Integral Equations, Calculus of Variations and their Applications
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Mod 06 Lecture Number 23
Classical Fredholm Theory Fredholm Second and Third Theorem

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Hello friends, welcome to the today's lecture. Please recall that in previous class we have, previous lecture we have discussed the solution method for this kind of Fredholm integral equation of y of x equal to f of x plus lambda, a to b K of x of t y of t d t and we have

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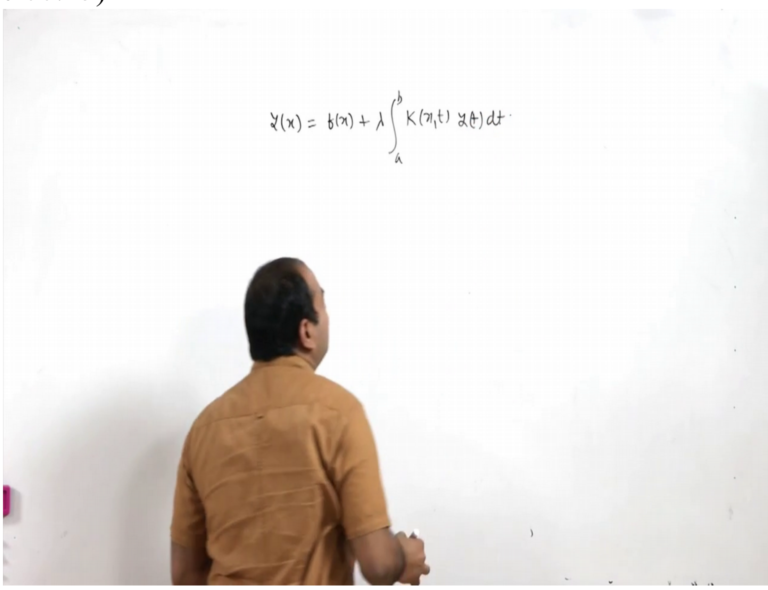

seen that the solution can be written as y of x equal to f of x plus a to b lambda gamma x of t lambda f of t d t

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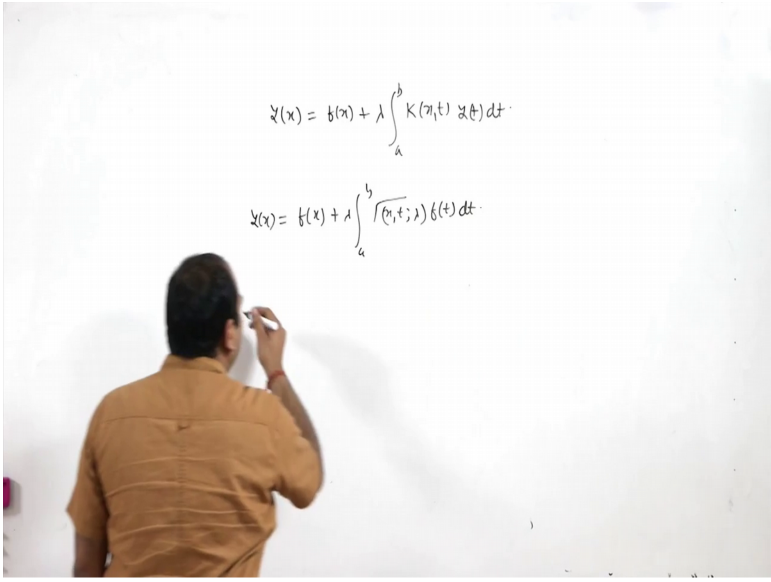

and where gamma x t lambda is given as D of x t lambda divided by D lambda.

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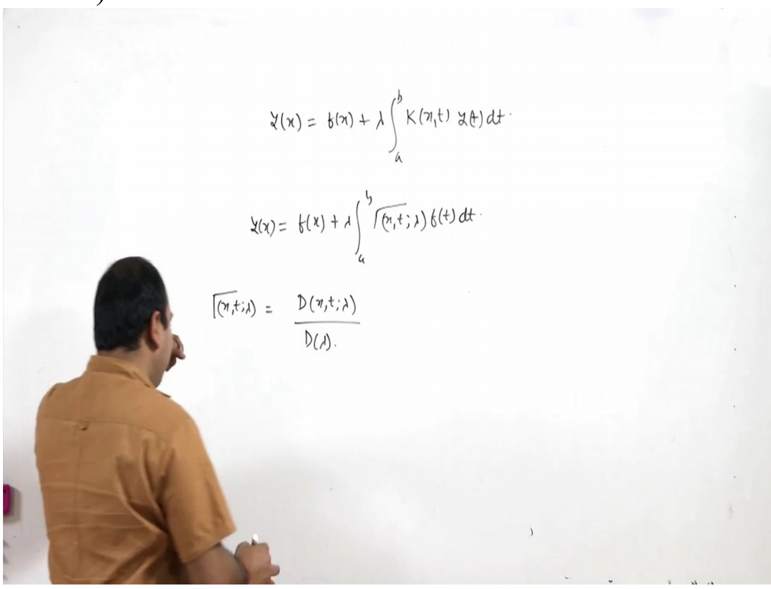

So it means that the solution of the given Fredholm integral equation can be given by y of x where gamma x t lambda is a resolvent kernel given by this ratio of two infinite series.

But the problem arises when this lambda, this lambda is coming out to be the root of D lambda. Or you can say the solution is valid when D of lambda is not equal to zero for this lambda.

