Integral Equations, Calculus of Variations and their Applications
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Mod 06 Lecture Number 24
Method of successive approximations

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Hello friends I welcome you to my lecture on the method of successive approximation Let us consider a Volterra integral equation of the second

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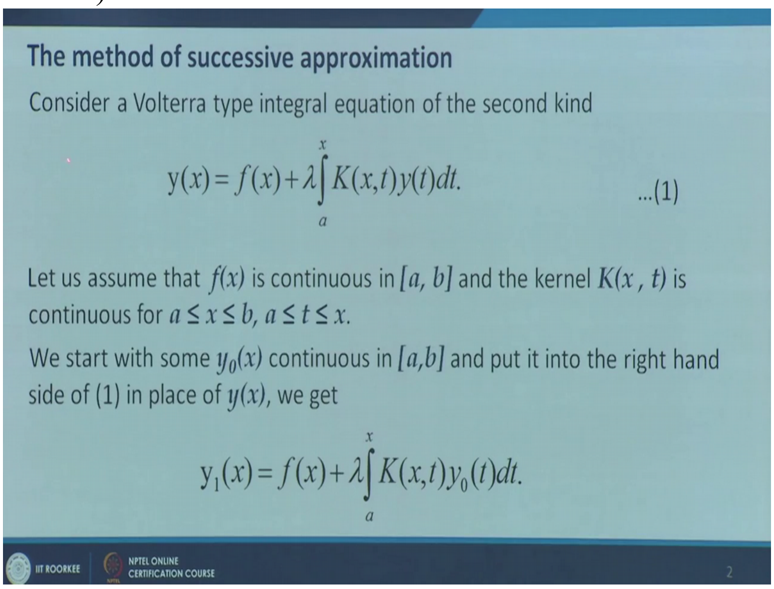

kind y x equal to f x plus lambda times integral a to x K x t y t d t and let us assume the conditions that f x is continuous on the closed interval a to b and the kernel K x t is continuous for a less than or equal to x less than or equal to b and a less than or equal to t or less than or equal to x in that region Now what we will do here is that we shall start with some y naught x initial approximation to the solution y x of the given differential equation

So we shall start with some y naught x as the initial approximation which we will take as continuous function in a b and put into the right hand side of the equation 1 in place of y t In place of y x when we put this then we will get y 1 x equal to f x the first approximation to the solution y x we get as y 1 x is equal to f x plus lambda times a 2 x K x t into y naught t d t Now after we have found y 1 x here we then put y 1 x again in the equation 1 to get the next approximation that is y 2 x

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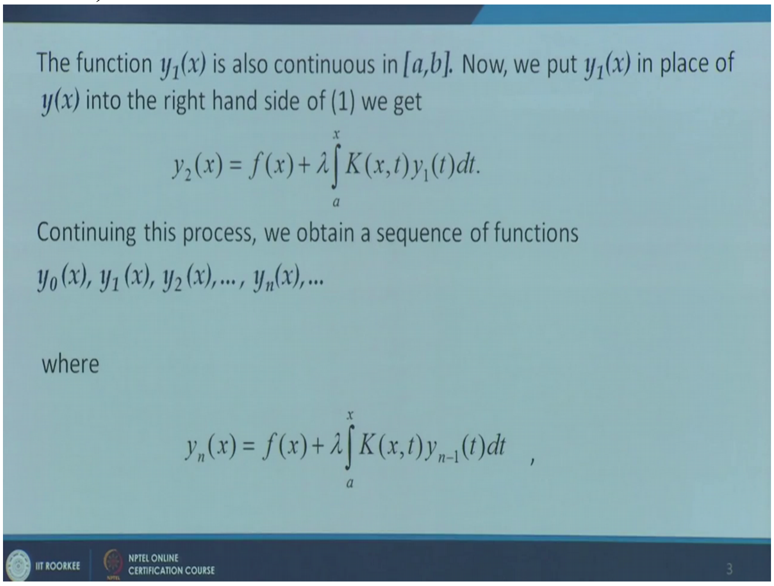

So since y 1 x is continuous in the closed interval a b we can put y 1 x in place of y x in the right hand side of equation 1 and obtain the second approximation y 2 x to the solution y x of the given integral equation So y 2 x is f x plus lambda times integral a to x K x t into y 1 t d t Now we can continue this process and then continuing this process we shall obtain a sequence of functions y naught x y 1 x y 2 x y n x and so on There the expression for y n x will be given by y n x equal to f x plus lambda times integral a to x K x t into y n minus t d t

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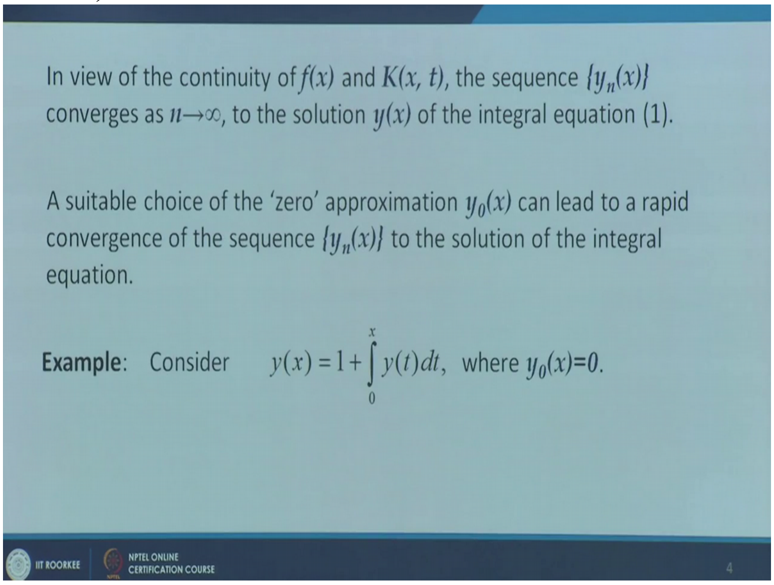

Since we have assumed K x t to be continuous f x to be continuous and K x t to be continuous in the region a less than equal to x less than or equal to b and a less than or equal to t less than or equal to x we it tells out that the sequence y n x converges as n goes to infinity to the solution y x of the integral equation given integral equation

Now a suitable choice of the initial approximation which we call as zero approximation y naught x can lead to a rapid convergence of the sequence So if we suitably choose the zero approximation y naught x the we can reach the solution of the integral equation faster So that is very essential how to choose the zero approximation y naught x Now let me show you how we apply this method to a given integral equation y x is equal to let us consider y x is equal to one plus integral zero to x y t d t and let us assume y naught x to be the equal to zero that is we are assuming zero solution to the given integral equation So what we will do to obtain the first approximation

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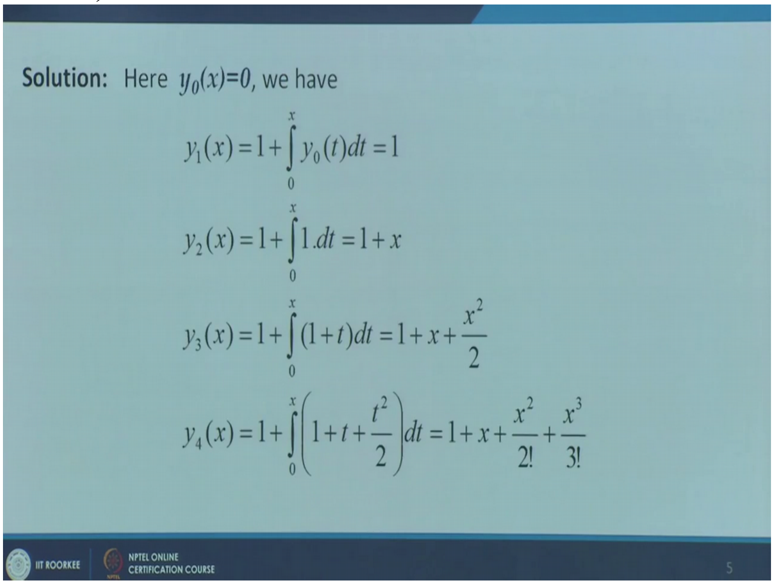

y 1 x let us put y naught x equal to zero in the right hand side of the

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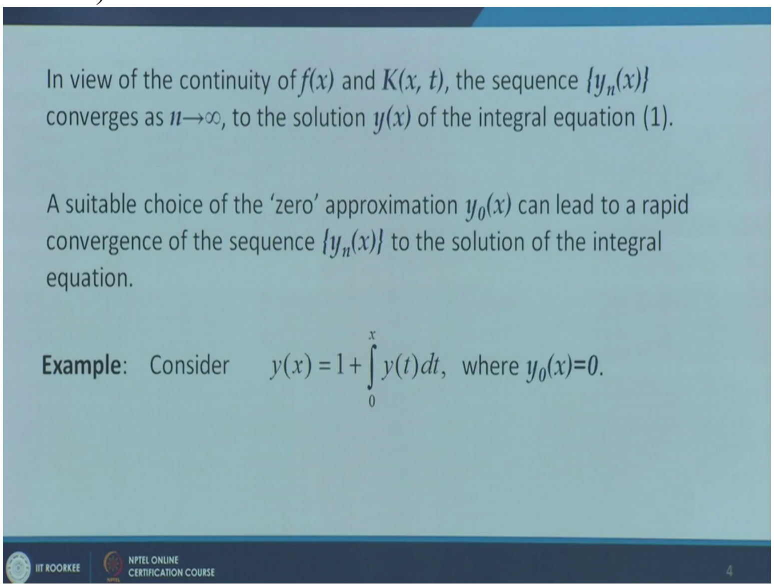

given integral equation So here we put y naught t So y 1 x will be 1 plus integral zero to x y 1 t d t and y naught t d t y naught t d t is equal to zero so we will get y 1 x

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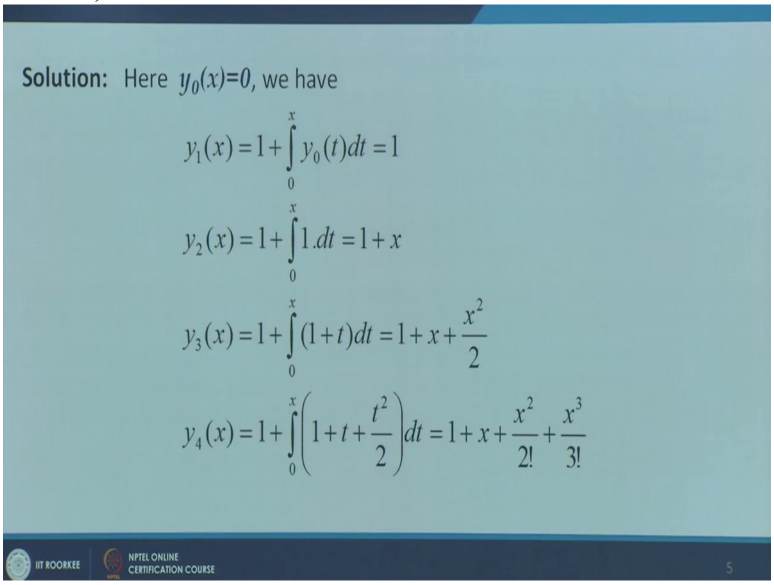

equal to 1 And then we put y 1 x in the right hand side of the given integral equation to obtain y 2 x y 2 x is then 1 to integral zero to x 1 d t so this is 1 plus x Then y 2 x is put in the right hand side of the given differential equation to obtain y 3 x and y 3 x is 1 plus x plus x square by 2 which is again put on the right hand side of the given integral equation to obtain y 4 x y 4 x is then 1 plus x We can write x square by 2 as x square by 2 factorial and we also have the x cube by 3 factorial

