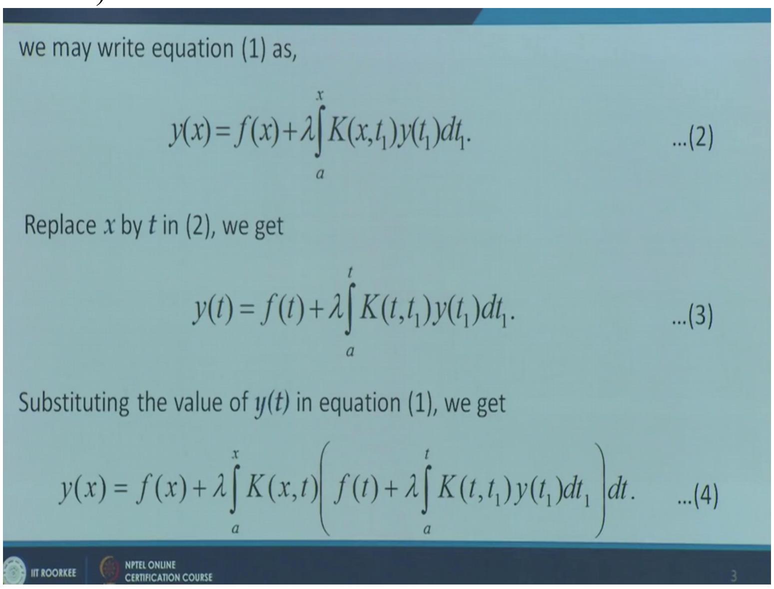

So this is our equation 1

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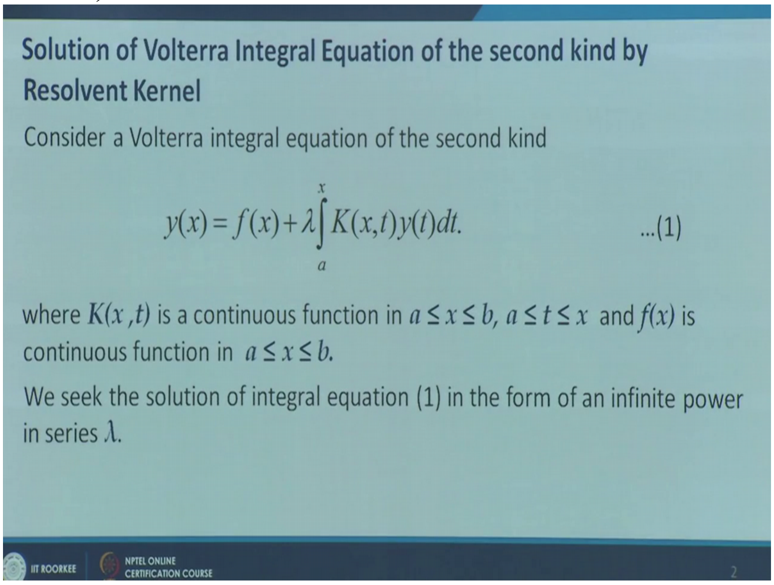

In equation 1 we substitute the value of y t so then what do we have

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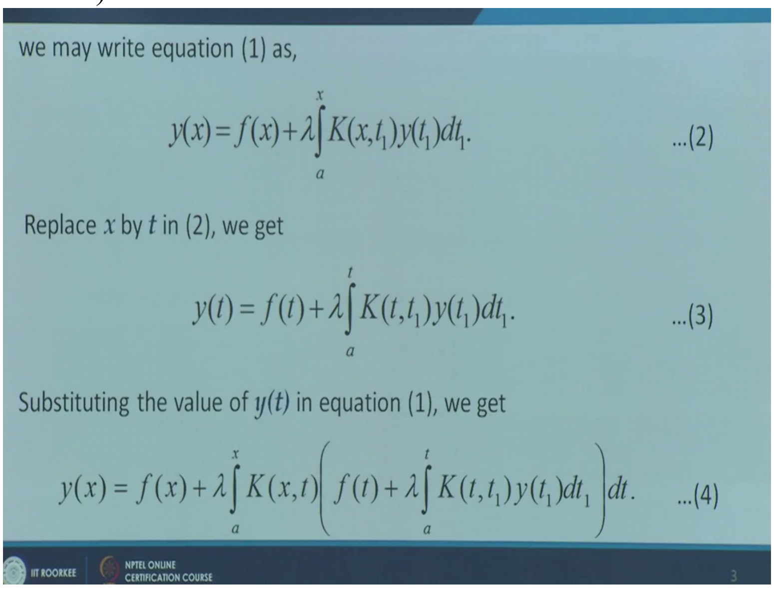

y x equal to f x plus lambda a to x K x t y t is replaced by f t plus lambda integral a to t K t t 1 y t 1 d t 1 d t

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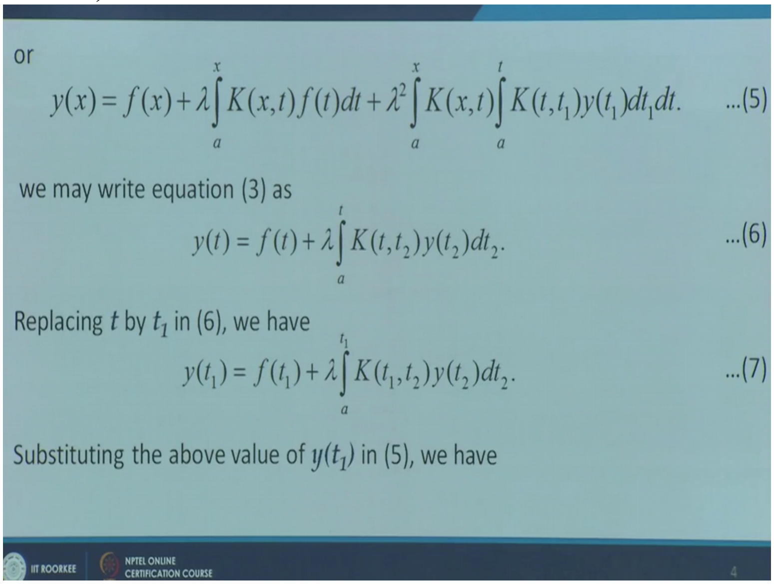

y x equal to f x plus lambda a to x K x t y t is replaced by f t plus lambda integral a to t K t t 1 y t 1 d t 1 d t Or we can write it as y x equal to f x plus lambda a to x K x t f t d t plus lambda square a to x K x t and then a to t K t t 1 y t 1 d t 1 d t Now let us write equation 3

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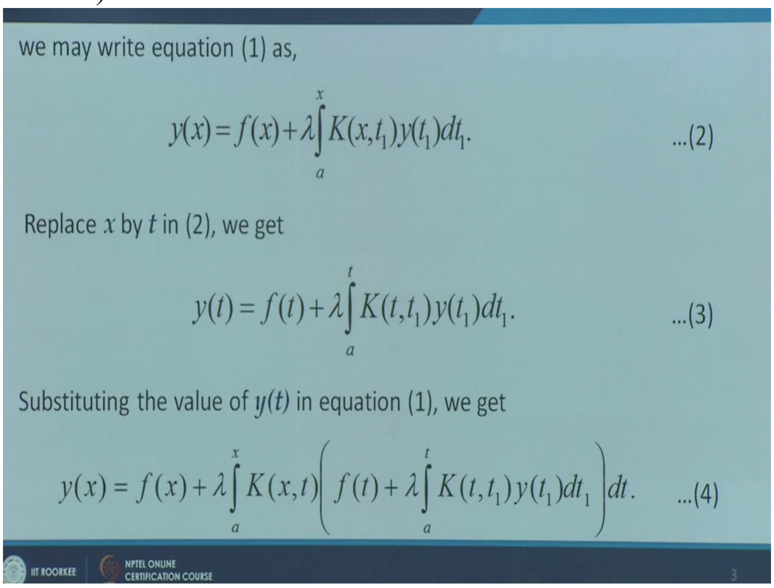

as this equation 3 as we can write as

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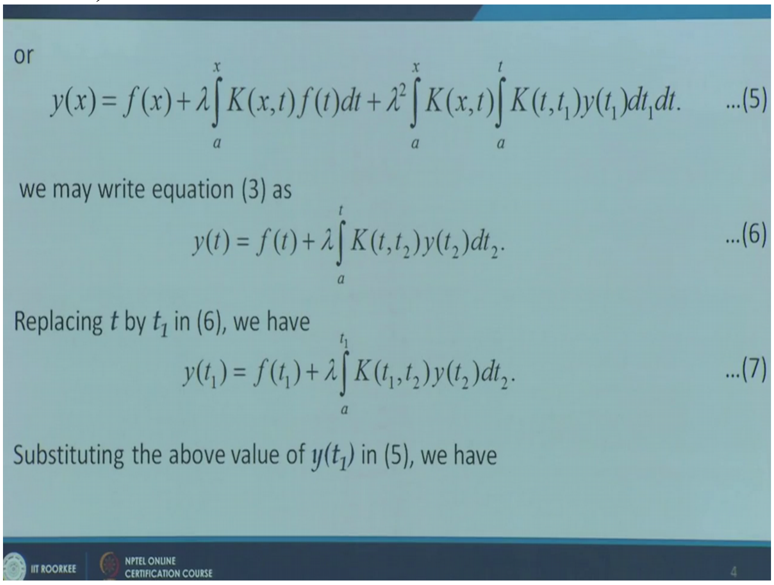

y t equal to f t plus lambda a to t K t 2 y t 2 d t 2 Instead of t 1 we can replace t 2 there So we have y t equal to f t plus this Now in this let us replace t by t 1 Then we shall have y t 1 equal to f t 1 plus lambda integral a to t 1 K t 1 t 2 y t 2 d t 2 And then we substitute this value y t 1 in the equation 5 And after substitution

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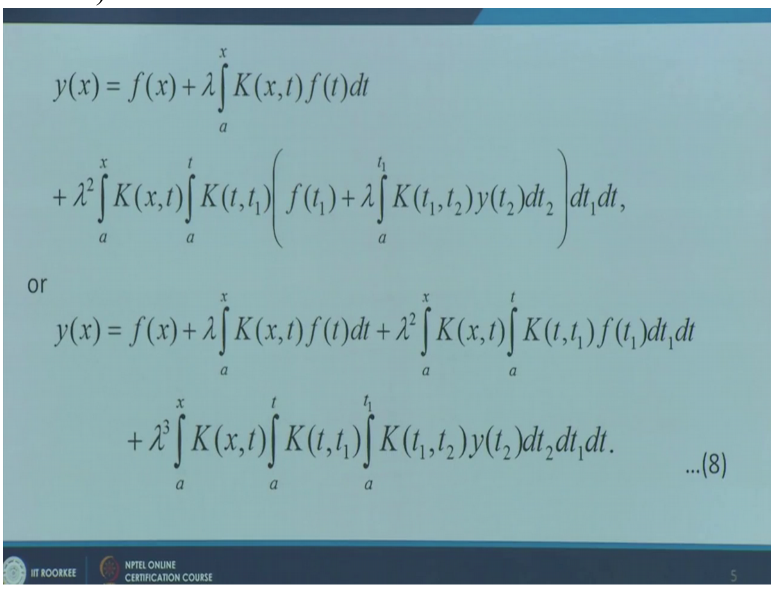

what we get is y x is equal to f x plus lambda a to x K x t f t d t plus lambda x square times integral a to x K x t a to t K t t 1 and this is the value of y t 2 This is the value of y t 1 so f t 1 plus lambda a to t 1 K t 1 t 2 y t 2 d t 2 d t 1 d t and simplifying we get y x equal to f x plus lambda a to x K x t f t d t plus lambda square a to x K x t a to t K t t 1 f t 1 d t 1 d t and lambda cube a to x K x t a to t K t t 1 a to t 1 K t 1 t 2 y t 2 d t 2 d t 1 d t