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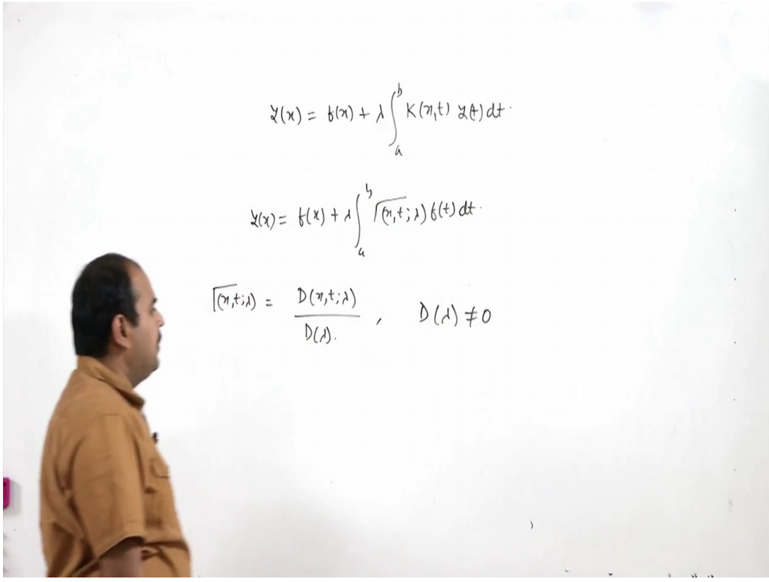

But when D lambda is equal to zero then how we can define our solution? Then please

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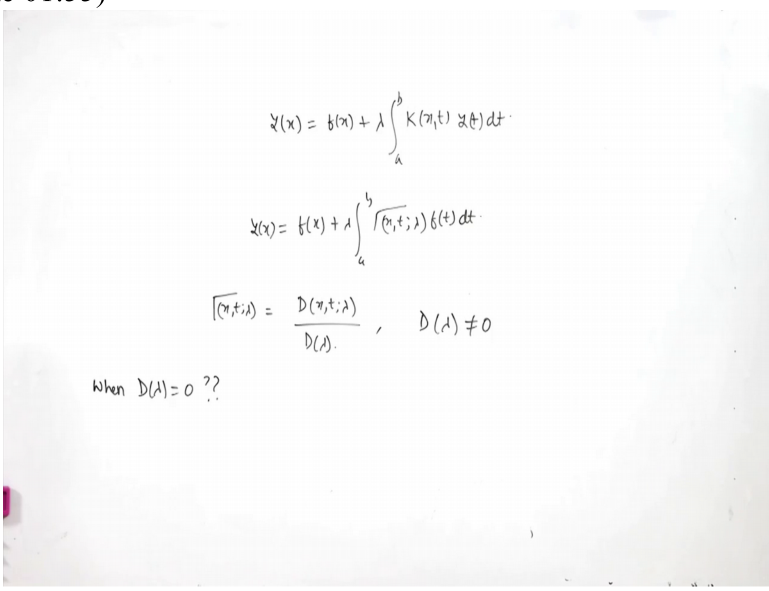

recall, then our D once we know the D lambda then your D of x t lambda is defined as, by this following relation that K of x t D of lambda plus a to b K of x of s D of x s of t lambda, s t lambda d t here. We know that when

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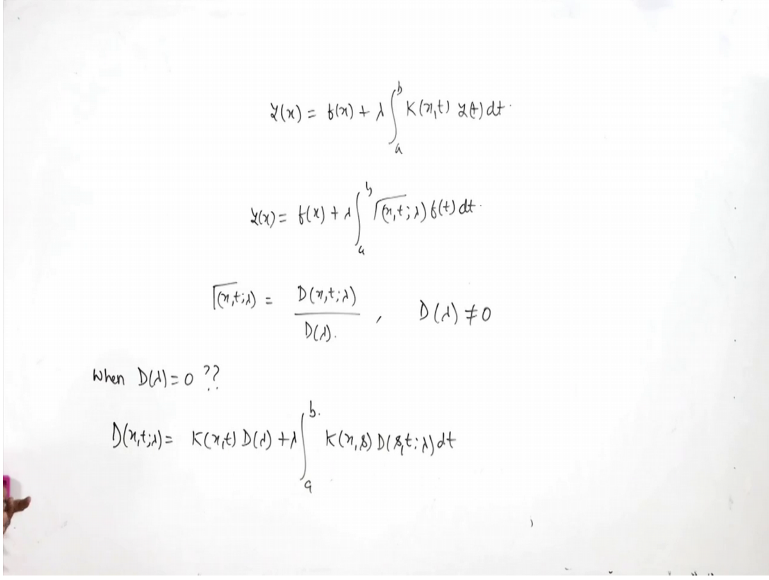

once we have expression for D lambda that is Fredholm determinant then we have calculated the Fredholm minor that is D x t lambda with the help of this integral equation. This is a relation hold by D of x t lambda and then we have seen that if D of lambda, this we have already done that if D lambda is given by c naught plus summation minus lambda to power m upon factorial m, c m m is from 1 to infinity, then with the help of