So this way when we carry on

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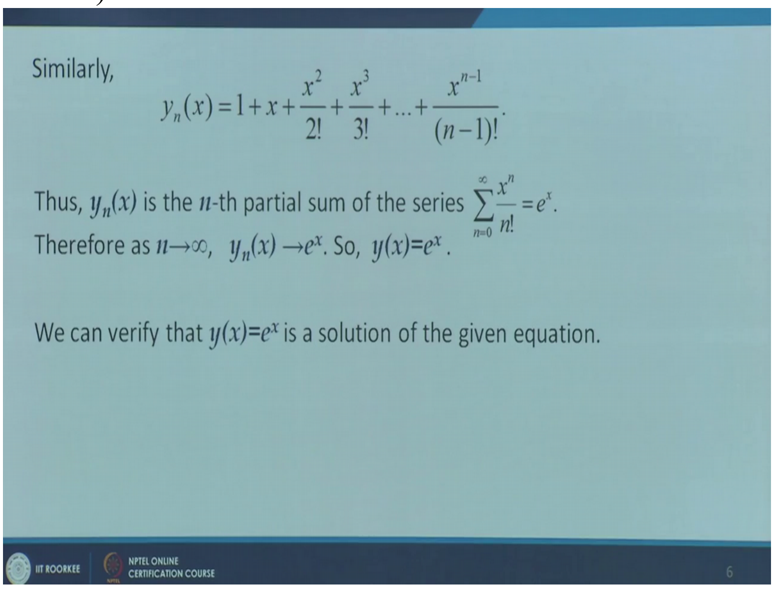

so similarly y n x is equal to 1 plus x plus x square by 2 factorial plus x cube by 3 factorial and so on x to the power n minus 1 over n minus 1 factorial Thus we notice that y n x is the n th partial sum of the Maclaurin series for e to the power x The Maclaurin series for e to the power x is given by sigma n equal to zero to infinity x to the power n by n factorial And hence as n goes to infinity y n x the n th partial sum converges to the sum of the series that is e to the power x And so we can say that y x is equal to e to the power x And we can also verify that y x is equal to e to power x is a solution of the given integral equation You can see that

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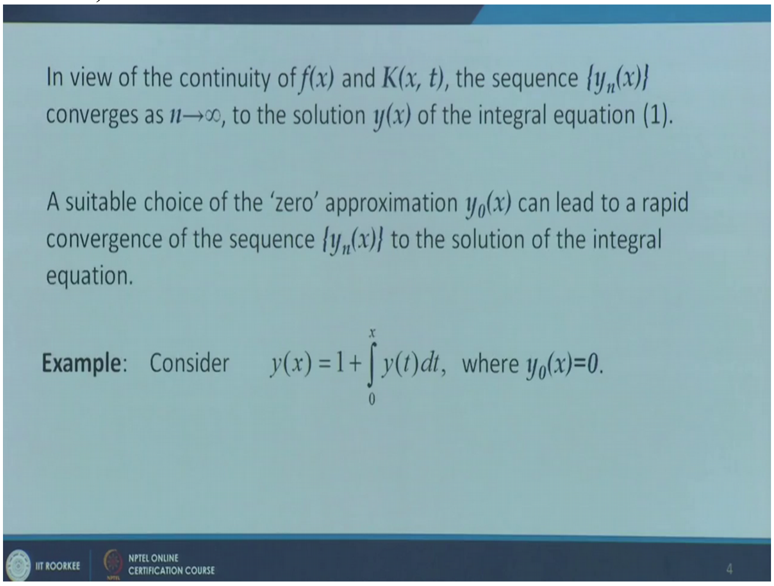

so here you put y x let us see We have y x equal to 1 plus integral 0 to x y t d t

Let us verify that y x is equal to e to power x is a solution of this So now let's take

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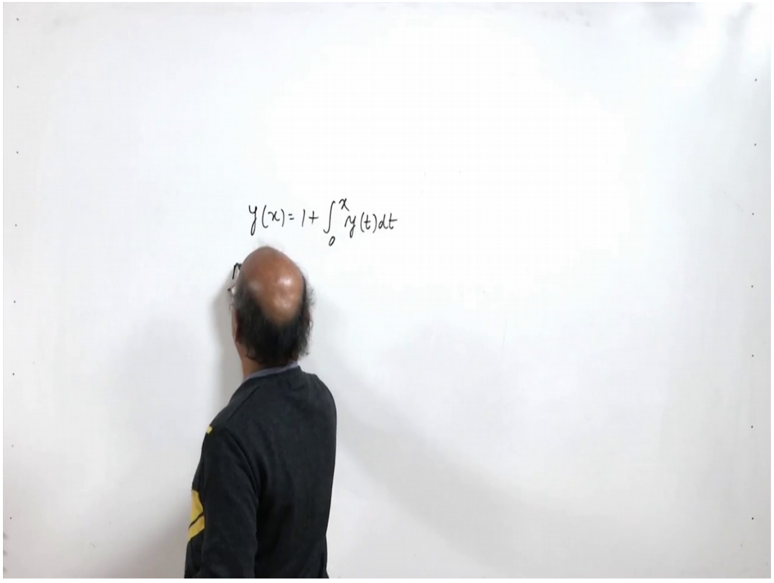

the right hand side This is 1 plus integral zero to x Let us put y t as e to the power e to the power t here So we get 1 plus e to the power t zero to x which is 1 plus e to the power x minus 1 So what we get is e to power x So left hand side in the left hand side y x is e to the power x so the left hand side and right hand side are both equal Ok

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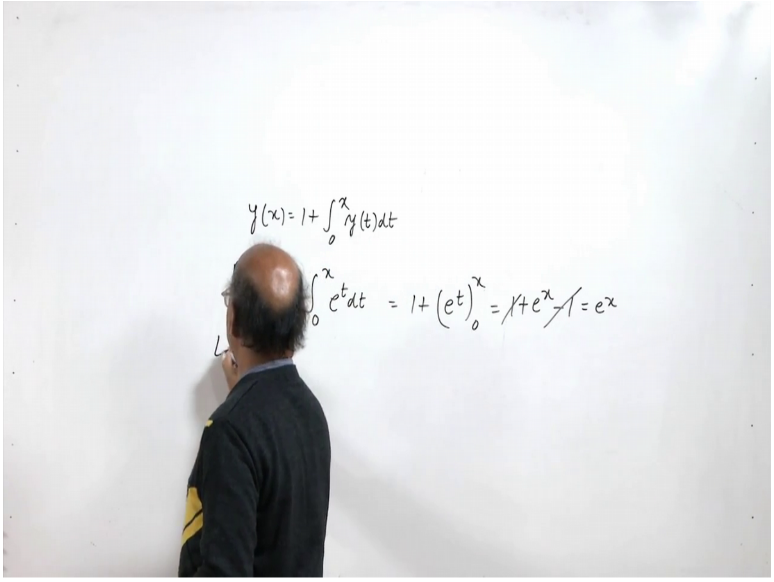

L H S is equal to y x equal to e to power x so thus L H S is equal to R H S when y x is equal to e to the power x So e to the power x is the solution of the integral equation

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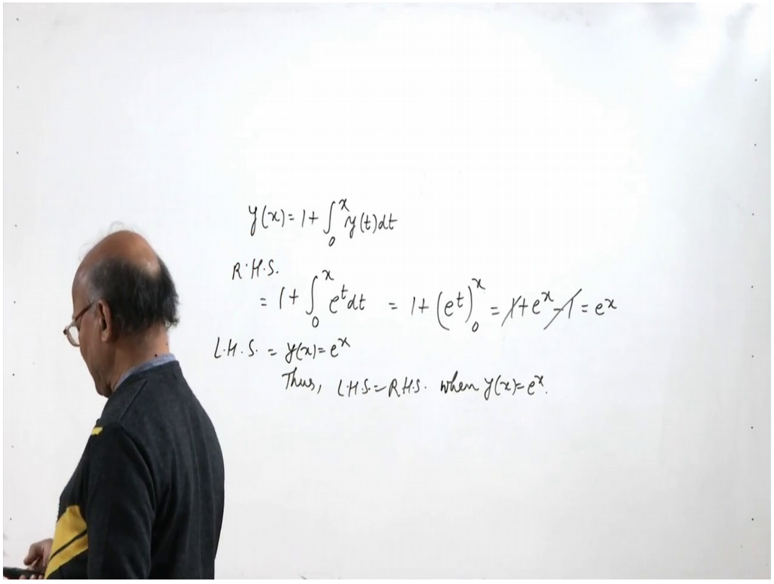

y x plus y x equal to 1 plus integral zero to x y t d t

Now let's take one more example So we have y x equal to 1 plus integral

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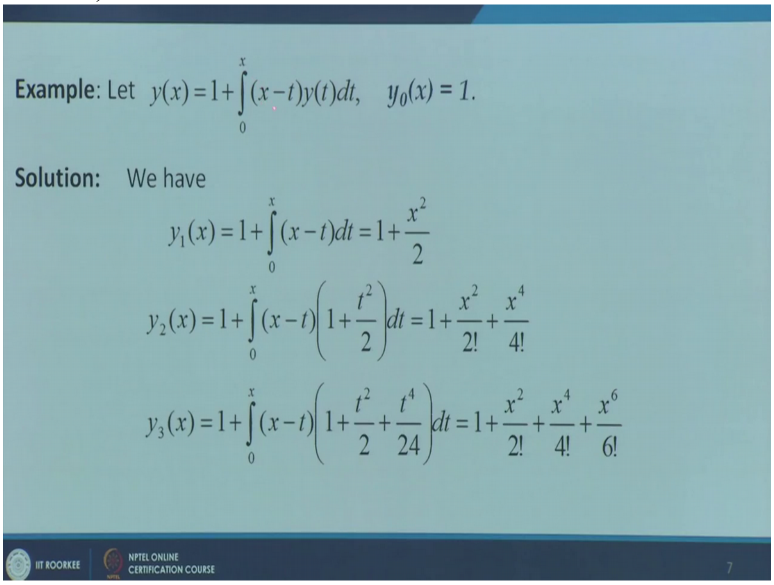

0 to x x minus t y t d t where we are given the zero approximation y naught x as equal to 1 So for all values of x over the range of values of x we are assuming y naught to be equal to 1 So here y 1 x the first approximation y 1 x is given by 1 plus integral zero to x y naught t d t and y naught t will be equal to x minus t into y naught t d t y naught t is equal to 1 so we get after integration we get 1 plus x square by 2 And then we put y 1 x in the given integral equation For y t we put y 1 t in the right side to get y 2 x y 2 x is 1 plus integral 0 to x x minus t into 1 plus t square by 2 d t So after integration and simplification we get 1 plus x square by 2 factorial plus x 4 by 4 factorial We can again continue this and we can find y 3 x by putting y 2 t in place of y t here So y 3 x will come out to be 1 plus x square by 2 factorial plus x 4 by 4 factorial plus x 6 by 6 factorial