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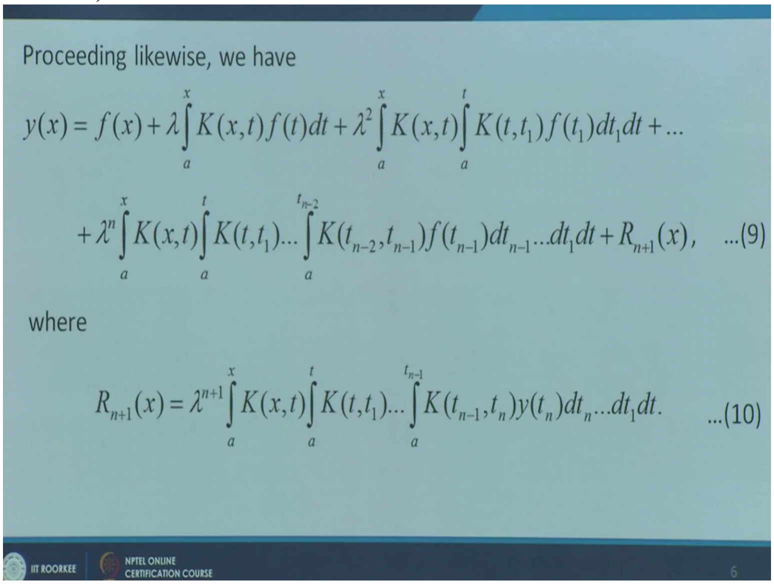

Proceeding in this manner we then have y x equal to f x plus lambda integral a to x K x t f t d t and so on lambda to the power n a to x K x t a to t K t t 1 a to t n minus 2 K t n minus 2 t n minus 1 f t n minus 1 d t n minus 1and so on d t 1 d t And this is R n plus 1 which is the remainder term R n plus 1 x is lambda to the power n plus 1 a to x K x t a to t K t t 1 a to t n minus 1 K t n minus 1 t n y t n d t n d t 1 d t

Now let us consider this infinite series Let us consider the instead of this let us consider the corresponding infinite series

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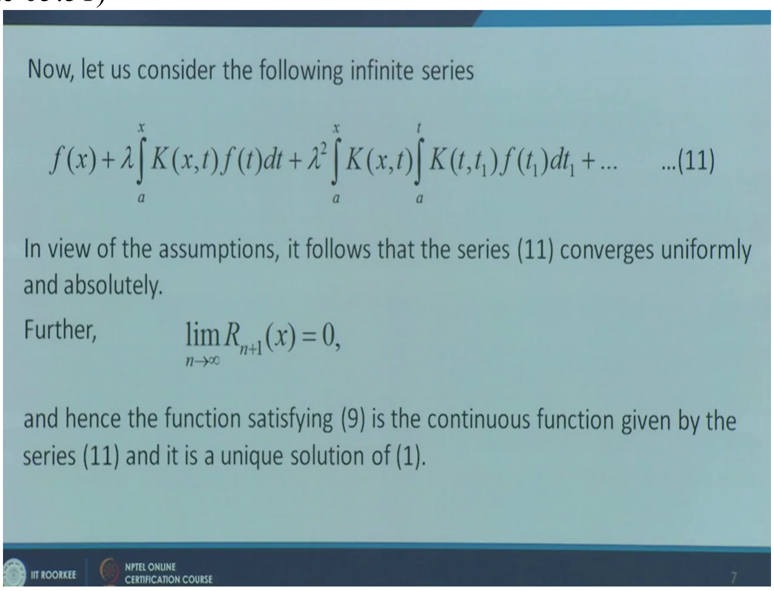

f x plus lambda a to x K x t f t d t plus lambda square K x t a to t K t t 1 f t 1 d t 1 and so on Then in view of assumption in view of our assumption it follows that this series converges uniformly and absolutely and further limit n tends to infinity

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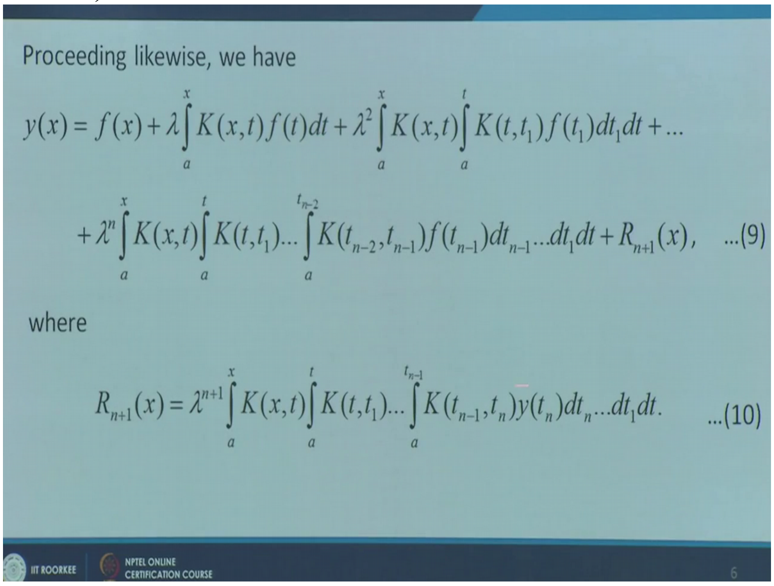

R n plus 1 x is equal to zero

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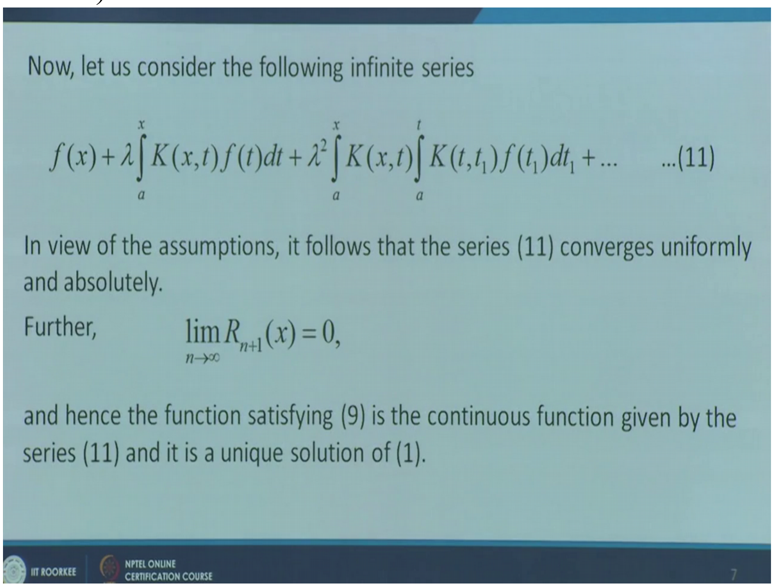

This is the proof of this is exactly similar as we prove it for the case of Fredholm integral equation of the second kind So and moreover the function which is satisfying 9

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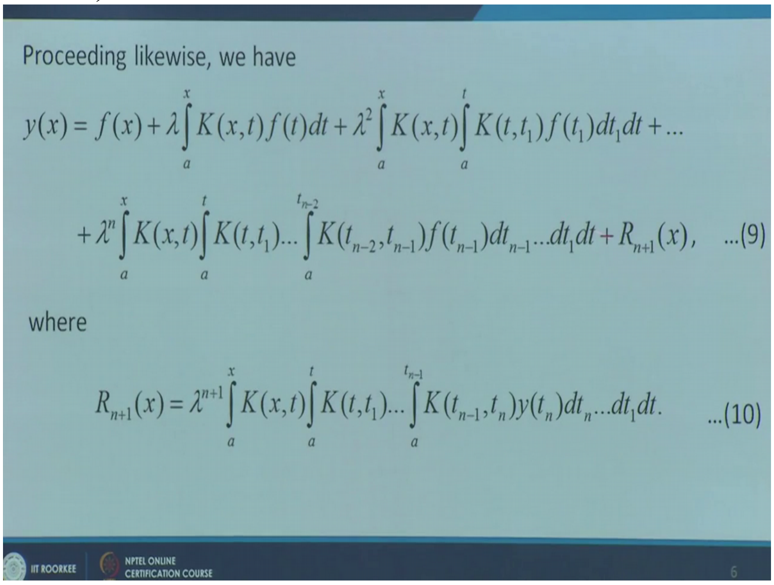

which is satisfying this equation 9 is

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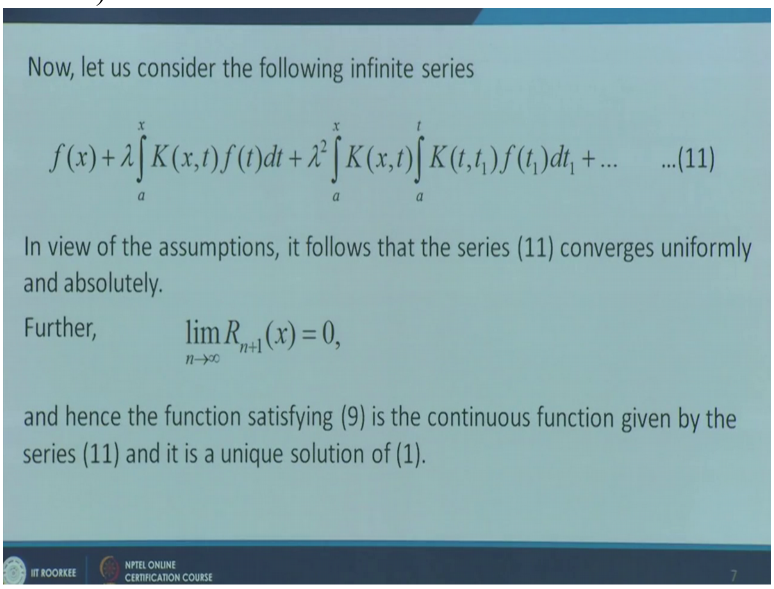

the continuous function which is given by the series 11 Because the series converges uniformly and absolutely so it will give us a continuous function So continuous function is given by this series 11 and moreover the solution of the integral equation is unique So the proof of these facts is exactly similar to the proof of the corresponding the Fredholm integral equation of the second kind So we are not repeating that proof here We have just used that one and from there it follows that the solution of the given Volterra integral equation is given by this infinite series and is a unique solution