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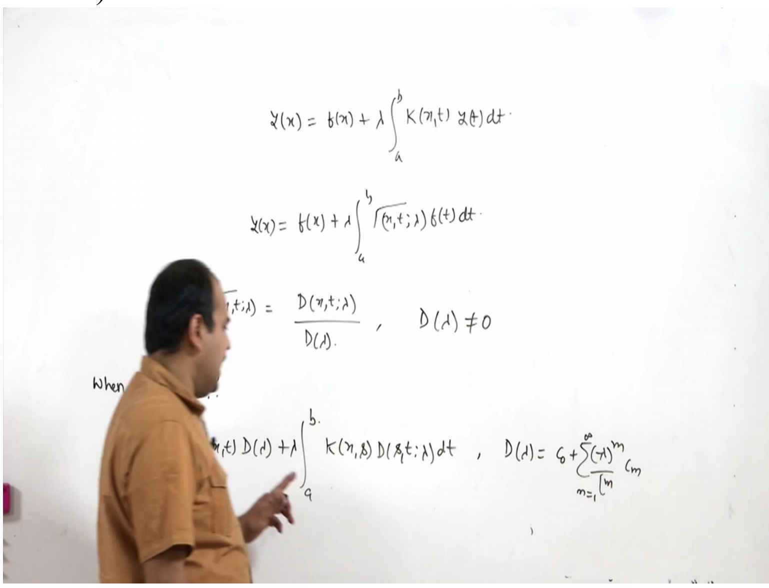

this integral equation we try to find out, we have already discussed that D of x t lambda you can calculate c naught of x of t plus summation m is equal to 1 to infinity minus lambda to power m upon factorial m c m x of t. So it means that this, with the help of this equation and the expression of D lambda we have to, we have calculated the expression for D x t lambda and also we have seen that if the kernel K is integrable and bounded then expression given as

D lambda is an uniform, uniform series in terms of lambda for all values of lambda. Similarly you can say that in the condition that this kernel is integrable and bounded then D x t lambda is also, this infinite series is also uniformly convergent for all values of lambda here. So this we have already discussed.

Now the case with the help of this equation only, we try to consider the case when D lambda is equal to zero. It means that let us say that you have a value lambda equal to lambda naught such that D lambda naught is equal to zero. So it means that here

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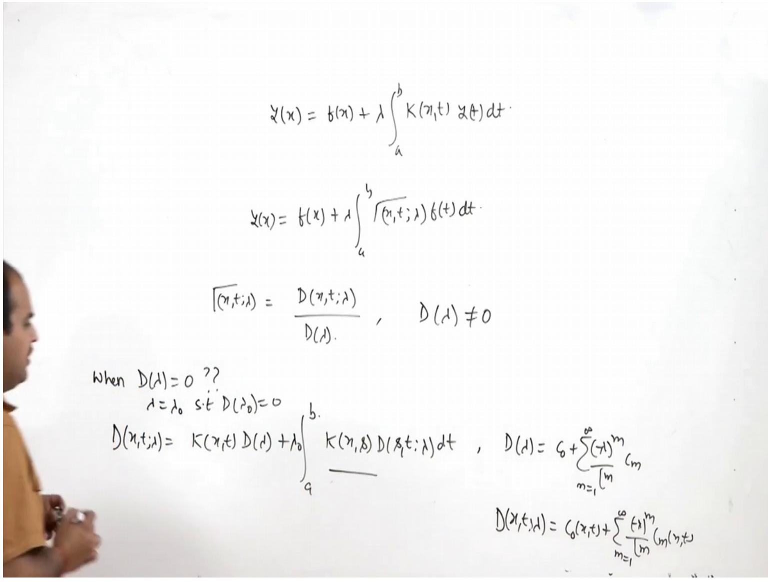

we are taking the value lambda naught. So here for this particular value, your D lambda naught is simply zero. I am just considering the value lambda equal to lambda naught. So this part is simply vanished. So you can write it D of x t lambda naught is equal to lambda naught a to b K of x of s D of s t lambda naught d of t.

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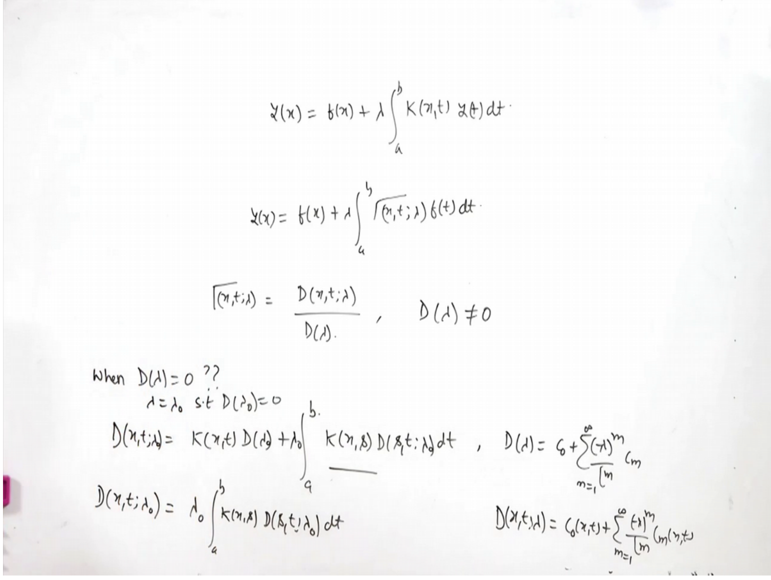

Is that Ok? Now I am saying that first we will consider the case when f of x is equal to zero, means that we are considering the homogenous part. So here we are considering that when D lambda is equal to zero and we are assuming that f of x is equal to zero. So it means that