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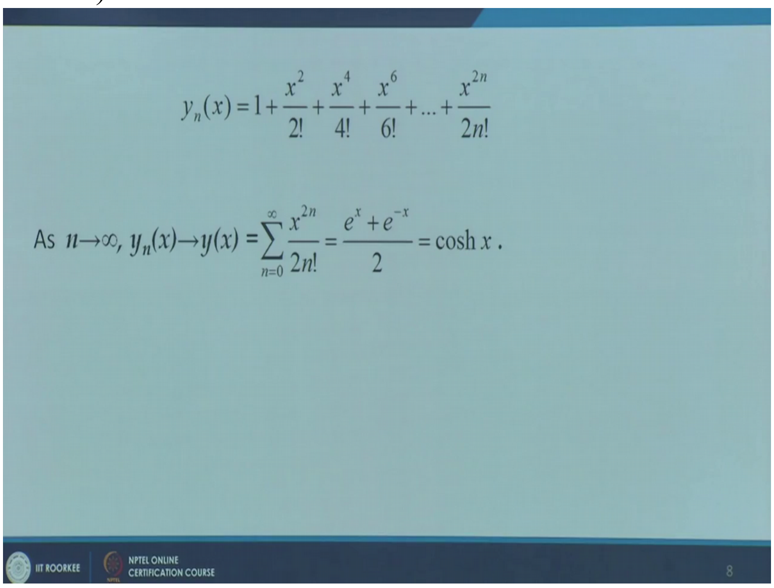

And generalizing we get y n x as 1 plus x square by 2 factorial x 4 by 4 factorial plus x 6 by 6 factorial and so on x to the power 2 n by 2 n factorial As n goes to infinity y n x goes to y x which is sigma n equal to zero to infinity x to the power 2 n by 2 n factorial And it is known that this is nothing but e to the power x plus e to the power minus x by 2 and we know that e to the power x plus e to the power minus x by 2 is the cos hyperbolic x So the solution of the integral equation

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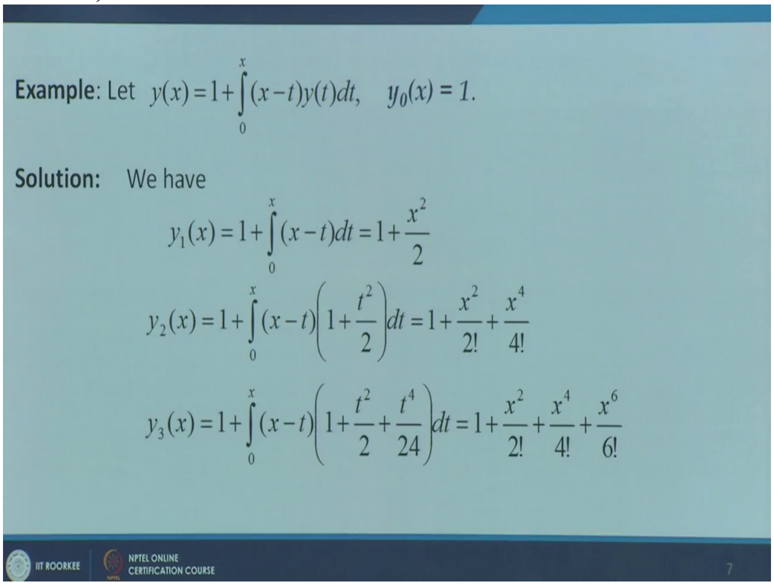

given integral equation is y x equal to cos hyperbolic x

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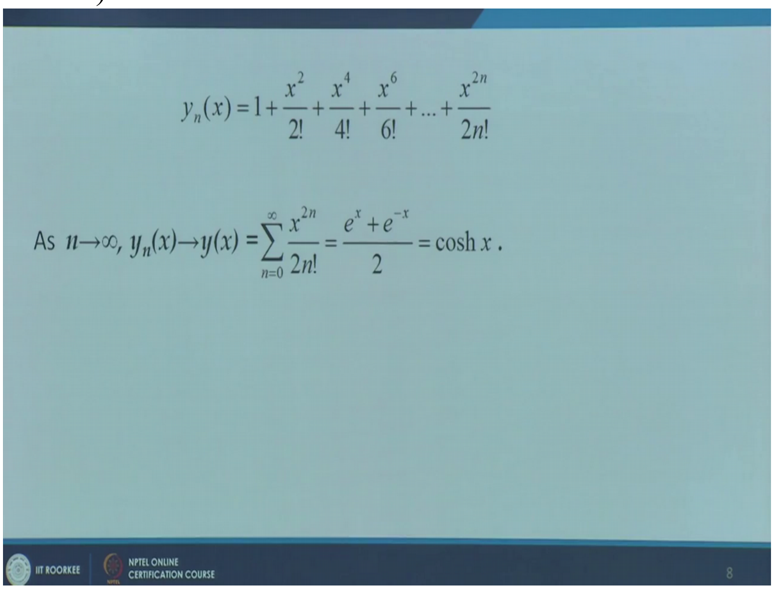

So this is how we apply this method

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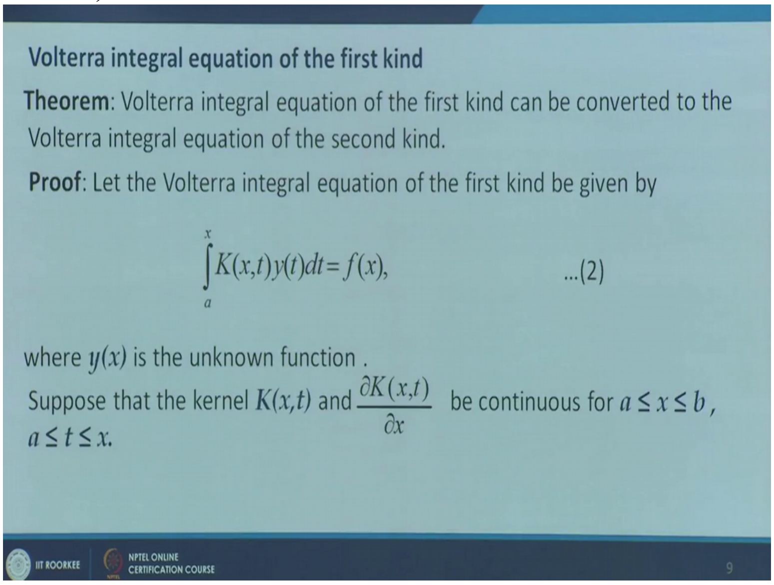

Now let us consider Volterra integral equation of the first kind

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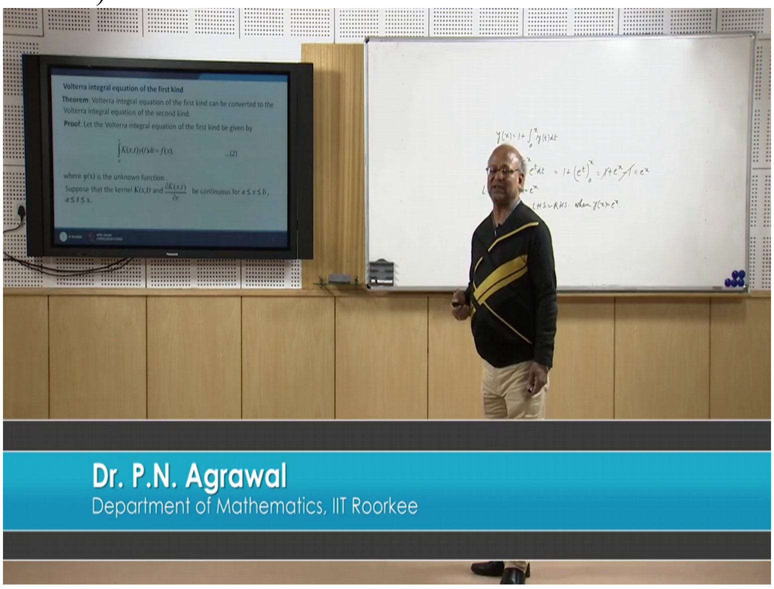

We shall see that Volterra integral equation of the first kind can be converted into Volterra integral equation of the second kind and we can obtain the solution there And that solution will also satisfy Volterra integral equation of the first kind So Volterra integral equation of the

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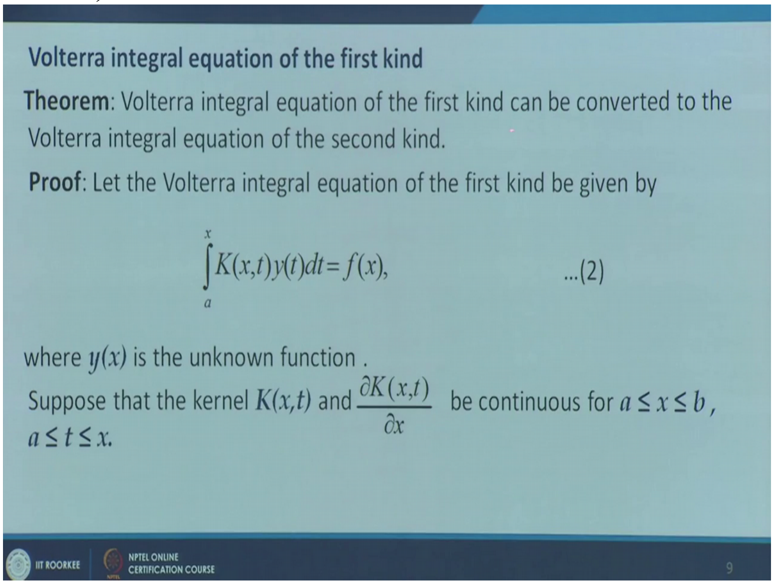

first kind can be written as integral a to x K x t y t d t equal to f x where y x is the unknown function Let us assume that the kernel K x t and its partial derivative with respect to x are continuous in the range in the region g a less than or equal to x less than or equal to b and a less than or equal to t less than or equal to x

Then

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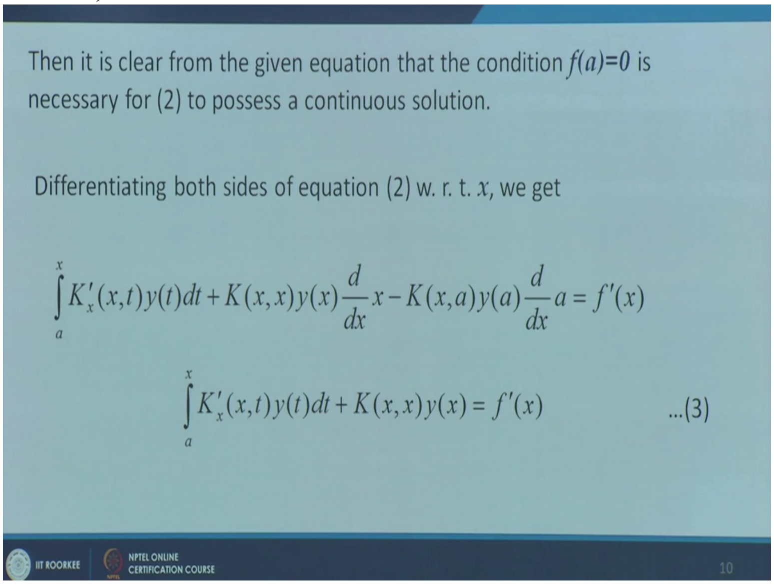

from the given first order

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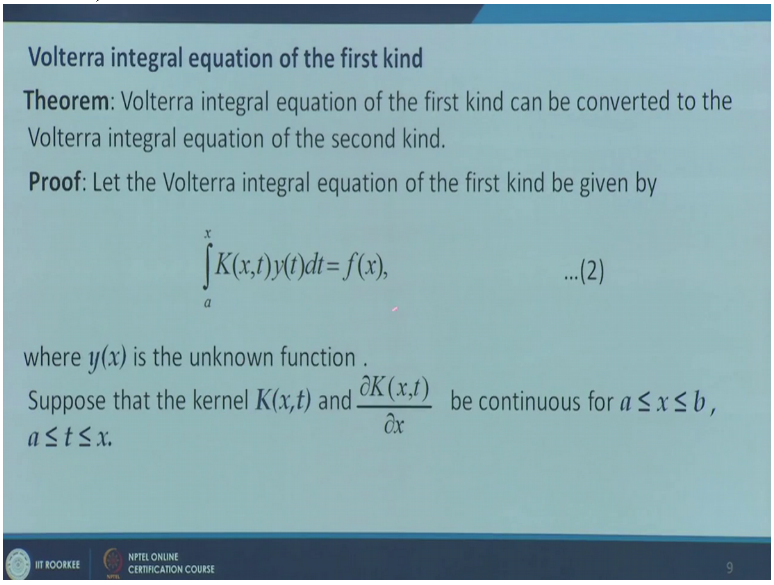