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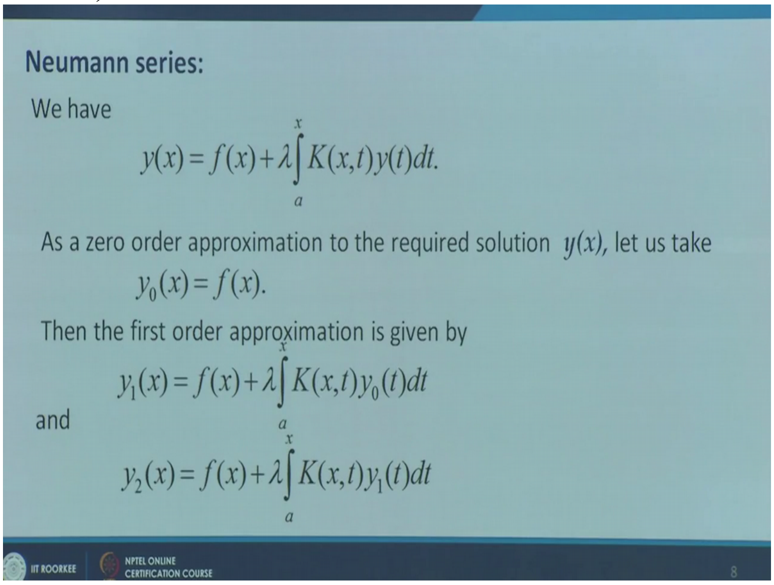

Now let us look at what is the Neumann series Let us begin again with the Volterra integral equation of second kind y x is equal to f x plus lambda a to x K x t y t d t To begin with the zero initial approximation or we can say zero order approximation to the required solution by x Let us assume that the initial approximation which we take as y naught x is given by the known function f x So y naught x let us take as f x and then the first order approximation of the given of the solution will be y 1 x equal to f x plus lambda a to x K x t into y naught t d t y 2 x will be because once y 1 x is calculated y 2 x will be f x plus lambda a to x K x t into y 1 t d t

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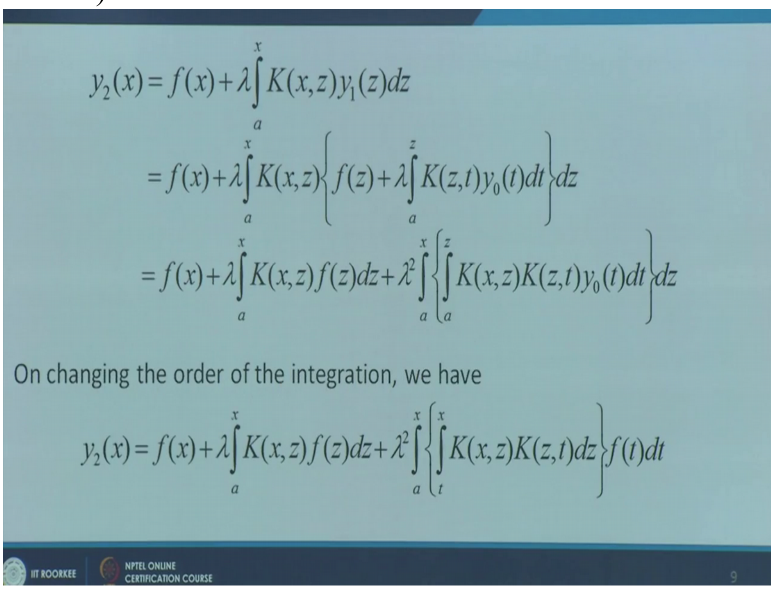

Now y 2 x can be written as f x plus lambda a to x K x z into y 1 z d z

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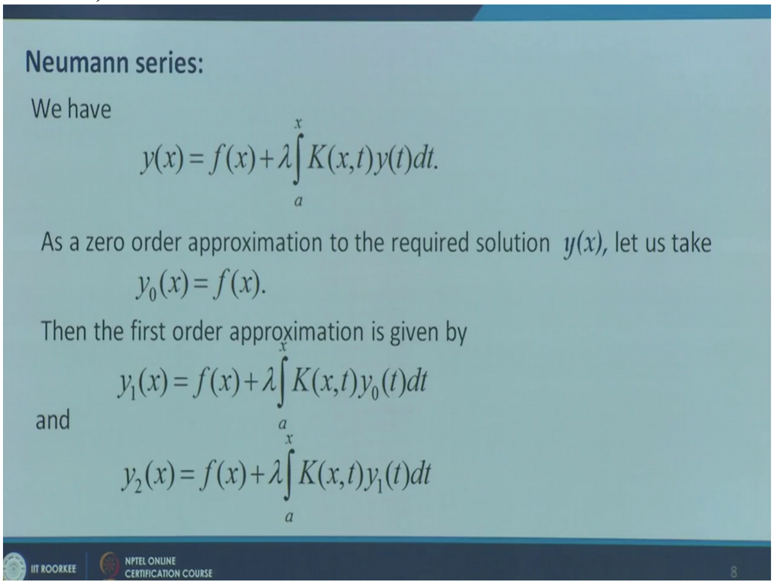

Here this y 2 x can be written as f x plus lambda a to x K x z into y 1 z d z and the value y 1 z can be then put from here

So we have

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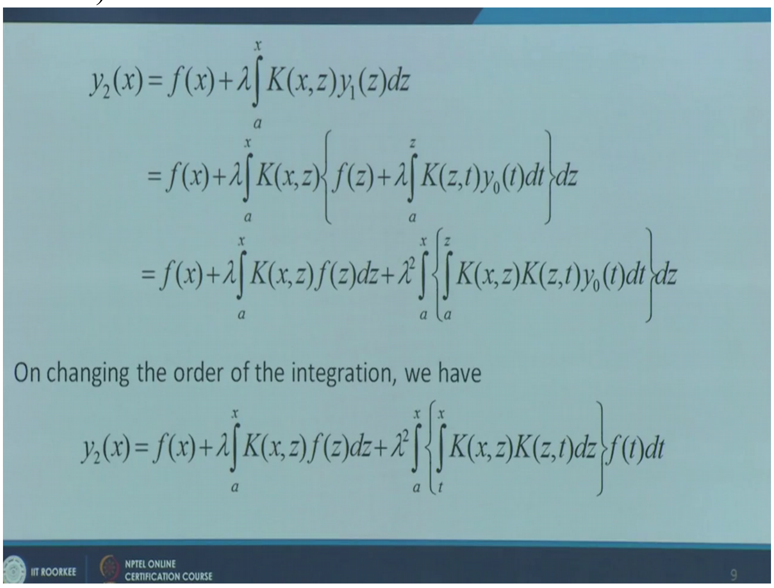

f x plus lambda a to x K x z and y 1 z is f z plus lambda a to z K z t into y naught t d t and then d z And then we can simplify this so f x plus lambda a to x K x z into f z d z plus lambda square times a to x a to z K x z into K z t into y naught t d t Now let us change the order of integration here in the third term on the right So on changing the order of integration we have y 2 x equal to f x plus lambda a to x K x z f z d z plus lambda square a to x t to x K x z K z t d z f t d t f t is y naught t

Now let us see how change the order of integration here So let us say this is t axis And this is z axis This is z equal to t

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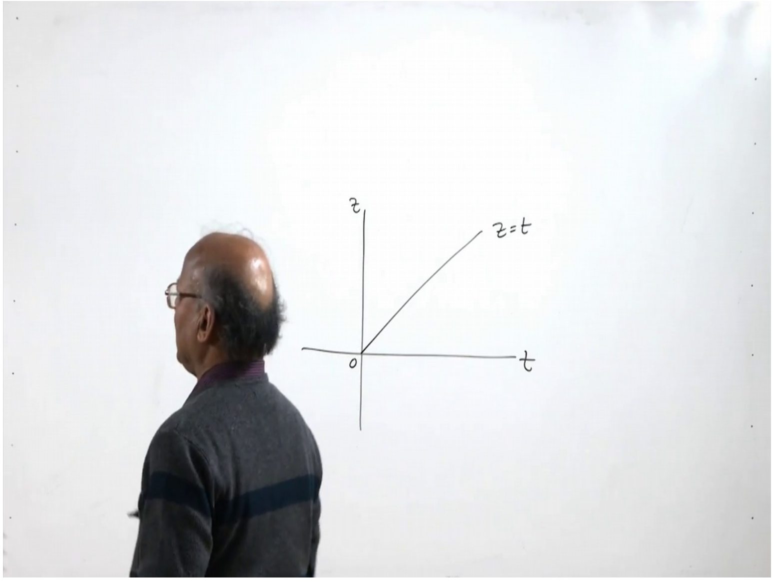

We have to we have to interchange the limits in the integral integral a to x a to z K x z into K z t y naught t d t d z

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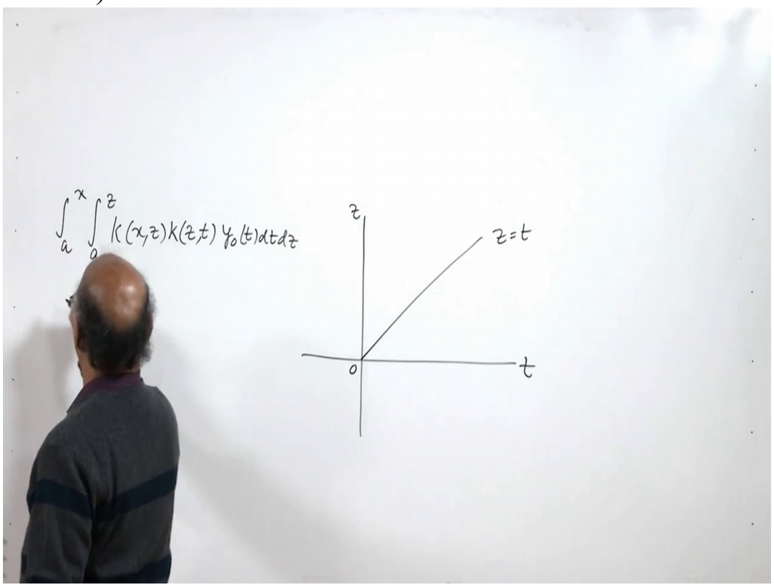

We shall change to a to x t to x K x z into K z t d z y naught t d t So let us see how we

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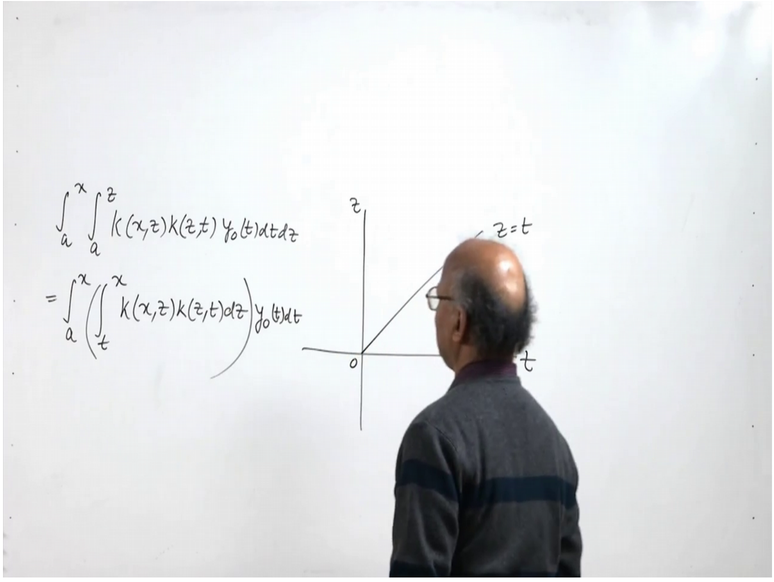

make this change by using the change of order of integration So we will first see what is the area represented by the limits of integration where t varies from a to z and z varies from a to x