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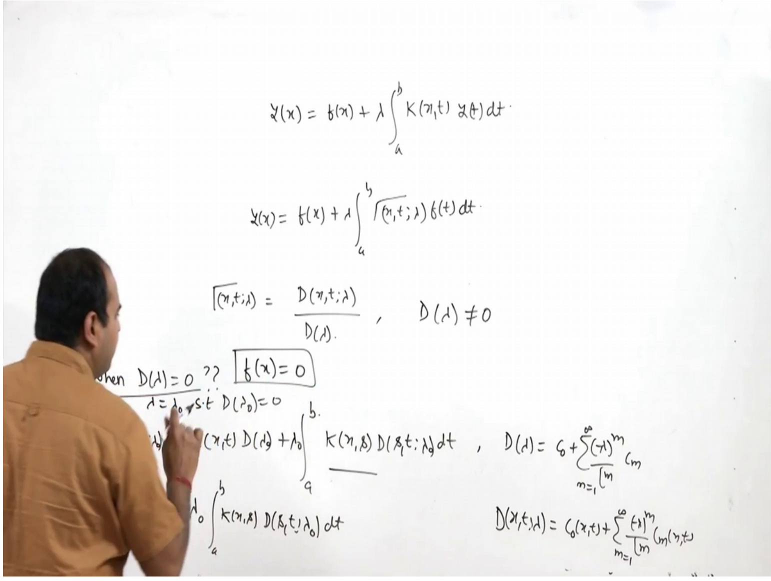

we are considering only the homogenous part. Then we say that in this case your equation is this; y of x equal to lambda a to b K of x of t y of t d t, right?

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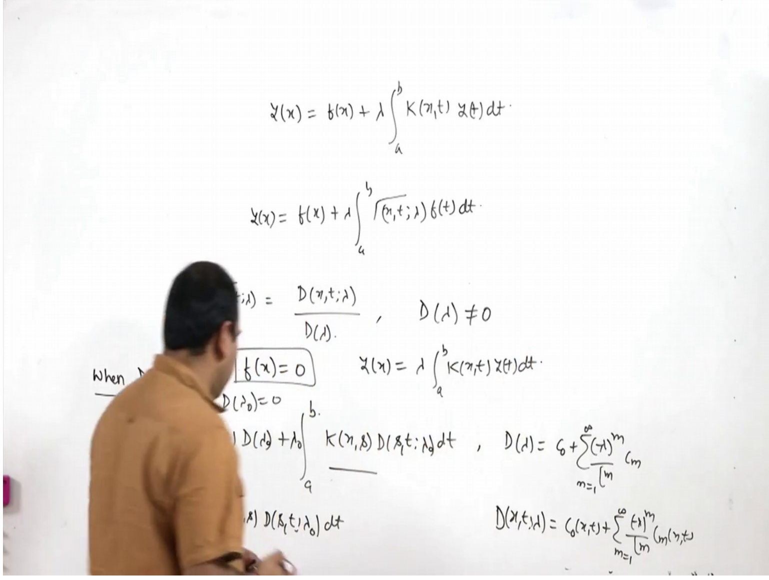

Ok. So this small correction here. This correction is this, that here this integral is with respect to s not with respect to t. So here integration is with respect to d of s. So here is d of s.

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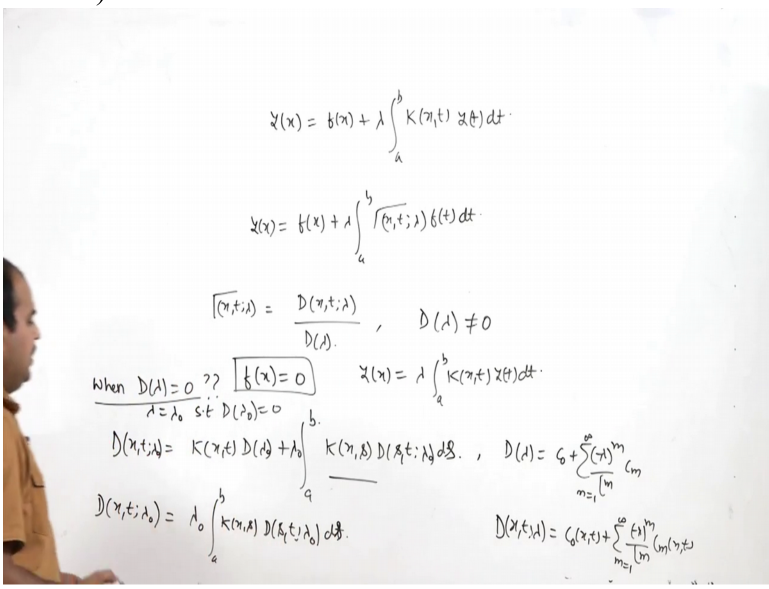

Ok, so here if we compare the equation and this equation, so if we compare these two equations you can see that here if I assume that D x t lambda naught is not non-zero, identically not equal to zero

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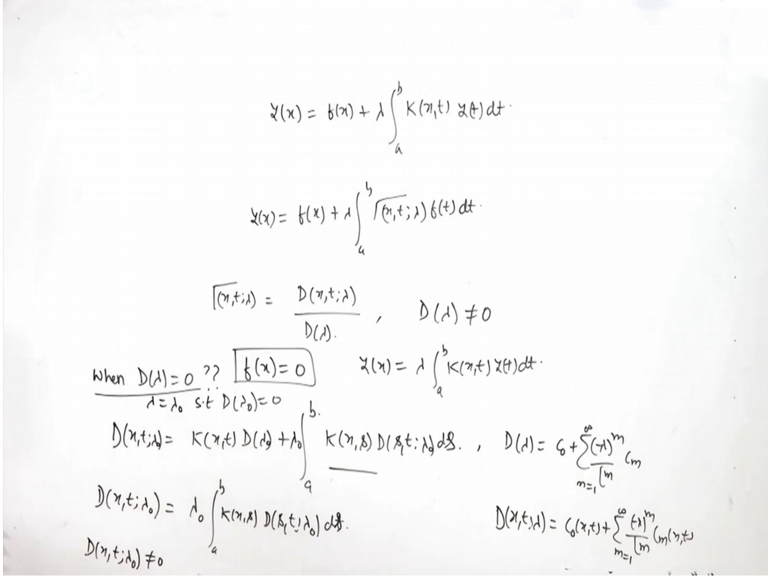