Volterra integral equation of the first kind it is clear that when x is equal to a f a must be equal to zero

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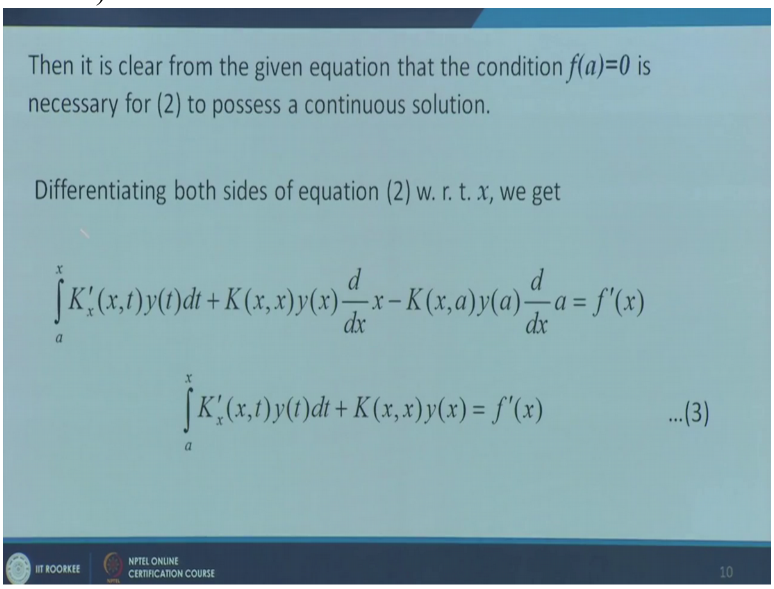

So it is clear from the given equation that the condition f a is equal to zero is necessary for the

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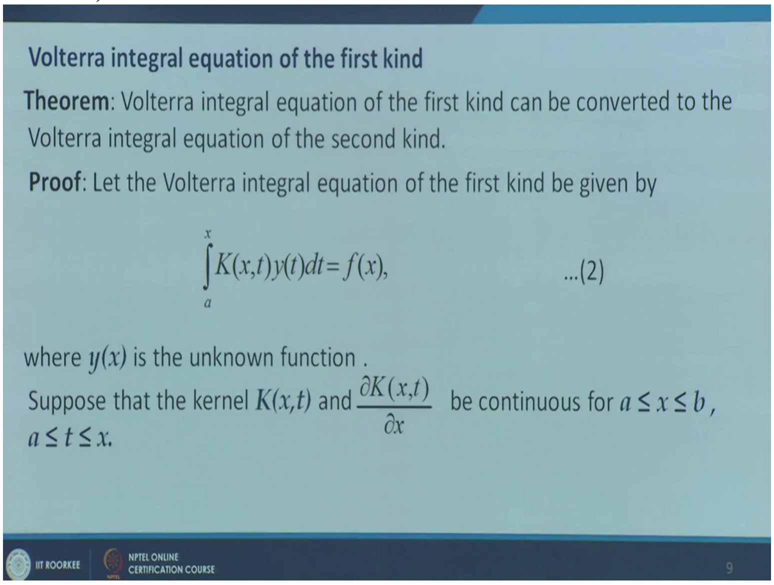

equation to possess a continuous solution

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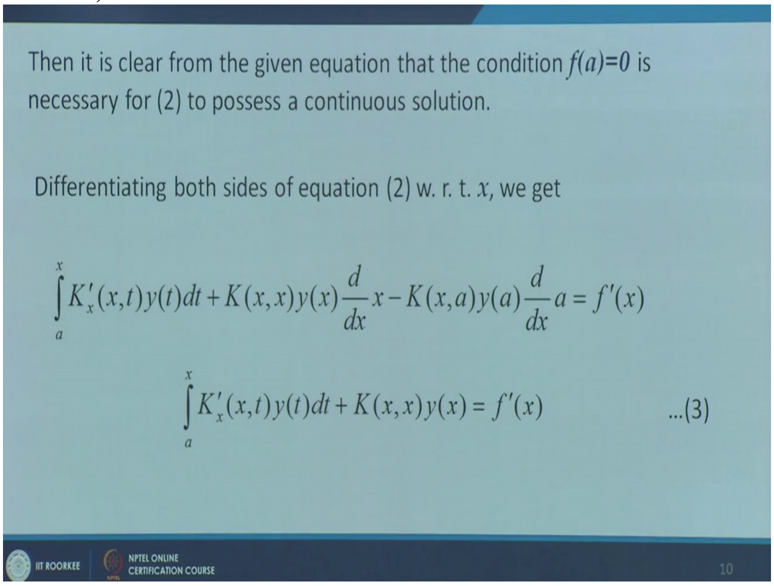

Now differentiating both sides of this equation 2 what we get is using the Leibniz rule for differentiation under sign of integration we will get integral a to x derivative of K x t with respect to x into y t d t then we put t in place of t we put x the upper limit So we get K x x into y x and then d over d x of x And then in the minus in the integrand we put t equal to a the lower limit So we get K x a into y a and then derivative of a with respect to x And right hand side will become f prime x Now since a is a constant d a by d x will be zero

So this equation will reduce to integral a to x K x dash x t into y t d t plus K x x into y x equal to f dash x because derivative of x with respect to x is 1 So now

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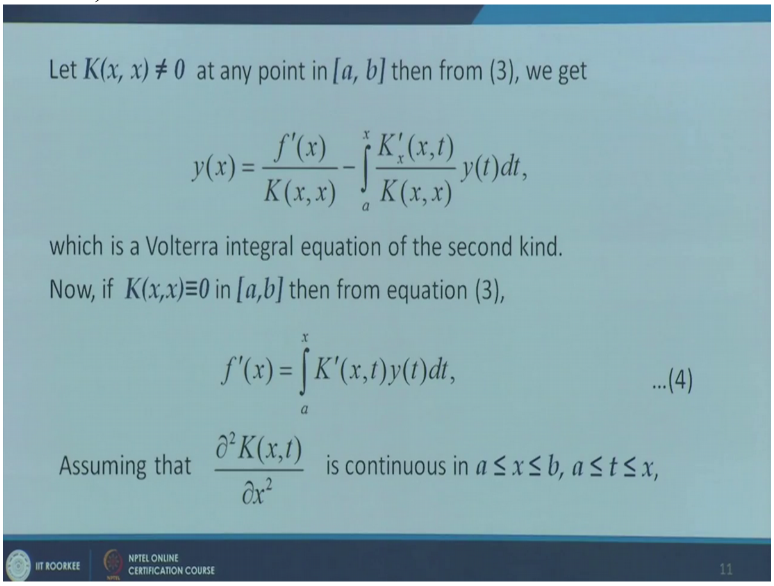

so let us assume that K x x does not vanish at any point in the interval a b then from equation 3 then from equation 3

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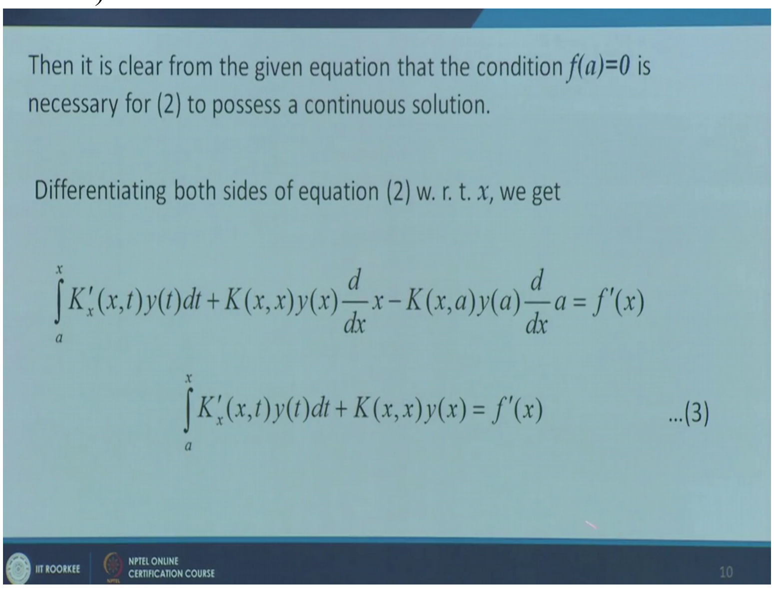

by on dividing by K x x both the sides we will have

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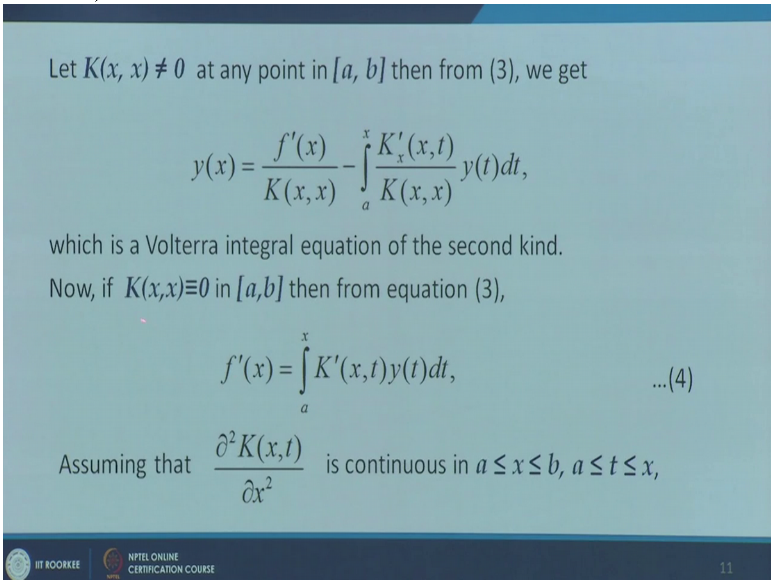

y x is equal to f dash x over K x x minus integral a to x K x dash x t over K x x into y t d t We are you can remember that K dash x t is the partial derivative of K x t with respect to x And it is a since K x t function is known f x function is known so f dash x over K x x is known and K dash x K x dash x t over K x x is also known So it is a Volterra integral equation of the second kind where we can take lambda to be minus 1 Now if it happens that K x x is zero in the interval a b then this equation this equation