So let us say this is t equal to a So

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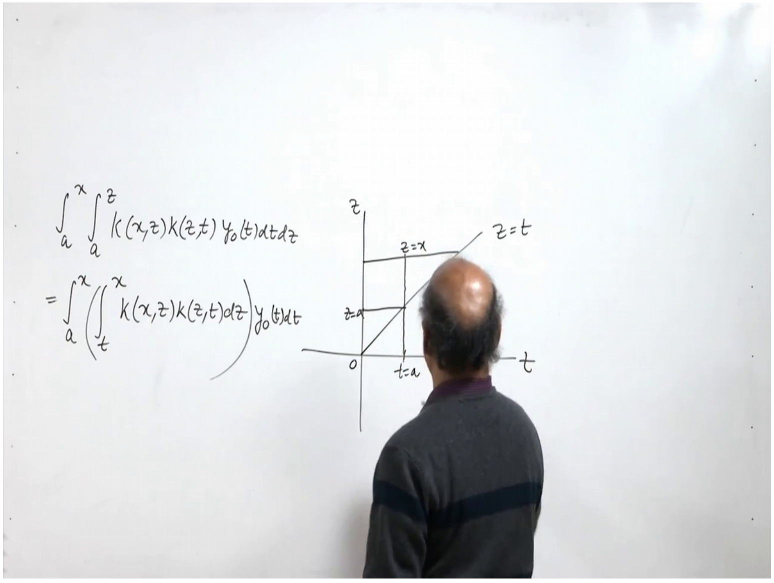

t varies from a to t equal to z This is t equal to z line and then we have z varies from a So this is z equal to a Then z varies from a to x So this is t equal to a So t varies from a to t varies from a to z and z varies from a to x z varies from a to x This is the region over which we are integrating

Now t varies from a to x Here t is a and t varies from

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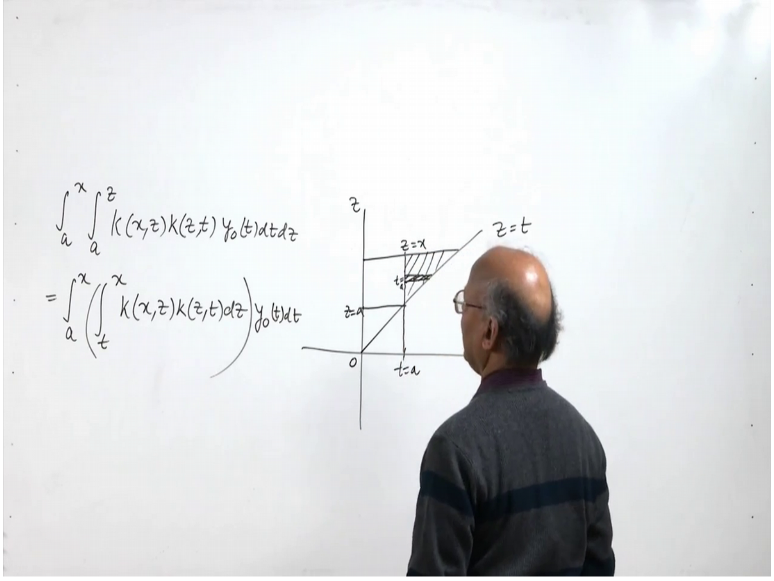

a to t varies from a to z So this is t equal to z and z varies from a to x So this means that we are taking a horizontal strip here We are taking horizontal strip here in the region shaded region now let us take a vertical strip So when we take a vertical strip in this region what are the limits of integration z varies from z varies from t to z equal to x So z varies

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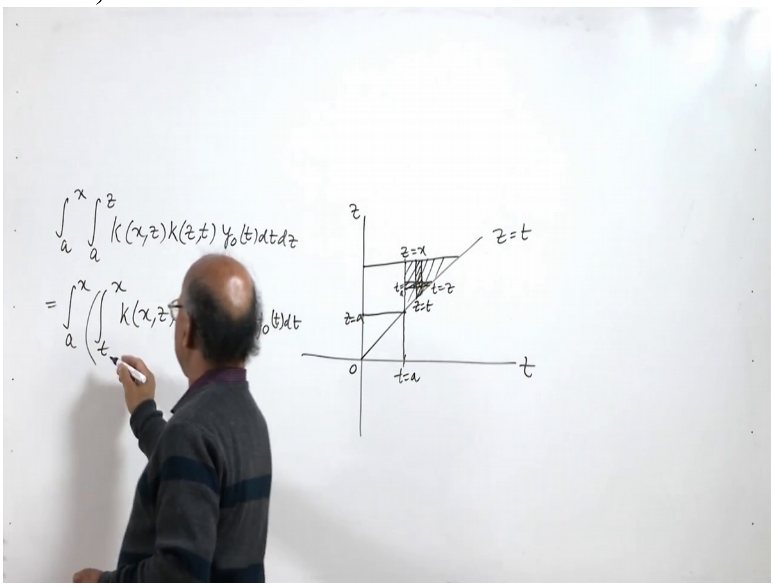

from t to x and t varies from a to t equal to x Because this point t equal to x so t varies from a to x and z t varies fromthis is d t oh oh so z varies from t to x z varies from t to x

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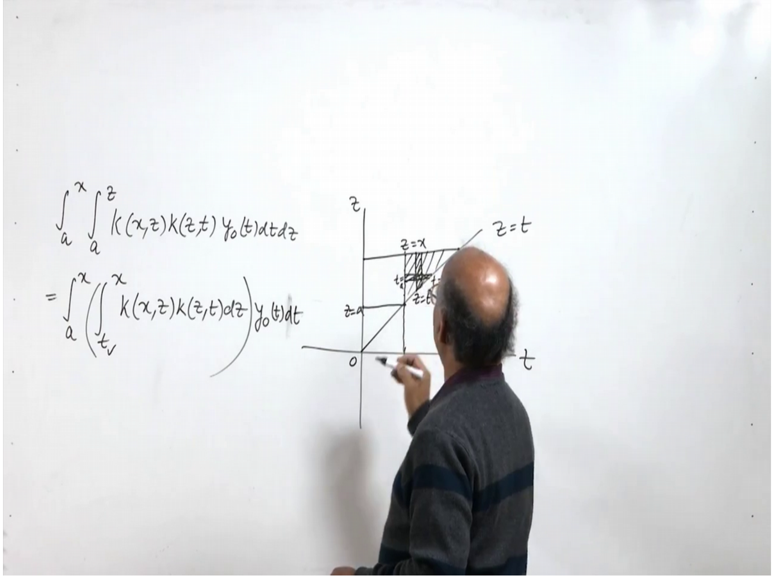

and t varies from a to x So this is how we change the order of integration and then we write it as lambda square integral a to x

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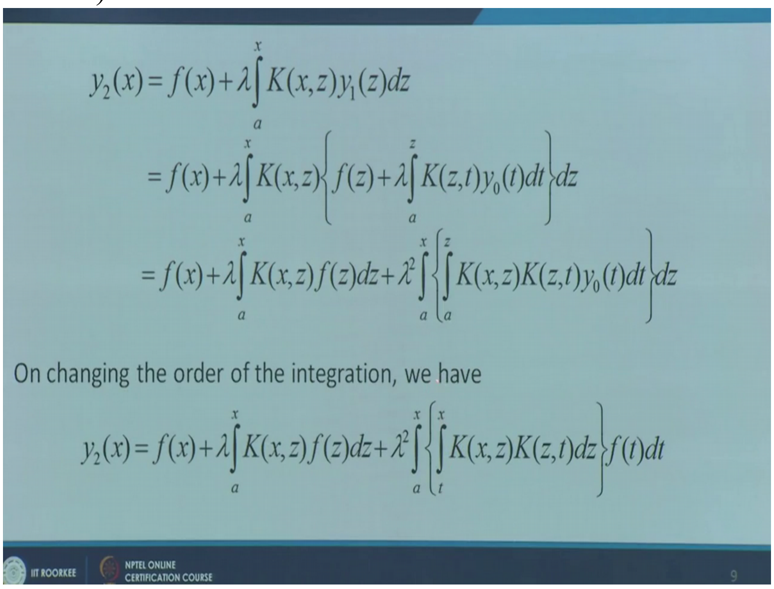

t to x K x z K z t d t and y naught t is equal to f t So then

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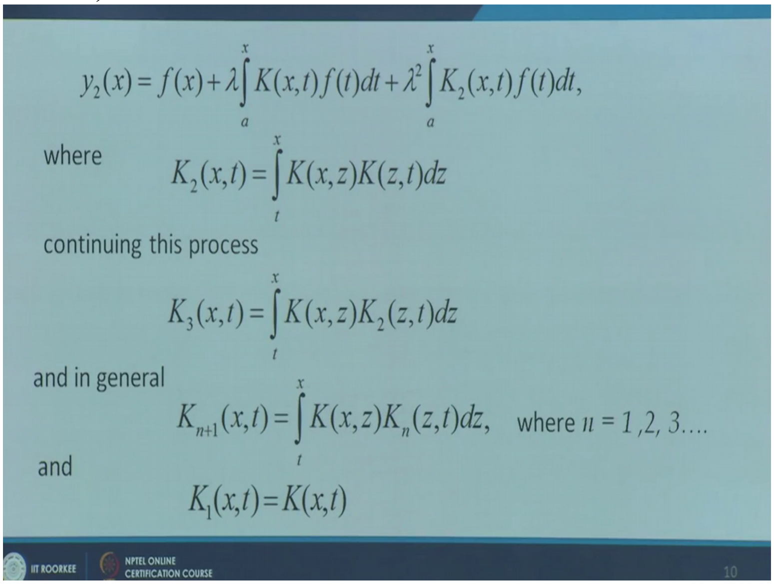

we write it as y 2 x equal to f x plus lambda times a to x f t d t plus lambda square times K 2 x t f t d t So K 2 x t is this integral is written like this