then I can assume that y of x is equal to

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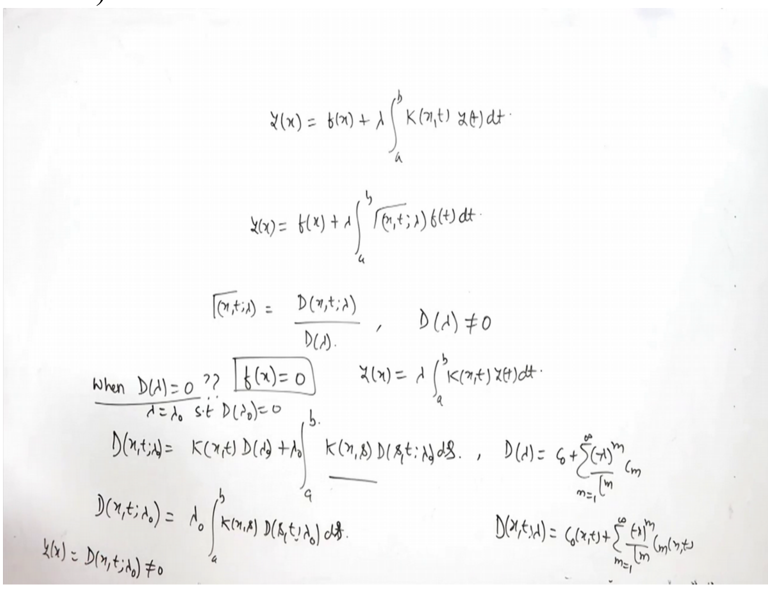

D of x t lambda naught is a solution of this particular equation for every value of t in the interval a to b. Is that Ok? So

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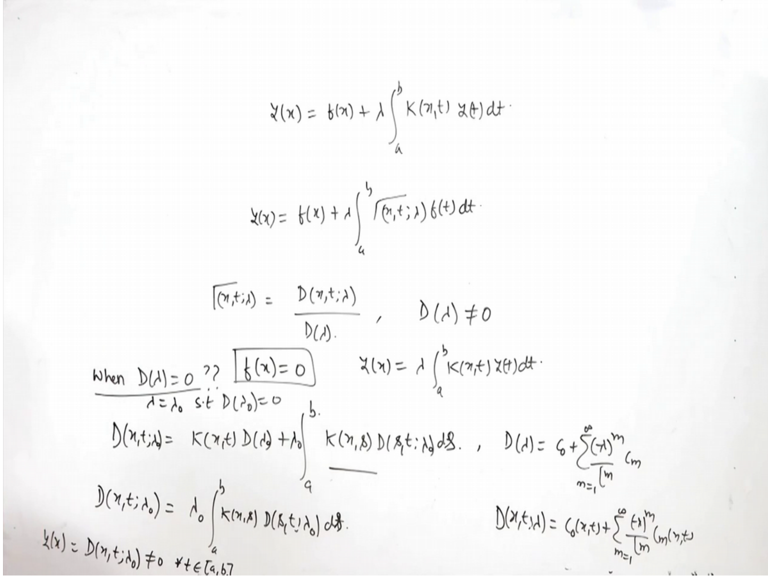

it means that this y of x equal to D of x t lambda naught is a solution of this equation for all t in this interval a to b, is it Ok? So only thing we are worried is that this expression whether this is zero or not. So this part, this decision, we can write it in this particular part that if D lambda

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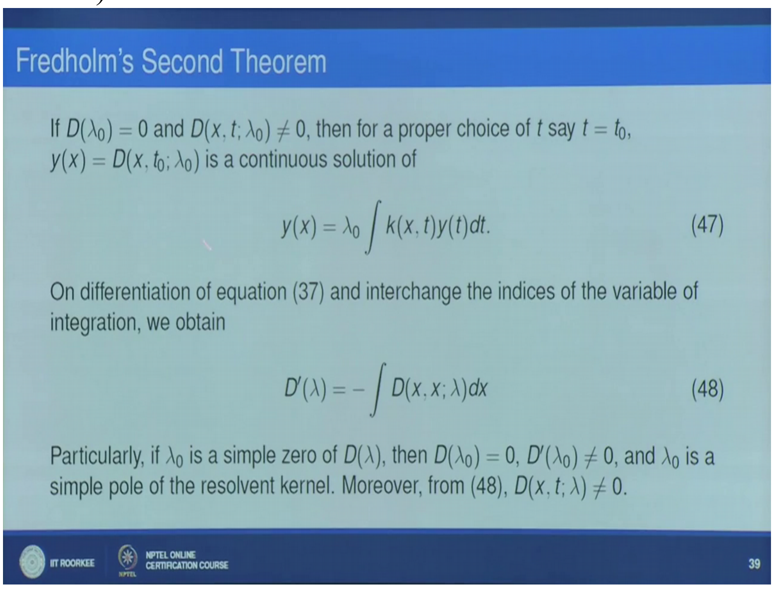

naught is equal to zero, so here we are assuming that lambda naught is a root of this Fredholm determinant that is D lambda naught equal to zero and we also assuming that D x t lambda naught is not equal to zero, not identically equal to zero then we can choose any value of this t and we can say that y of x as D of x t naught lambda naught is a solution of this equation. That is y of x equal to lambda naught a to b K x t y t d t. So that is clear from this. So here if you compare these two equations, this and this and we can say that y of x