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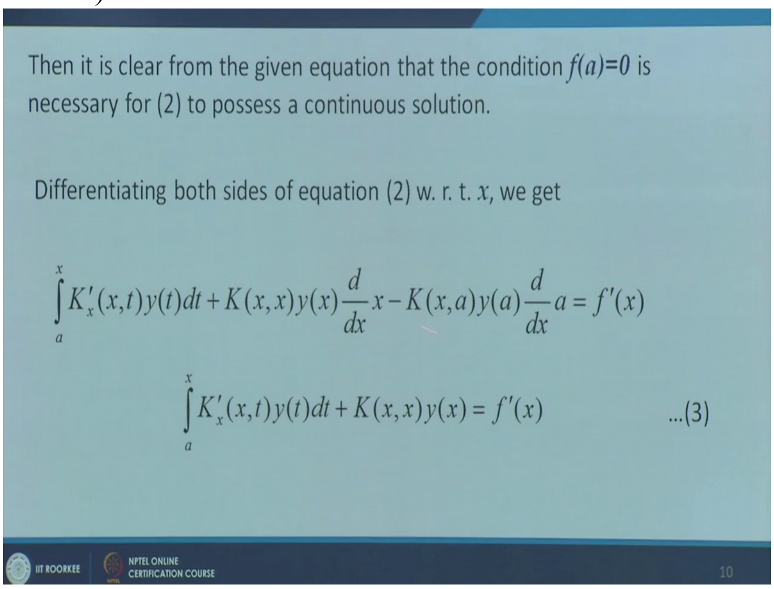

will reduce to integral a to x K x dash x t into y t d t equal to f dash x So if K x x is zero for all values of x in the interval a b then we again get the Volterra integral equation of the first kind

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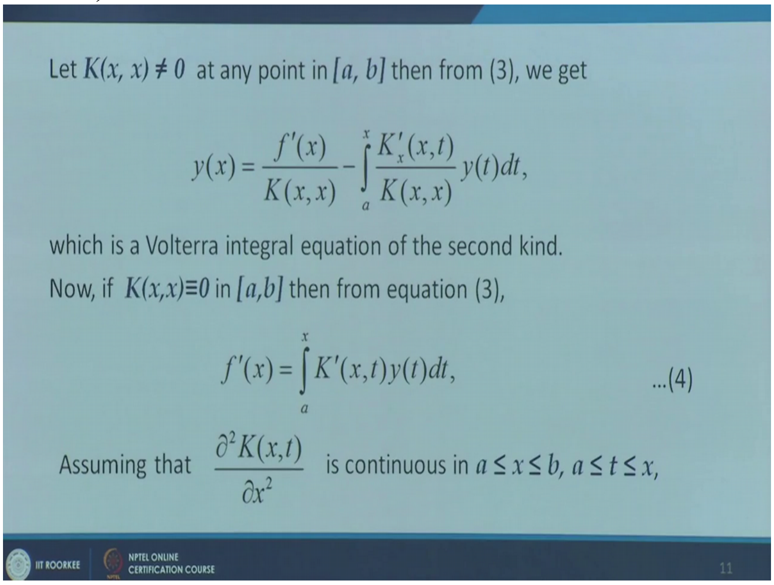

So what we will do is now let us assume that this second derivative of K x t with respect to x is continuous in the domain a less than equal to x less than or equal to b a less than equal to t less than or equal to x then

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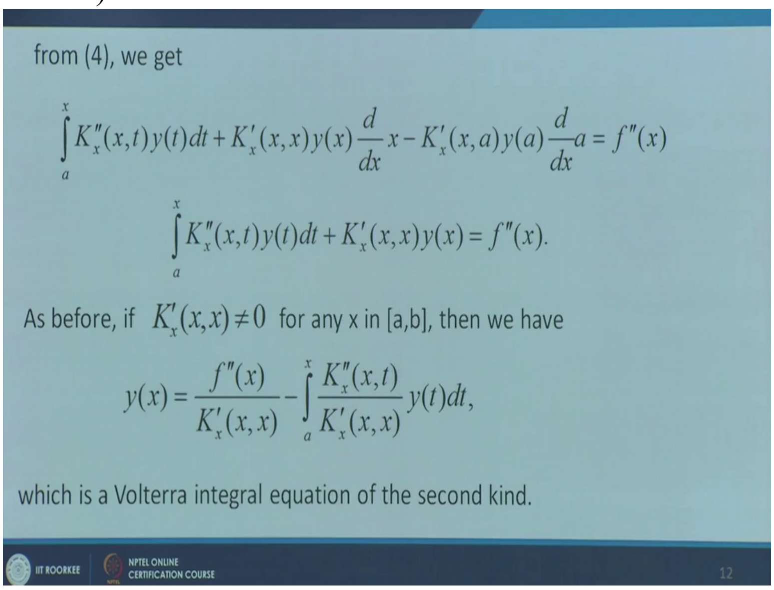

we differentiate again with respect to x while using Leibniz rule so we shall have integral a to x second order partial derivative of K x t with respect to x into y t d t plus K x dash x x into y x d x by d x minus K x dash x a y a into d a by d x equal to f double dash x Now a is a constant So d a by d x will be zero and for this equation will reduce to integral a to x K x double dash x t y t d t plus K x dash x x y x equal to f double dash x Now as before if the first order partial derivative of K x t with respect to x at t equal to x is non-zero for any x in the closed interval a b then we can divide this equation by K x dash x x and arrive at Volterra integral equation of the second kind y x equal to f double dash x over K x dash x x minus integral a to x K x double dash x t over K dash x x K x dash x x into y t d t

And we can solve it by the by the methods of solving Volterra integral equation of the second kind Now

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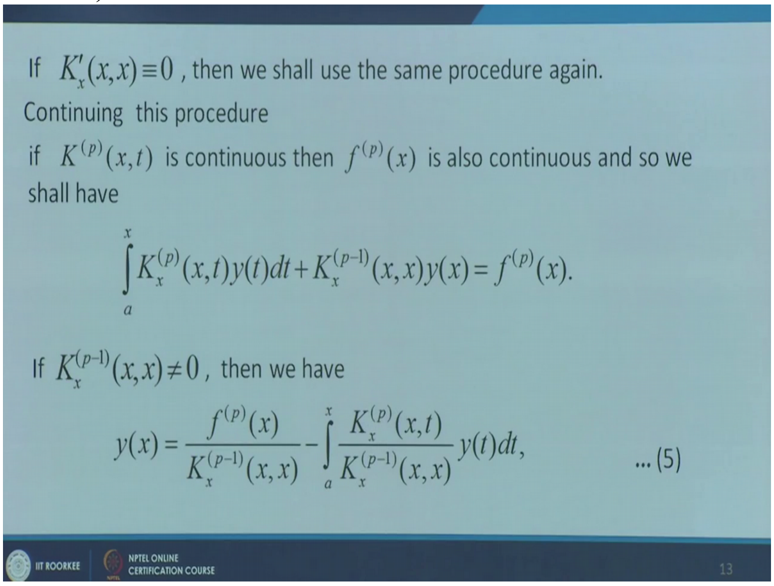

if again if K x dash x x is identically zero Then we shall use the same procedure So if it so happens that we have to continue this procedure and the p th derivative of K x t is continuous the pth derivative of K x t with respect to x is continuous in the domain a less than equal to x less than equal to b a less than equal to t a less than equal to x then f p x is also continuous and so we shall have integral a to x K p x t y t d t plus K x p minus 1 x x y x equal to f p x Now when K p K x p minus 1 x x is not zero for any x in the closed interval a b we shall dividing this equation by K x p minus 1 x x And arrive at the Volterra integral equation of the second kind

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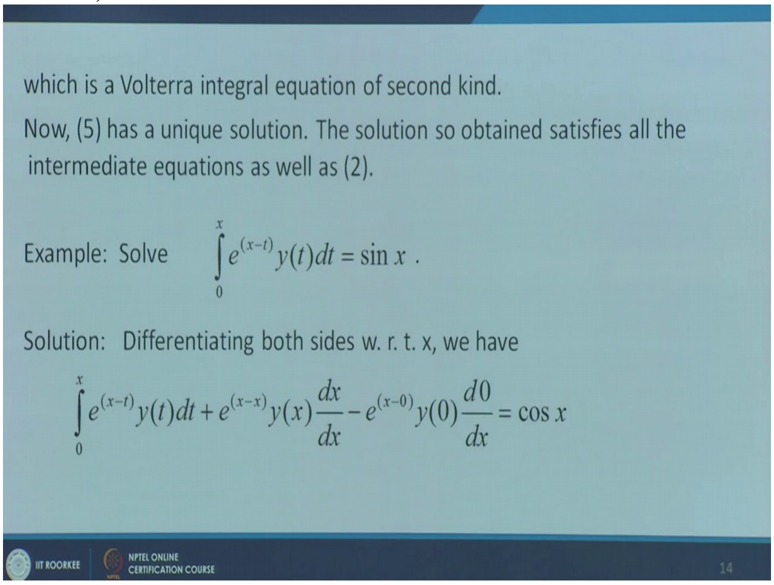

So this equation has

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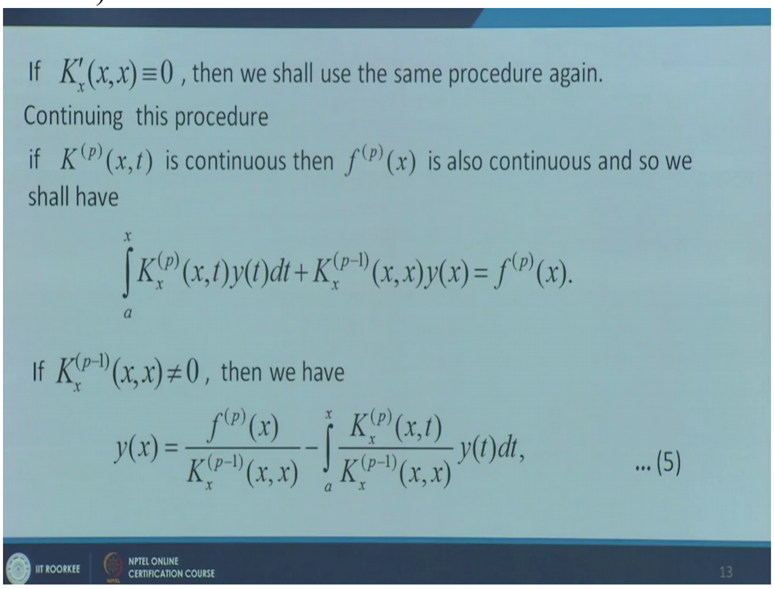

a unique solution equation 5 and

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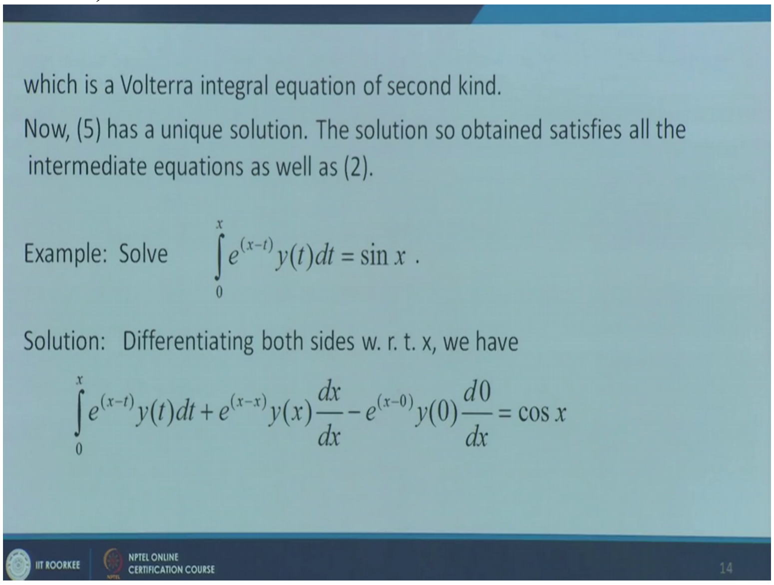

the solution so obtained satisfies all the intermediate equations which we have got as well as the given integral equation that is equation number 2 Now so this is how we apply use this method So let us see how we can solve Volterra integral equation of the first kind Suppose we have this Volterra integral equation of the first kind