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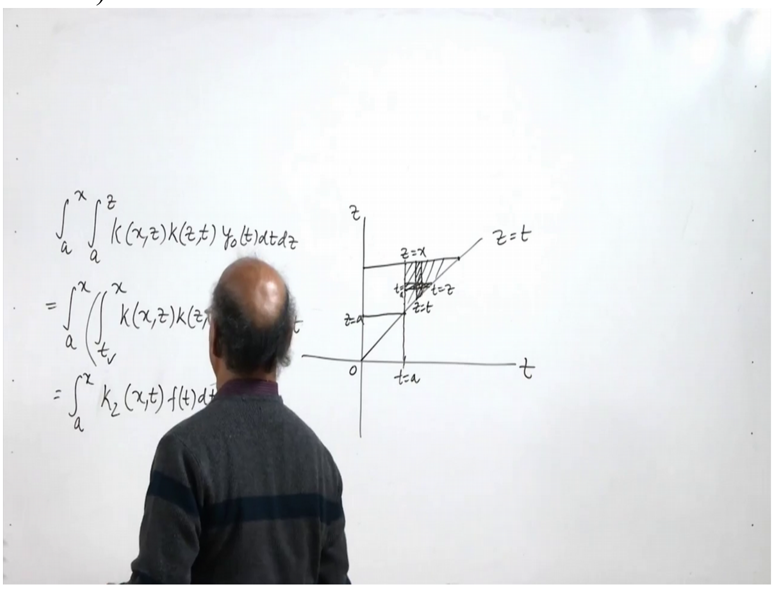

where K 2 x t is integral t to x K x z into K z t d z

Now we can carry on this process further So continuing this process

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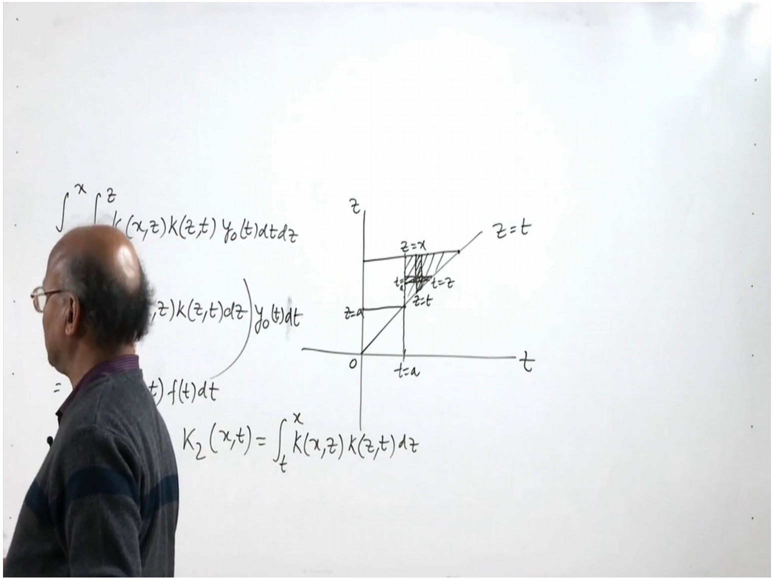

we will have

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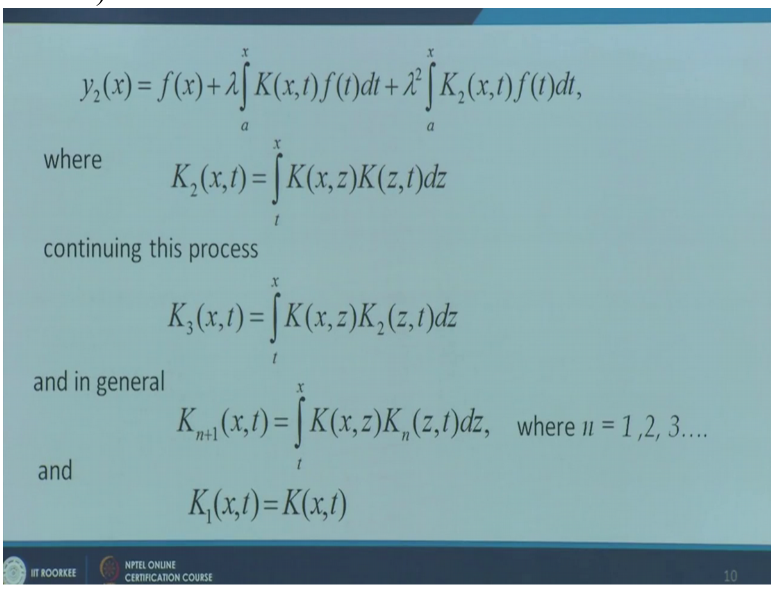

to define K 3 x t and K 3 x t will be defined as integral t to x K x z into K 2 z t d z and in general n plus 1 of the iterated kernel K n plus 1 x t K n plus 1 x t is defined as integral t to x K x z into K n z t d z where at n equal to 1 K 1 x t is taken as the given kernel K x t So this is kernels K n x t are called as iterated kernels So K 1 x t is the given kernel K x t and K n plus 1 x t is given by this integral t 2 x K x z K n z t d z These functions

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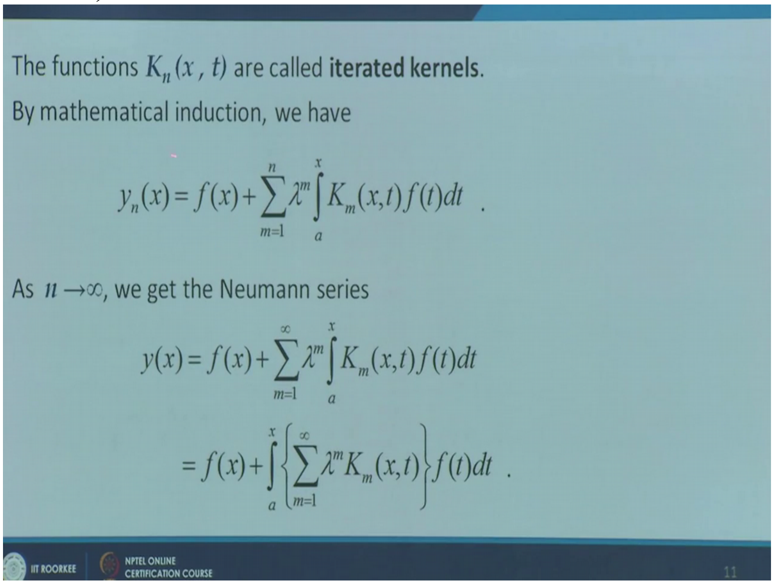

K n x t are called as iterated kernels

Now by mathematical induction we can now write as the nth approximation to the solution y x as y n x equal to f x plus sigma m equal to 1 to n lambda to the power m a to x K 1 x t f t d t

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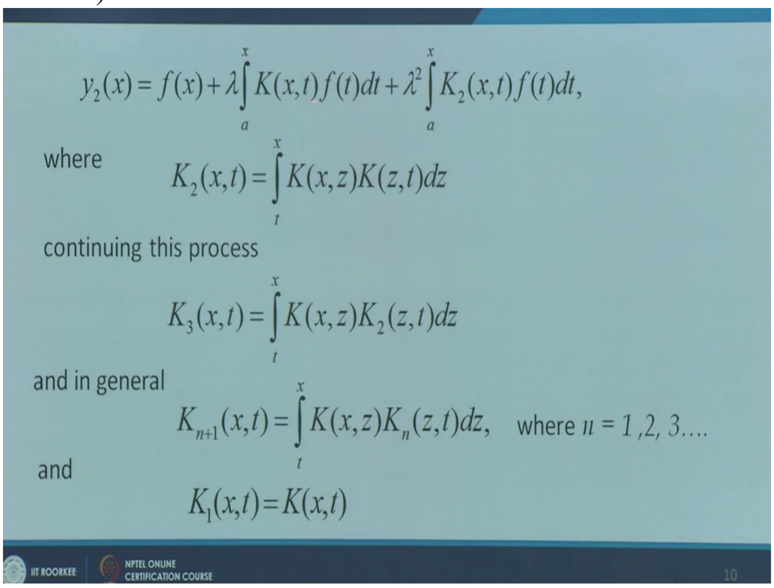

You can see here y 2 x is f x plus lambda times a to x this is K 1 x t because K x t is K 1 x t K 1 x t into f t d t plus lambda square times integral a to x K 2 x t f t d t So this is second approximation to the solution So nth approximation to the solution will be

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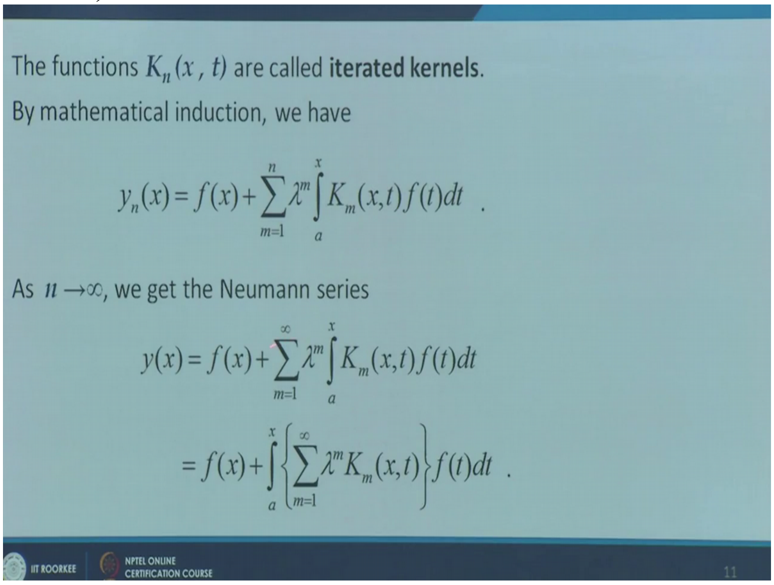

given by this expression Now as n goes to infinity we get the Neumann series y x equal to f x plus sigma m equal to 1 to infinity lambda to the power m integral a to x K m x t f t d t Now

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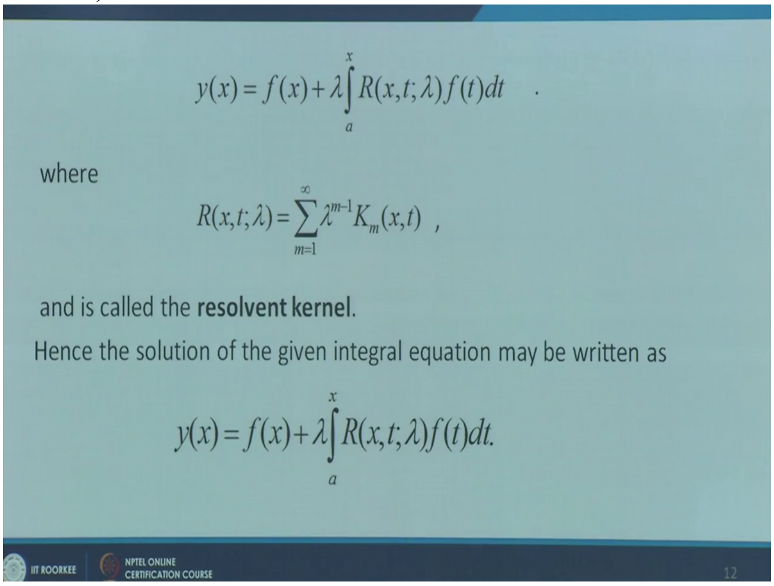

since sigma m equal to 1 to infinity lambda to the power m K m x t is uniformly convergent The term by term integration is allowed So we can interchange the integration and summation here

And therefore we can write f x plus integral a to x sigma m equal to 1 to infinity lambda to the power m K m x t f t d t Now which is further written as y x is equal to f x plus lambda integral a to x R x t lambda f t d t where R x t lambda is integral sigma m equal to 1 to infinity lambda to the power m minus 1 K m x t and this is called as this R x t lambda is called as the resolvent kernel