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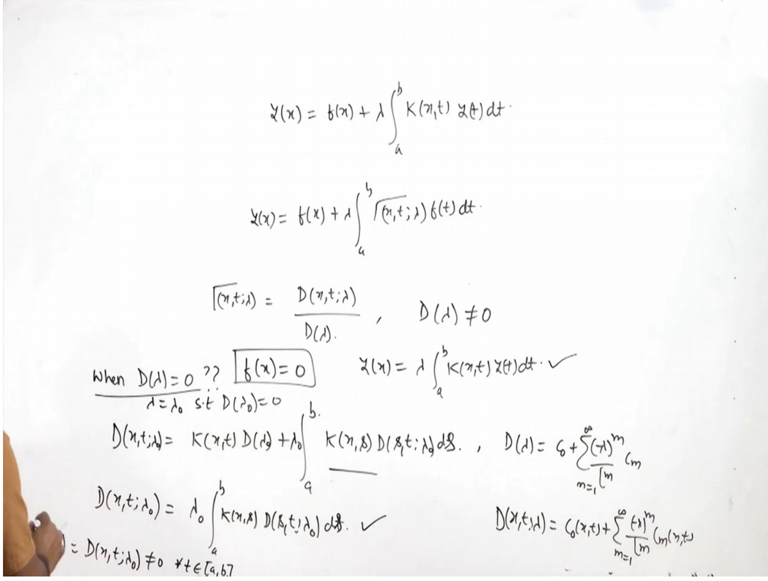

equal to D x t lambda which is not identically equal to zero is a solution of this equation. Now the thing is that, the condition that D x t lambda naught is not equal to zero,

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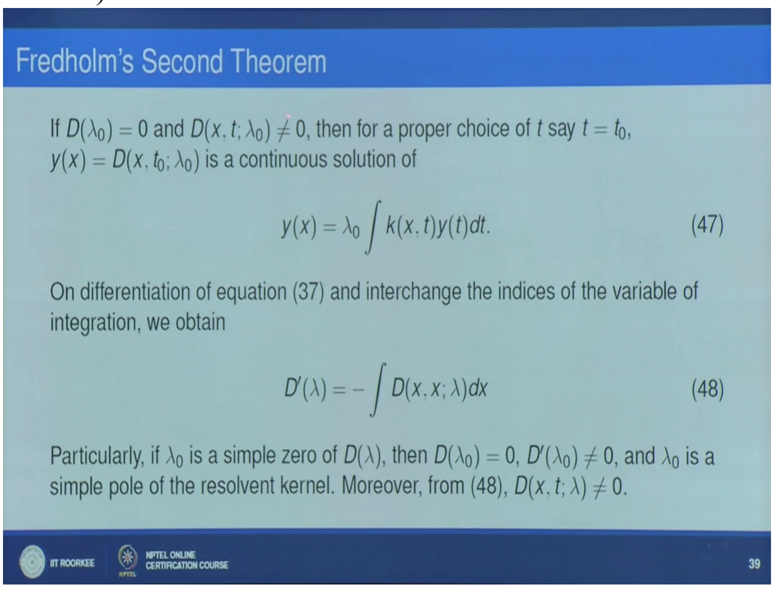

how we can say, rectify this. So this condition that D x t lambda naught is not identically equal to zero can be related to the special Fredholm determinant D lambda in this way; that if you look at the expression of D lambda and if we differentiate that then we have this relation. So just let me see what is this relation.

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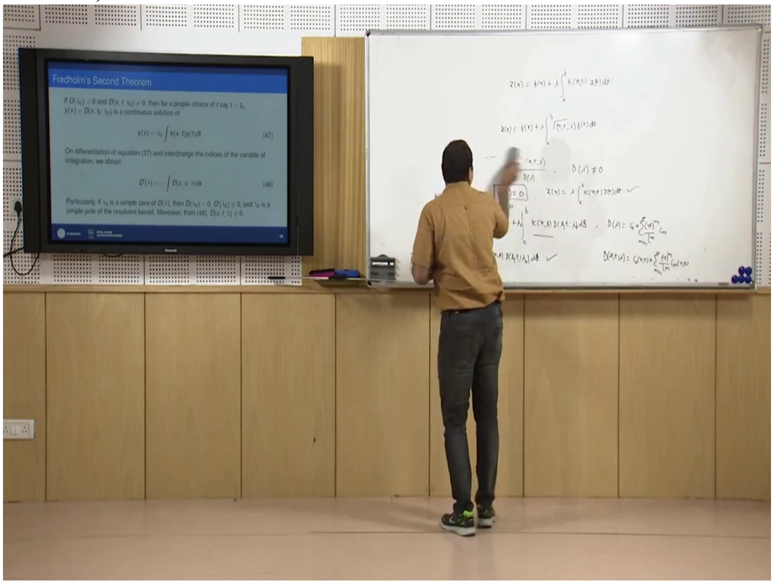

So here we have this D lambda that is small c naught plus summation m equal to 1 to infinity minus lambda to power m small c m upon factorial m. So if we differentiate this,