So we have y integral zero to x e to the power x minus t into y t d t equal to sin x we can see here that f x is equal to sin x and K x t is equal to e to the power x minus t So f x is a continuous function and K x t is also a continuous function Now let us differentiate this equation with respect

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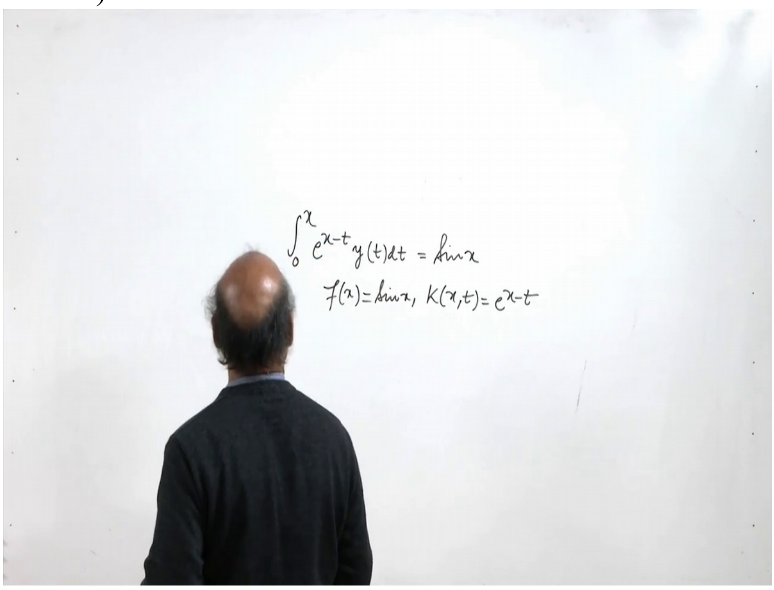

to x by using Leibniz rule So let us recall the Leibniz rule d over d x of integral G x to H x f x t d t let us say capital F x t where we can take this is integral G x to H x partial derivative plus F x x and then instead of x we have H x minus F So this is the formula So let's apply

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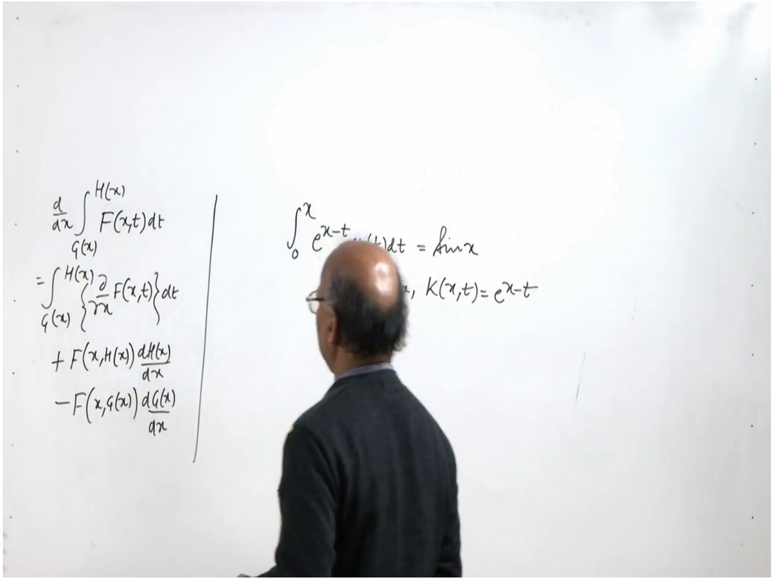

here so when we differentiate with respect to x we have which is cos x Ok Now let's apply Leibniz rule So this is integral zero to x we put the upper limit we get this So this gives you zero to x partial derivative with respect to x gives e to the power x minus t y t d t And here we get y x equal to cos x And which can written as

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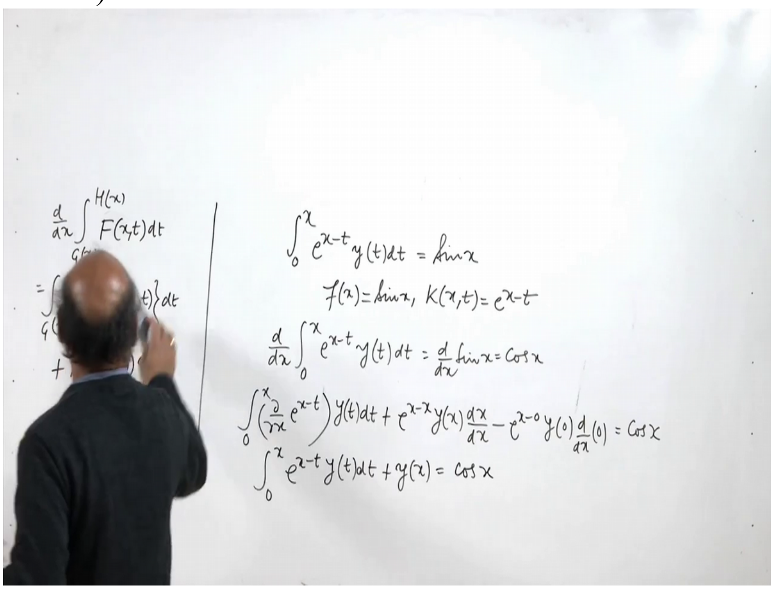

so you can see that the coefficient of y x is 1 which is not zero Of so we can write it as y x equal to cos x minus integral zero to x e to the power x minus t into y t d t which is a Volterra integral

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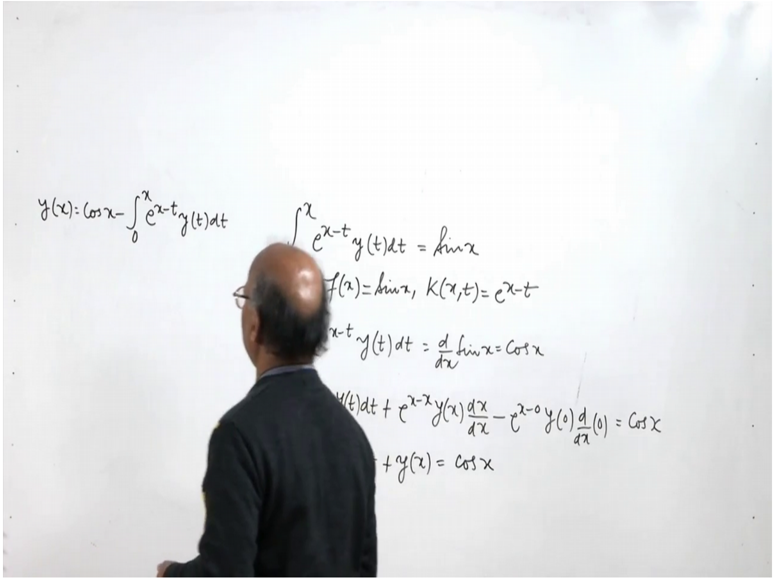

equation of the second kind

Now we shall solve it by the method of finding the resolvent kernel So here we notice that here f x is equal to cos x lambda is equal to minus 1 and K x t is equal to e to the power x minus t So let us find the iterated kernels

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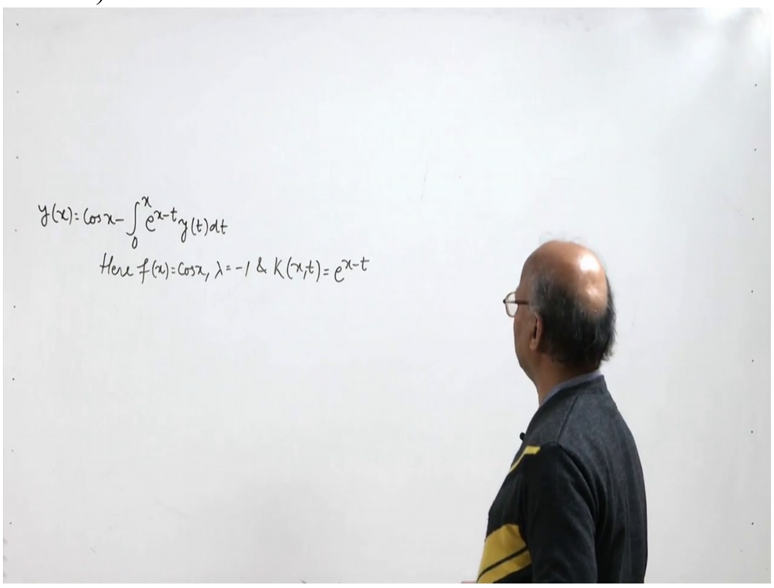

first So we have K 1 x t is equal to K x t is equal to e to the power x minus t then K 2 x t is equal to integral t to x K x z into K 1 z t d z Ok So this is equal to integral t 2 x K x z is e to the power x minus z and then K 1 z t is same as K z t but K z t will be equal to e to the power z minus t d z e to the power x minus z into e to the power

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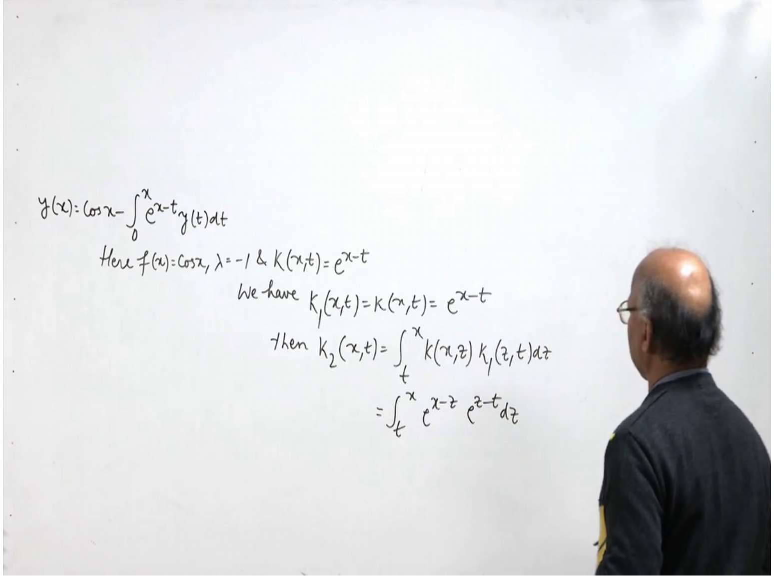

z minus t is e to power x minus t So integral t to x Ok now e to power x minus

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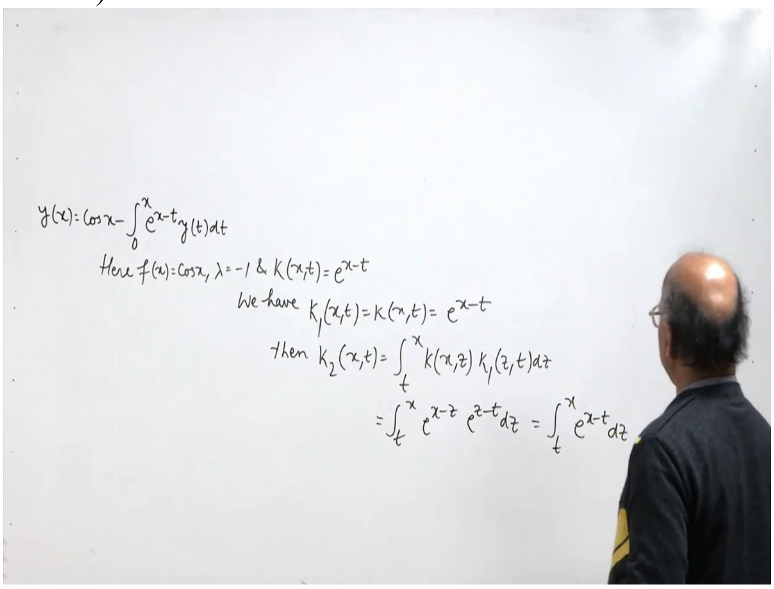

t is independent of z so I can write it outside and then we integrate and find the value of K 2 x t as e to the power x minus t into x minus t