And then so the solution of the given integral equation may be written as

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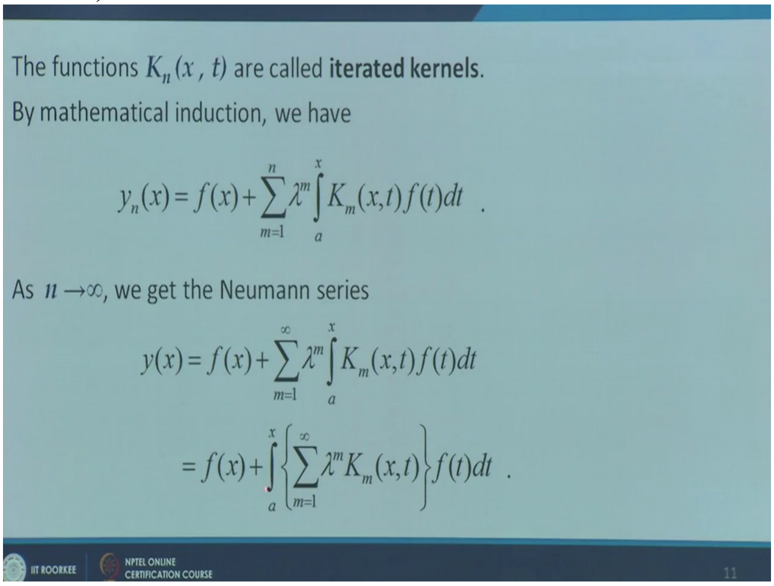

y x equal to f x plus integral a to x R x t lambda into f t d t

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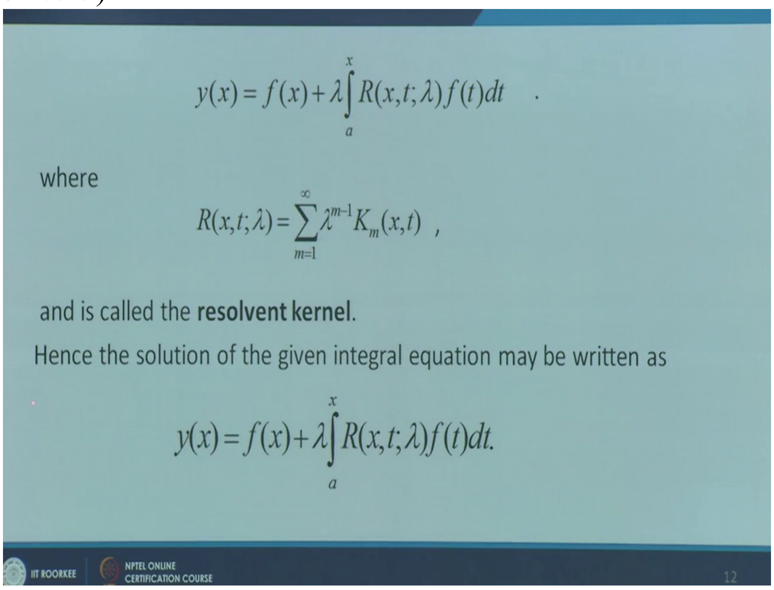

This is the solution of the given Volterra integral equation of the second kind So we will find for the given Volterra equation of second kind we shall find the value of R x t lambda and then integrate it over the interval a to b a to x and then we will get the solution of the given integral equation R x t lambda can also be written as sigma m equal to zero to infinity lambda to the power m So

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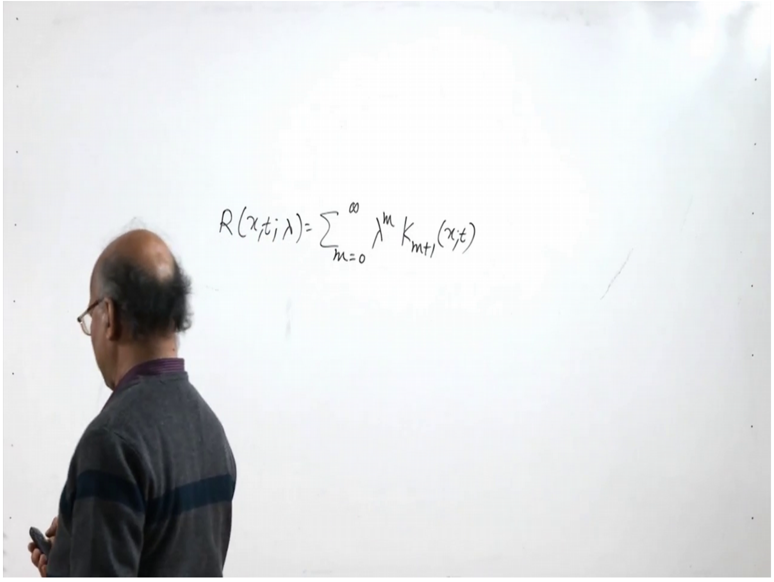

R x t lambda is equal to by replacing m minus 1 by m we get here

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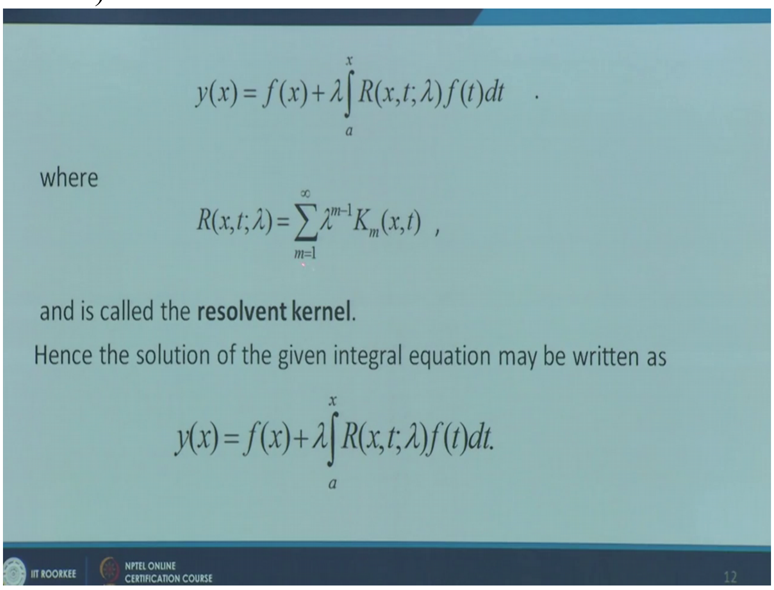

sigma m equal to zero to infinity lambda to the power m K m plus 1 x t So let us see how we will use this method to solve a Volterra integral equation of the

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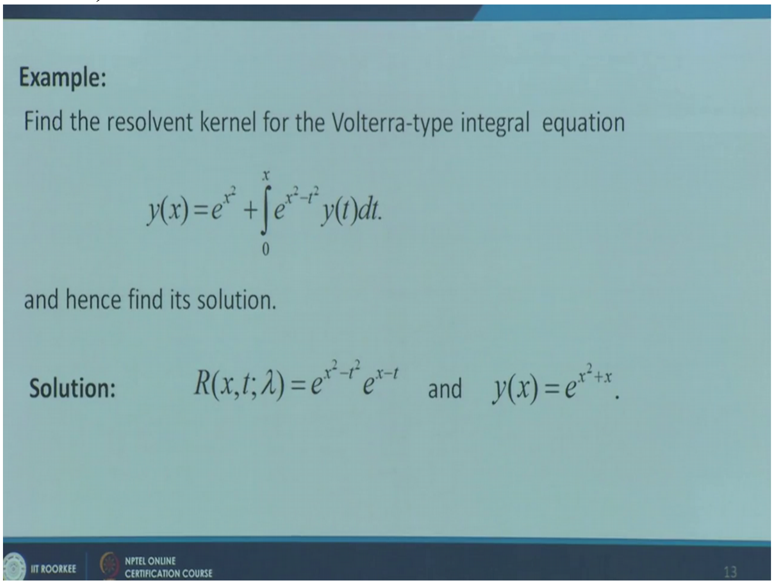

second kind

So let us begin with this integral equation of second kind Volterran type y x equal to e to power x square integral zero to x e to power x square minus t square y t d t So if you compare it with the standard Volterra integral equation of the second kind we see that y of f x is equal to e to the power x square K x t equal to e to the power x square minus t square and lambda equal to 1

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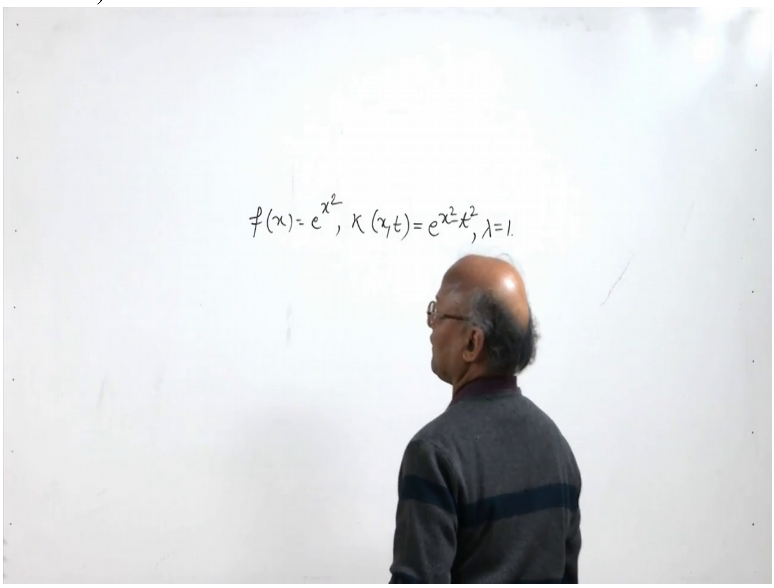

So we will find the value of the resolvent kernel and for the resolvent kernel we need to know the iterated kernels

So the first iterated kernel K 1 x t is defined as K x t So we get e to power x square minus t square Now let us find the next iterated

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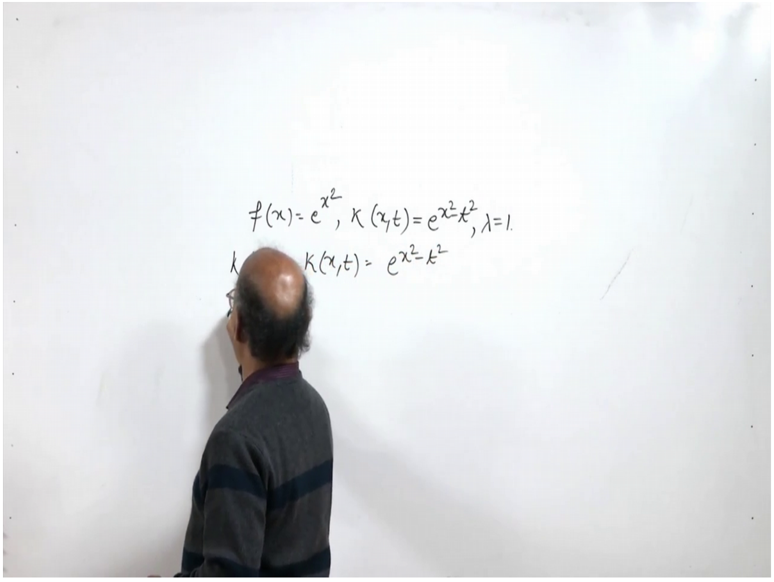

kernel K 2 x t It will be given by integral t to x K x z into K 1 z t d z And which is equal to integral t to x K x z into K z t d z This is equal to integral t to x e to the power x square minus z square into