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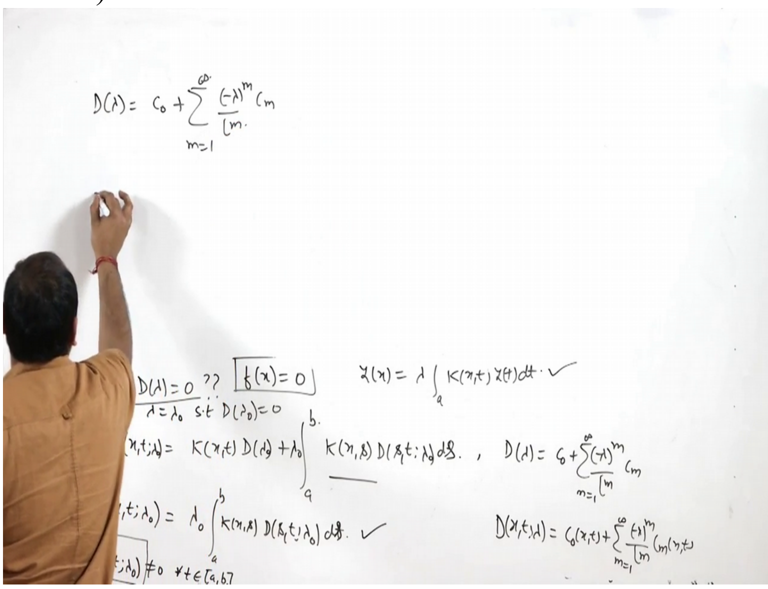

D dash lambda you will get minus m equal to 1 to infinity this part is simply vanished. So minus you will get, minus m minus lambda to power m minus 1 upon factorial m and c m. And if you simplify this you will get this as minus summation m equal to 1 to infinity this is m it will cancel out here minus lambda to power m minus 1 upon factorial m minus 1 small c m.

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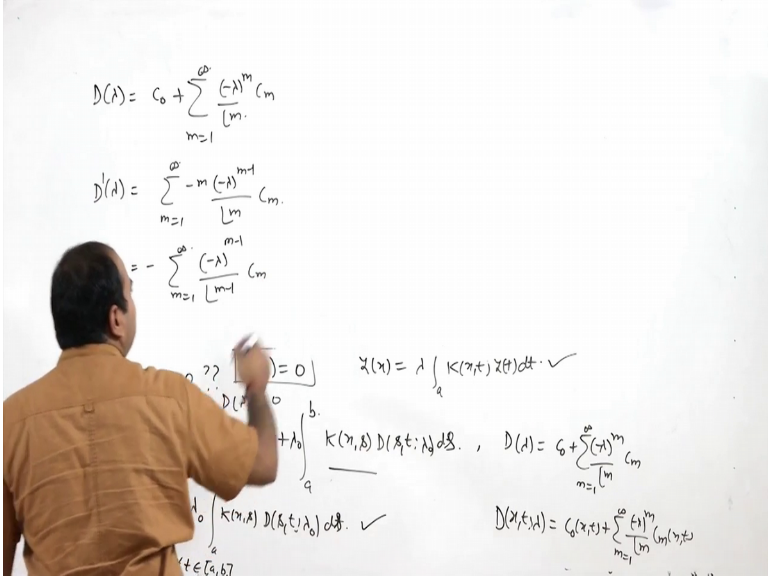

And we know that small c m is connected with capital C m minus 1 x comma x d x like this. So

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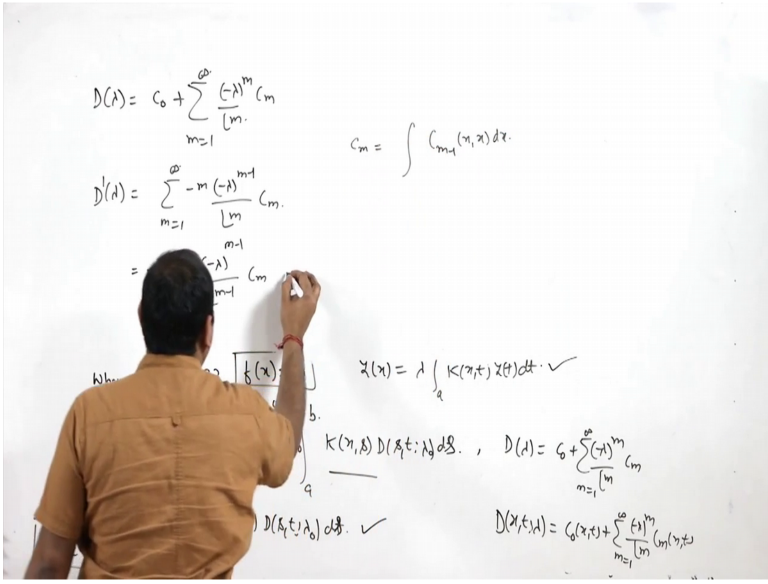

using this I can write it minus summation m equal to 1 to infinity minus lambda to power m minus 1 and upon factorial m minus 1, this c m I can write it as integral of c m minus 1 x comma x d of x. And if we simplify,

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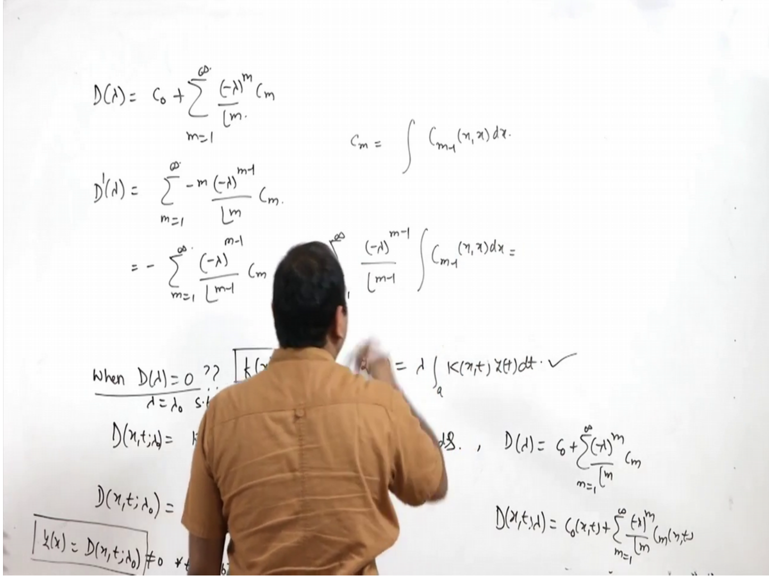

if you replace this dummy variable m minus 1 by say some n, you can write it here that it is minus n equal to zero to infinity minus lambda to power n upon factorial n and integral here c n x comma x d of x. Now this c infinite series is uniformly