Now let's find

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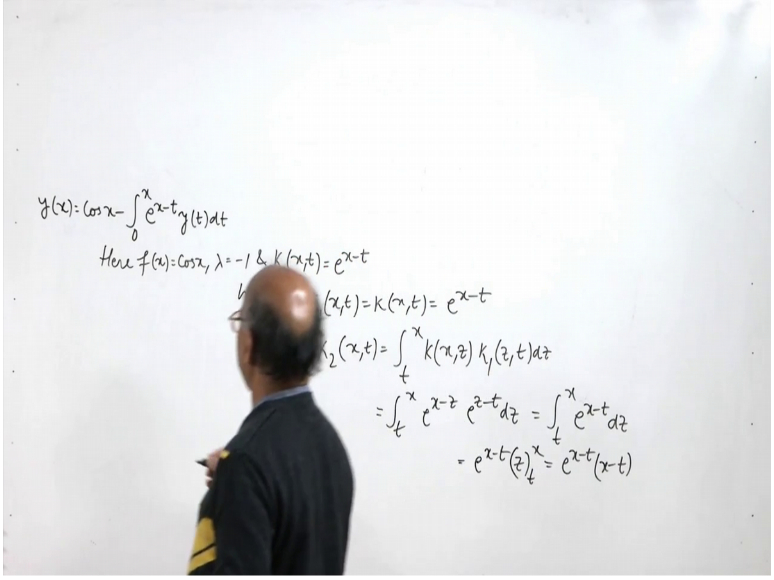

K 3 x t So

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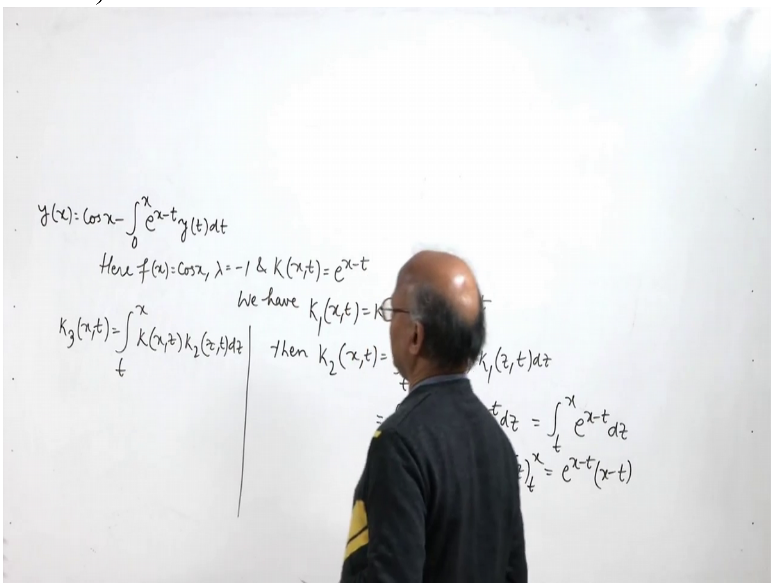

K 3 x t is integral t to x K x z K 2 z t d z which is equal to K 2 z t will be equal to

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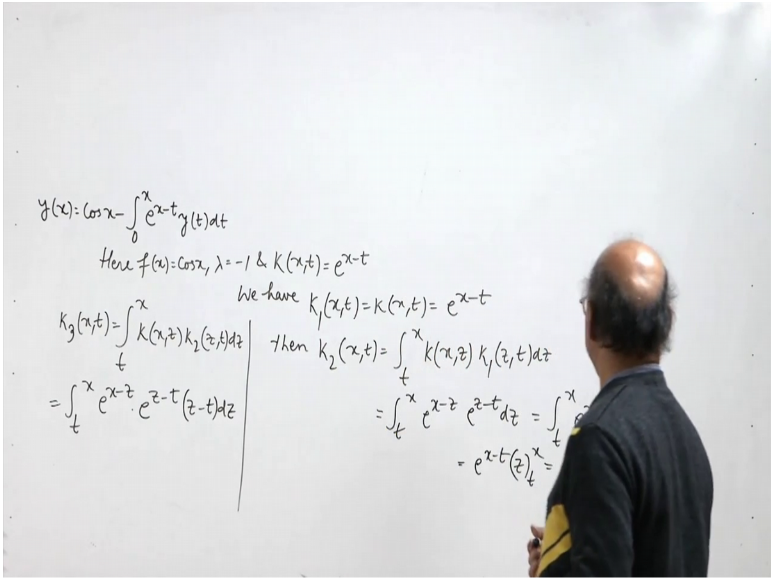

e to the power z minus t into z minus t And this is now e to the power x minus t integral t to x and z minus t d z When you

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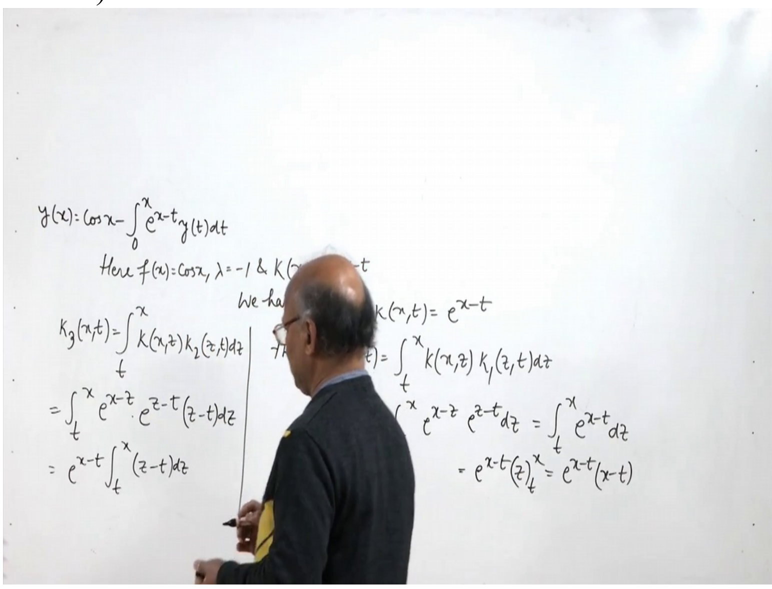

integrate this and substitute the limits what you get is e to power x minus t into x minus t whole square by 2 Now

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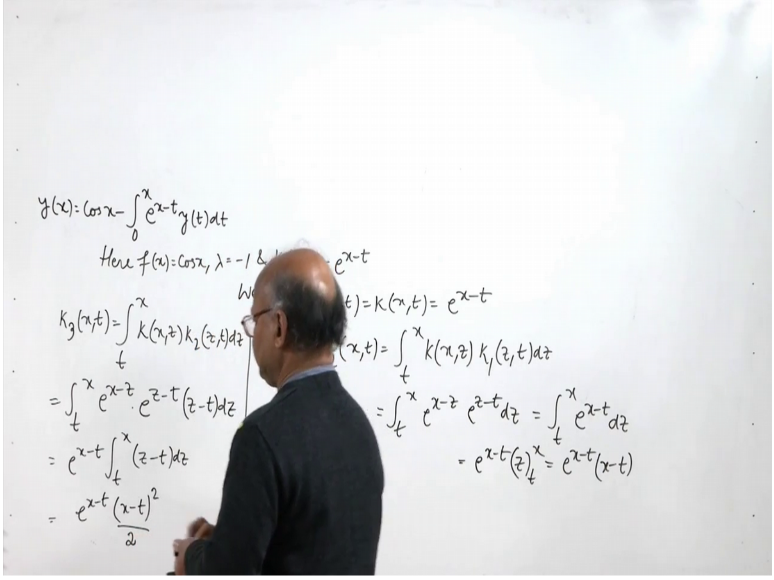

if we find K 3 x t K 3 x t will come out to be so similarly it will come out to be e to the power x minus t into x minus t whole to the power 3 divided by 6 which can be interpreted as e to the power x minus t into it is K 4 not K 3. K 3 we have already found So e to power x minus t x minus t raised to the power 3 divided by 3 factorial And this 2 we can interpret as 2 factorial

So generalizing we have

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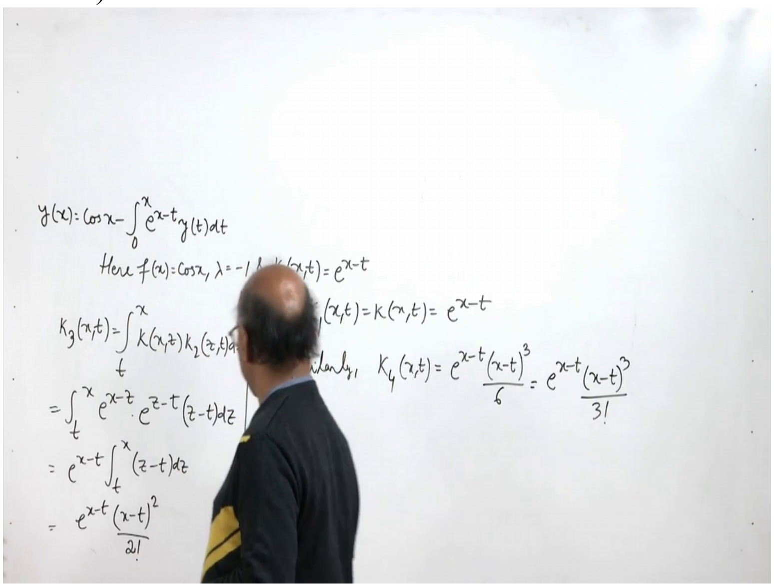

K n x t equal to e to the power x minus t x minus t raised to the power n minus 1 over n minus 1 factorial Ok