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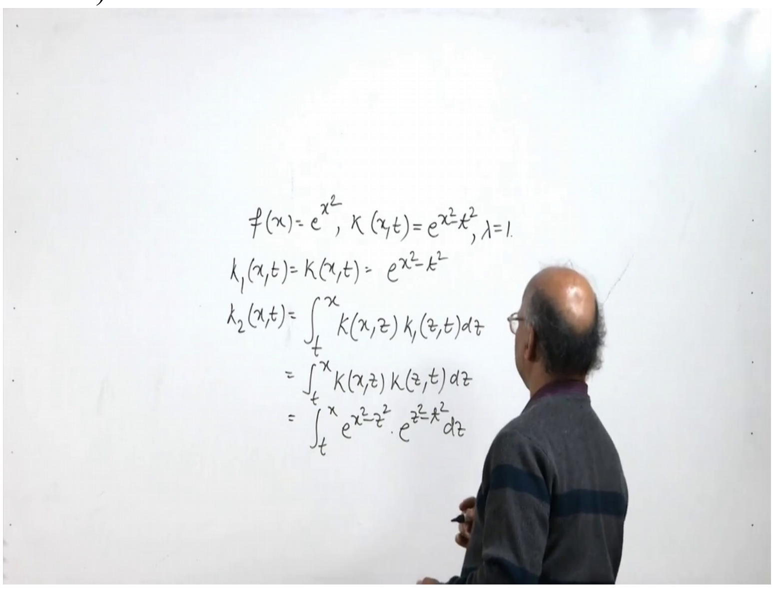

e to the power z square minus t square into d z e to the power x square minus z square into e to the power z square minus t square will become e to power x square minus t square which is independent of z So I can write it as and after integration we get e to the power x square minus t square into x minus t e to power x square minus t square into x minus t now let us find K 3 x t

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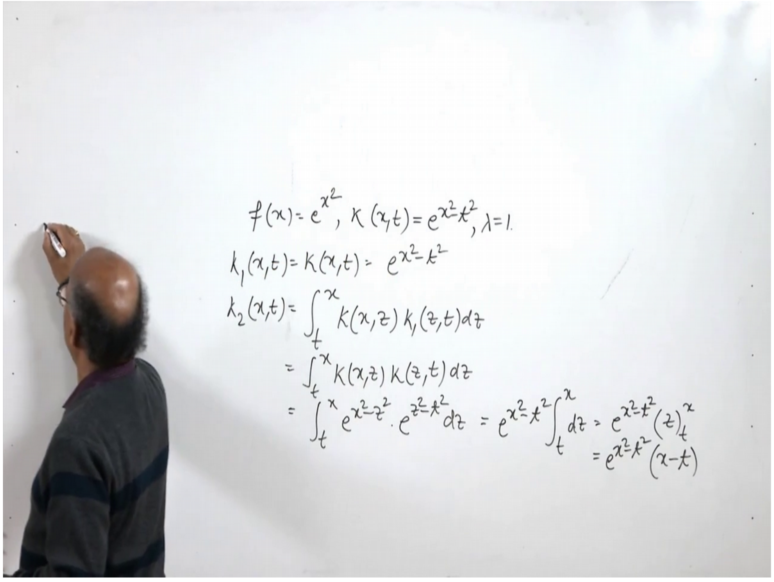

So this is again K x z into K 2 z t d z which is integral t to x e to the power x square minus z square into K 2 z t will be e to the power z square minus t square and then z minus t d z And this is nothing but e to power x square minus t square t 2 x z minus t d z which is e to power x square minus t square z minus t whole square by 2 So this will give us

If we find K 4 x t

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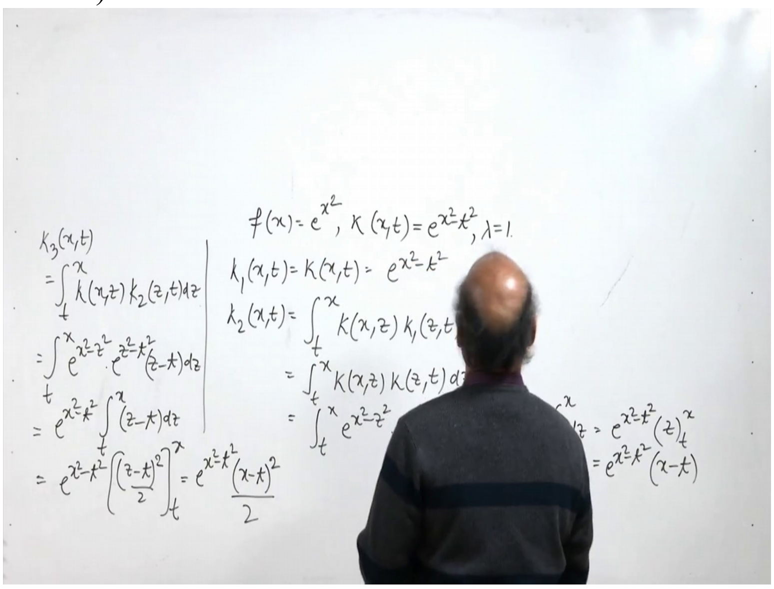

this will be integral t to x K x z into K 3 z t d z which is

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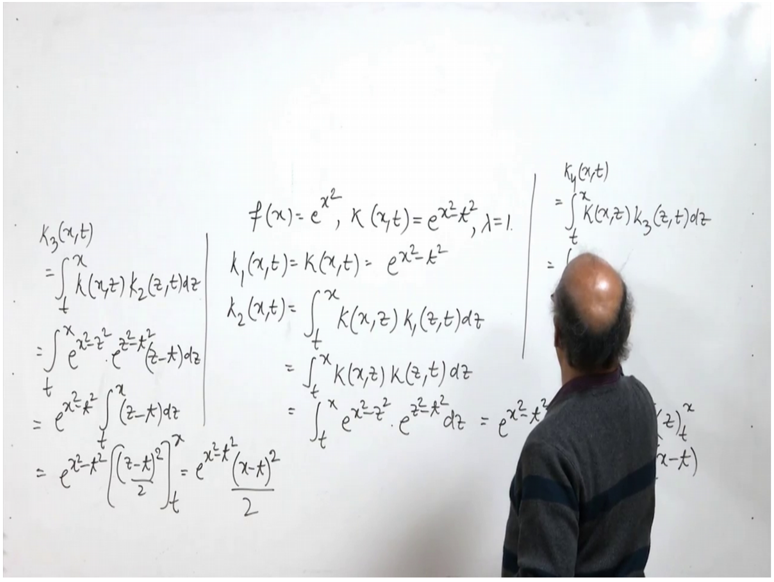

e to power x square minus z square e to power z square minus t square into z minus t whole square by 2 d z And this is e to the power x square minus t square and then we integrate so we get z minus t whole to the power 3 divided by 3 into 2 that is 6 So we get t 2 x here and this is e to the power x square minus t square into x minus t whole to the power 3 divided by 6 which we can write as 3 factorial Here this 2 can

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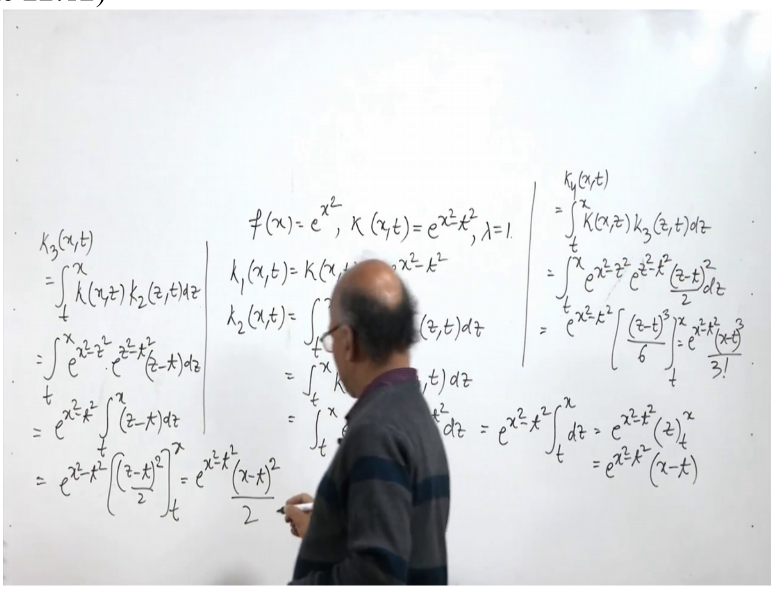

also be written as 2 factorial So generalizing so generalizing this result so by induction we have K n x t K n x t is equal to e to the power x square minus t square x minus t raised to the power n minus 1 over n minus 1 factorial

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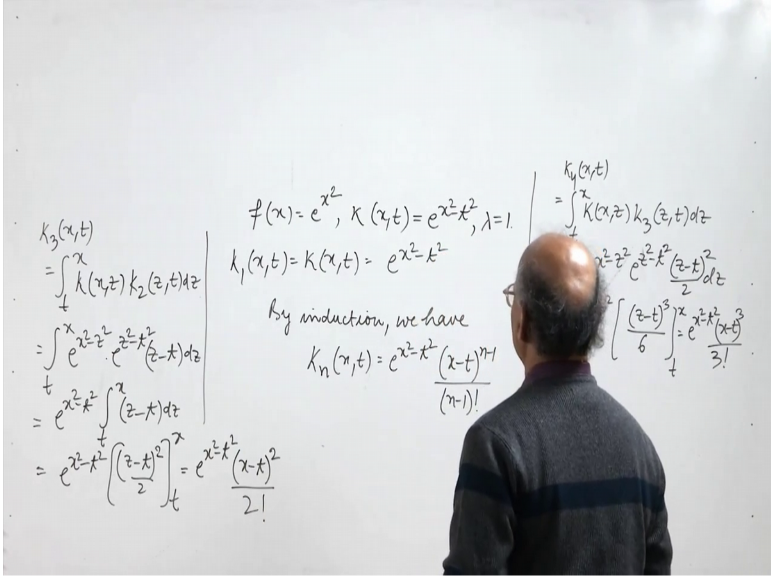

Now let us find the resolvent kernel Having found the iterated kernels this is true for n equal to 1 2 and so on

So having found the resolvent kernel let us now having find the iterated kernel let us find the resolvent kernel now So R x t lambda is equal to sigma n equal to zero to infinity lambda to the power n K n plus 1 x t lambda is given to be equal to 1 so this is summation n equal to zero to infinity lambda is 1 K n plus 1 will be K n plus 1 x t is e to the power x square minus t square x minus t raised to the power n divided by n factorial This is e to the power x square minus t square summation n equal to zero to infinity x minus t

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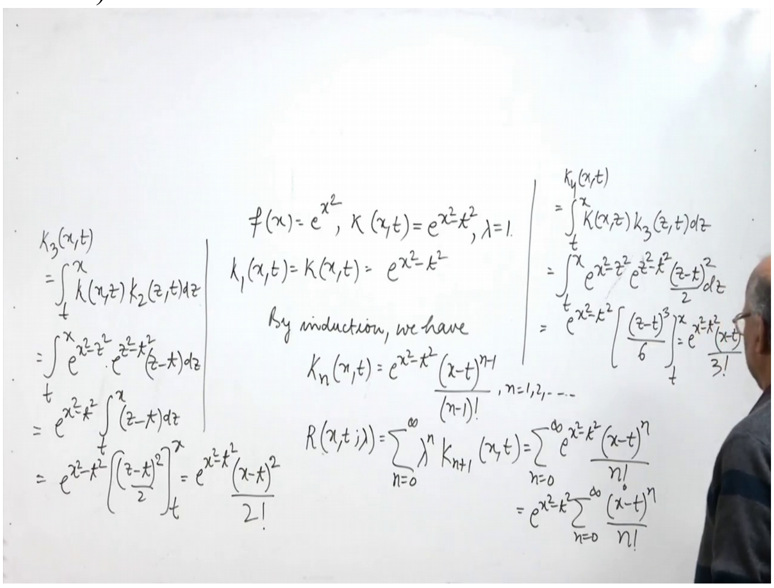

raised to power n over n factorial so this is e to power x square minus t square into e to the power x minus t

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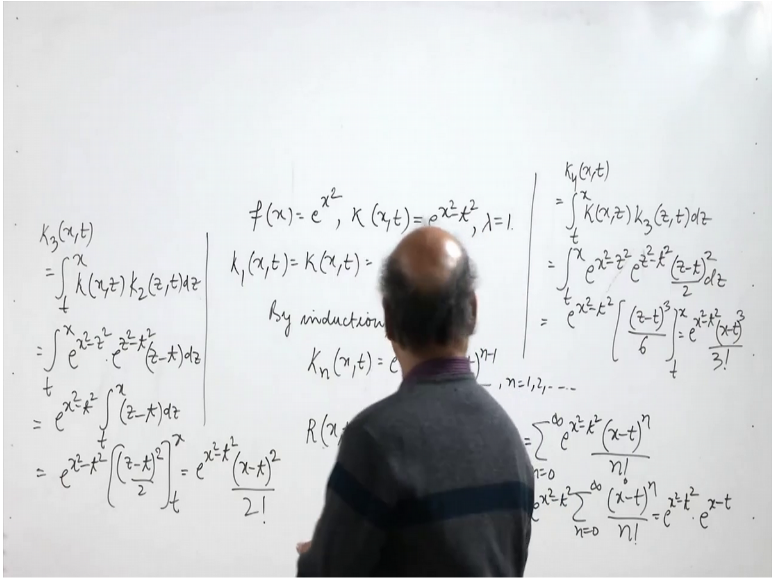

sigma n equal to zero to infinity x to the power n by n factorial is e to the power x So we get the resolvent kernel R x t lambda Now let us find the solution of the Volterra integral equation

So the solution is given by y x equal to f x plus lambda times integral zero to x R x t lambda into f t d t Let's put the value here

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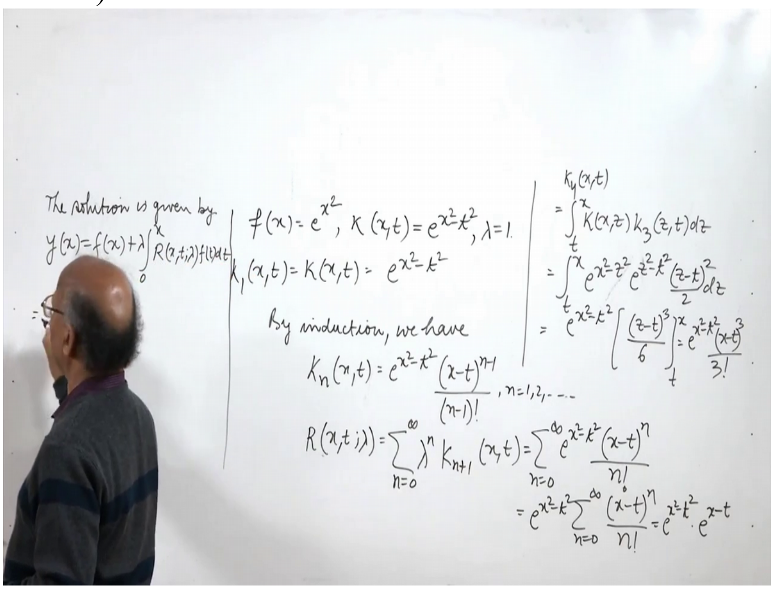

f x is given as e to the power x square plus lambda is 1 so integral zero to x R x t lambda is e to the power x square minus t square into e to the power x minus t into f t is e to the power t square

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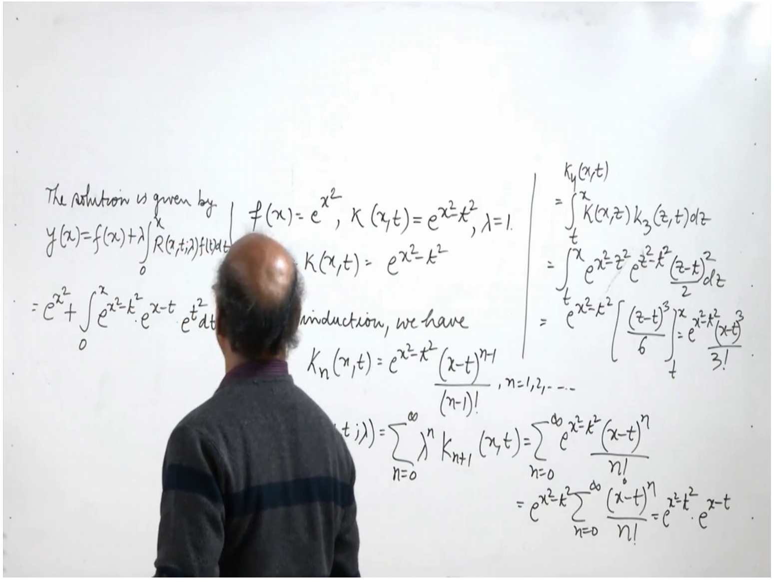

d t So we shall have this e to the power t square into e to power x square minus t square will become e to the power x square

We can write it outside and then we have e to the power x minus t so e to power x can also be written outside we have e to power x square plus x integral zero to x e to the power minus t d t we are left with

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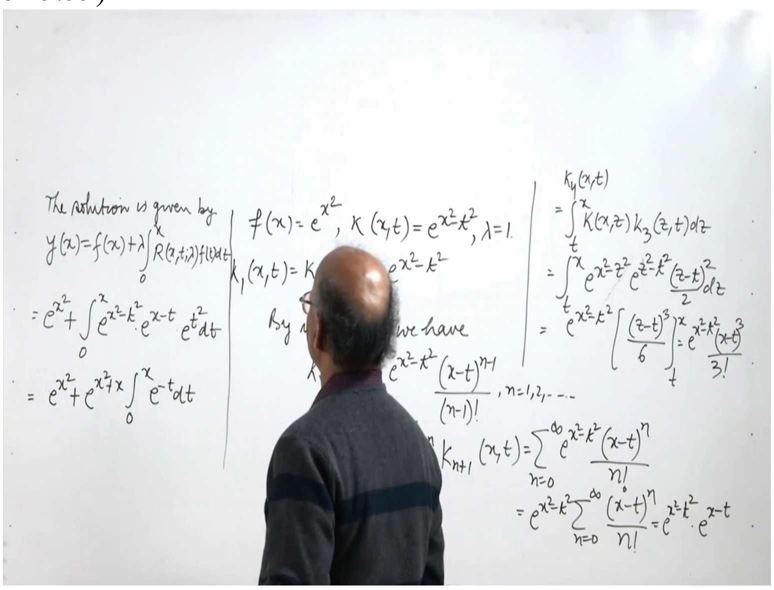

And this gives you e to the power x square e to the power x square plus x minus e to the power minus t zero to x This is e to the power x square this is 1 minus e to the power minus x And what we get is e to the power x square plus e to the power x square plus x minus e to the power x square So this cancels with this and we get e to the power x square plus x

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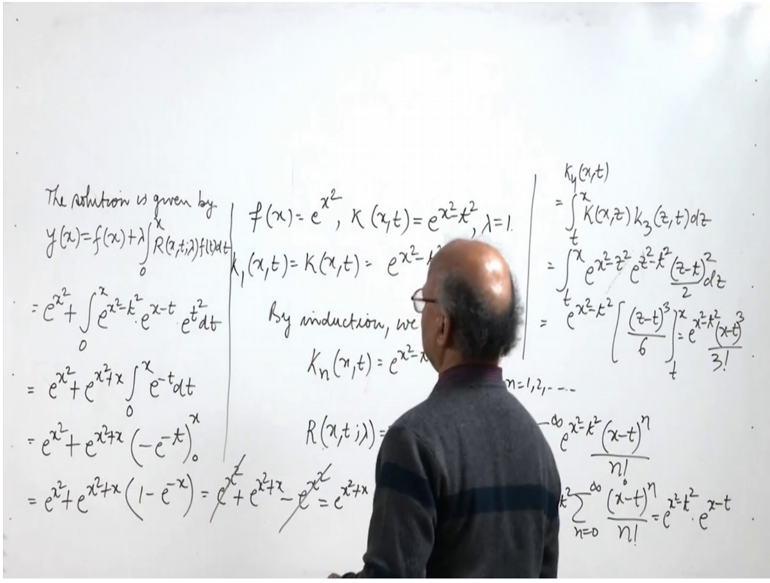

So we get the solution of the given Volterra integral equation of the second kind as

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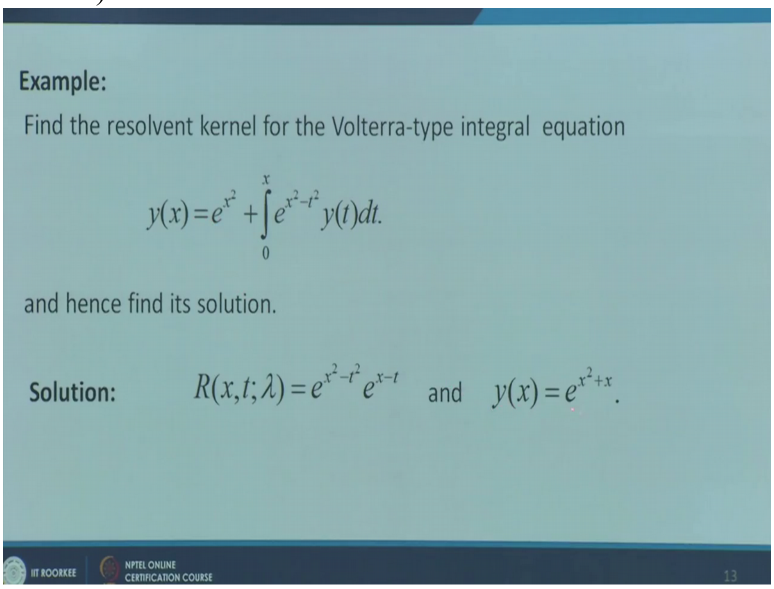

y x equal to e to the power x square plus x So this is how we find the solution of the given Volterra integral equation of the second kind by first finding the resolvent kernel and then substituting the value of the resolvent kernel in this equation

So for resolvent kernel we need to find the iterated kernels We begin with the first kernel iterated kernel which is K 1 x t K 1 x t is the same as K x t Then we find K 2 x t and K 3 x t and from their values we will be able to see what will be K n x t like So by induction process we then write the general value of K n x t and then that value of K n x t is put in the formula for R x t lambda to get the value of the resolvent kernel So this is how we solve this problem This is what I have to discuss in this lecture Thank you very much for your attention