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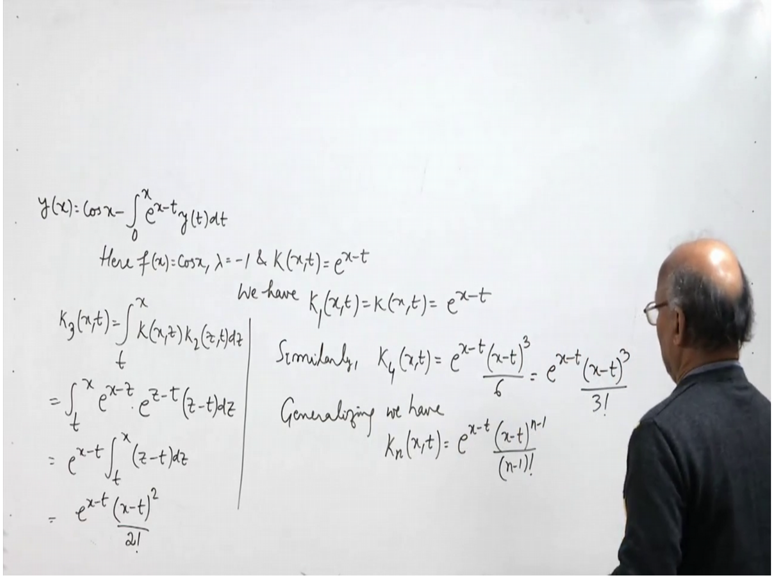

Now let's find R x t lambda So now R x t lambda is summation n equal to zero to infinity lambda to the power n K n plus 1 x t So let us put the value of lambda

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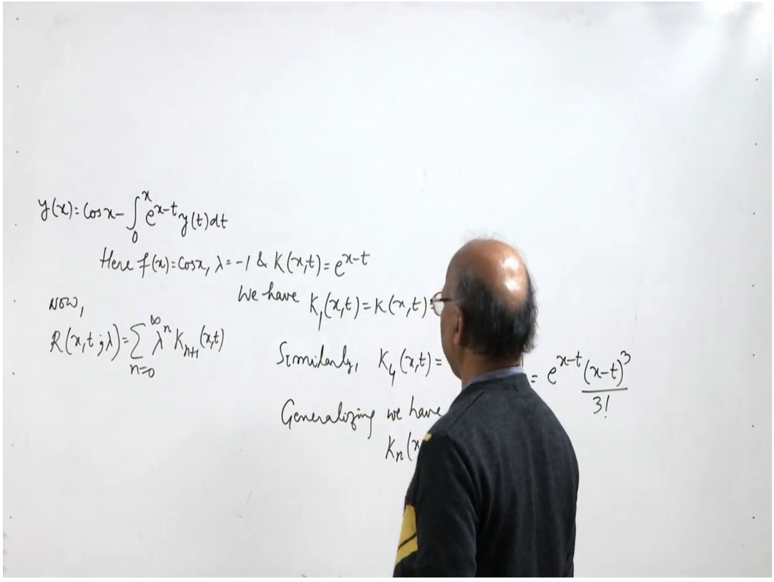

n K n plus 1 x t here and determine the sum of the series So summation lambda is equal to minus 1 So we have minus 1 to the power n n K n plus 1 x t is e to the power x minus t into x minus t to the power n divided by n factorial I can write it as e to the power x minus t times summation n equal to zero to infinity minus 1 to the power n I can multiply to x minus t to the power n and write t minus x to the power n divided by n factorial Now

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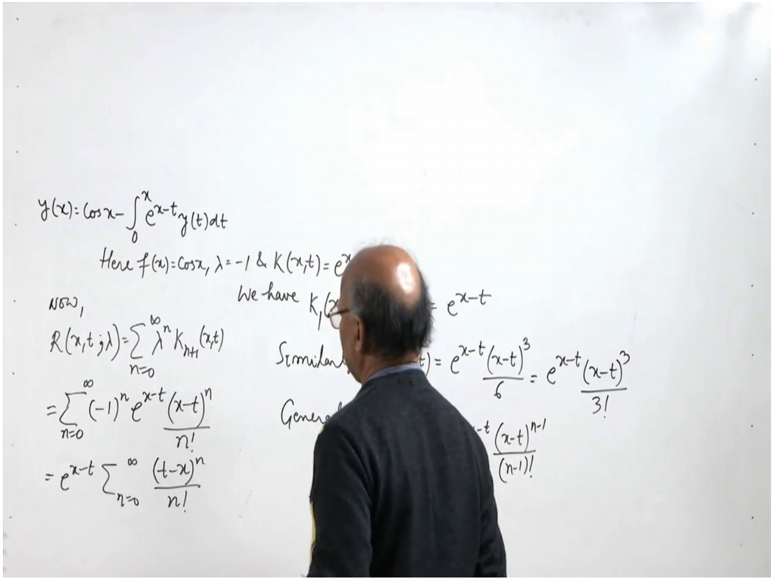

let's find R x t lambda So now R x t lambda is summation n equal to zero to infinity lambda to the power n K n plus 1 x t So let us put the value of lambda n K n plus 1 x t here and determine the sum of the series So summation lambda is equal to minus 1 So we have minus 1 to the power n n K n plus 1 x t is e to the power x minus t into x minus t to the power n divided by n factorial I can write it as e to the power x minus t times summation n equal to zero to infinity minus 1 to the power n I can multiply to x minus t to the power n and write t minus x to the power n divided by n factorial Now we know the sum of this series sigma n equal to zero to infinity t minus x for n by n factorial this is e to the power t minus x so e to the power x minus t into e to the power t minus x So we get e to the power zero which is equal to 1 So we have found the value of the resolvent kernel R x t lambda which is equal to 1 here Now we can proceed to find the solution of the Volterra integral equation of the second kind

So we have y x equal to cos x minus integral zero to x e to the power R x t lambda R x t lambda is 1 into f t that is cos t d t

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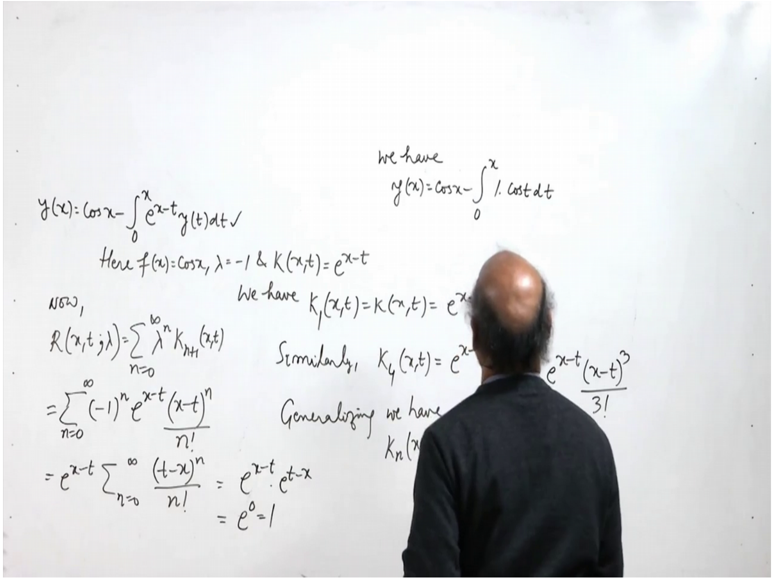

So we have cos x minus sin t zero to x

And which will give us cos x minus sin x So y x is equal to cos x minus sin x is the solution of the Volterra integral equation of the second kind y x

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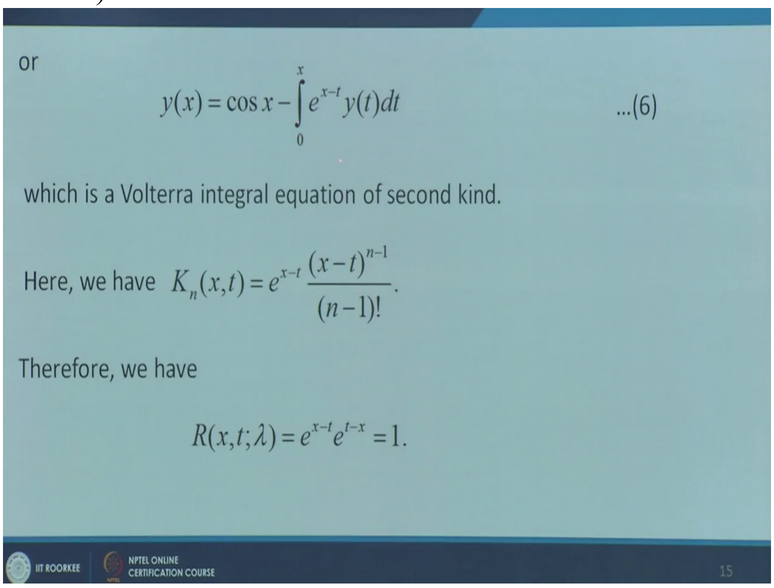

equal to cos x minus

integral zero to x e to power x minus t y t d t and from the method it follows that it is also the solution of the given integral

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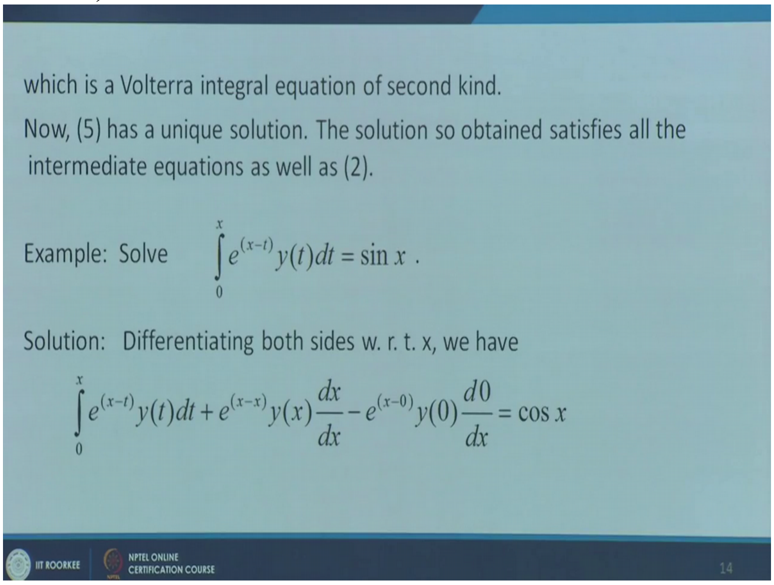

equation Volterra integral equation of the first kind So the solution of this Volterra integral equation of first kind is y x equal to cos x minus sin x That's what I want to say

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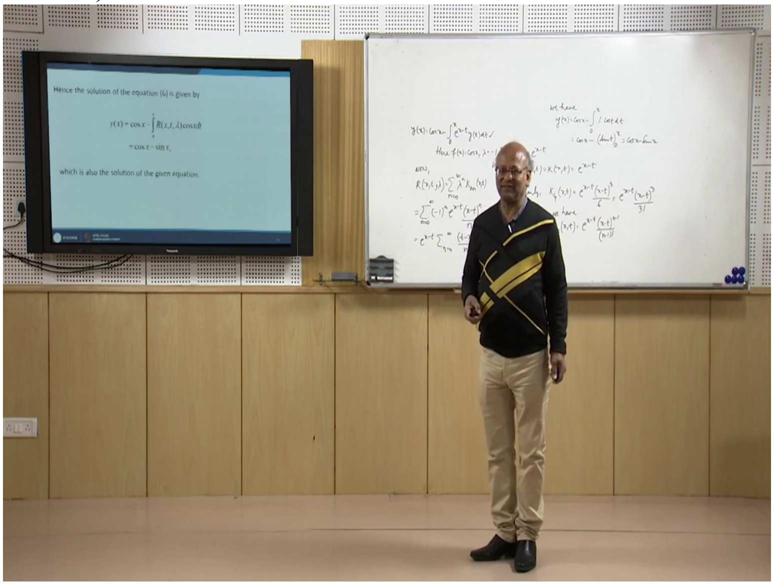

thank you very much for your attention.