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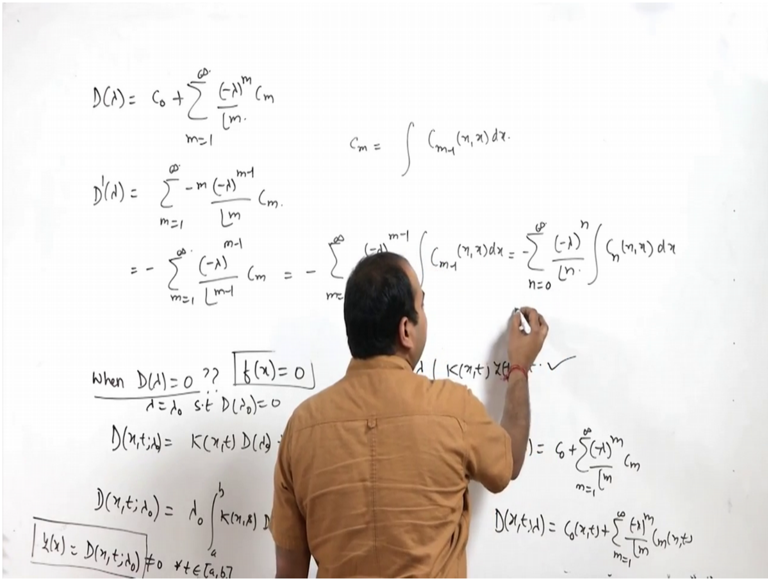

convergent so we can interchange this integral sign and you can write this minus integral of, and if you take inside it is nothing but your D of x comma x lambda d of x.

So it means that your

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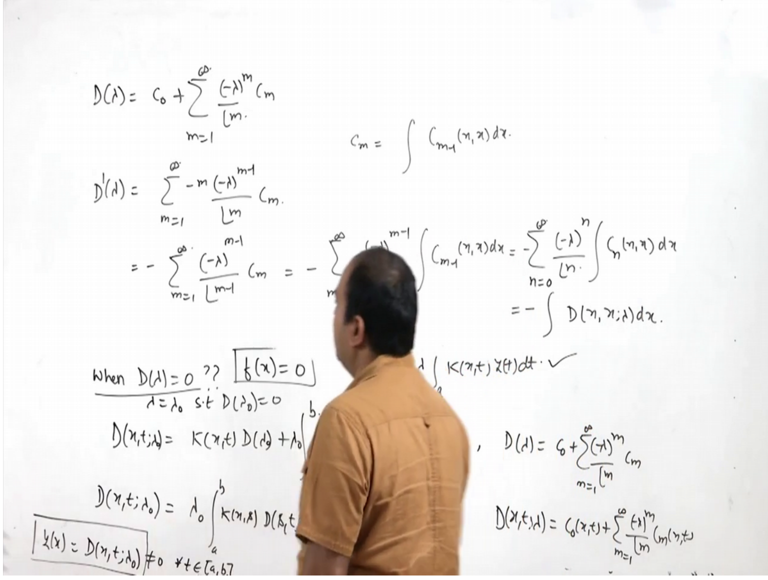

D dash lambda is related to D x x lambda in as this equation number 48. So it means that if we have

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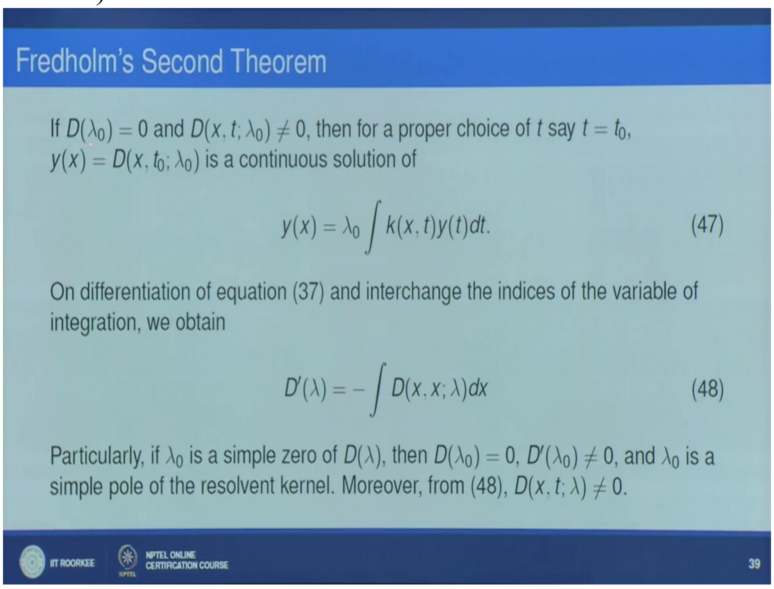

lambda naught in a way such that D lambda naught is zero but D dash lambda naught is non zero or we can say that lambda naught is a simple zero of D lambda then this portion D dash lambda naught is non-zero so we can say that integral of D x x lambda naught d x is not equal to zero. So it means that this expression D x x lambda is not identically equal to zero. So we can say that when lambda naught is a simple zero of D lambda then this y of x is equal to D x t naught lambda naught is a solution of the homogenous Fredholm integral equation of this, y of x equal to lambda naught K x t y t d t.

So in this case,

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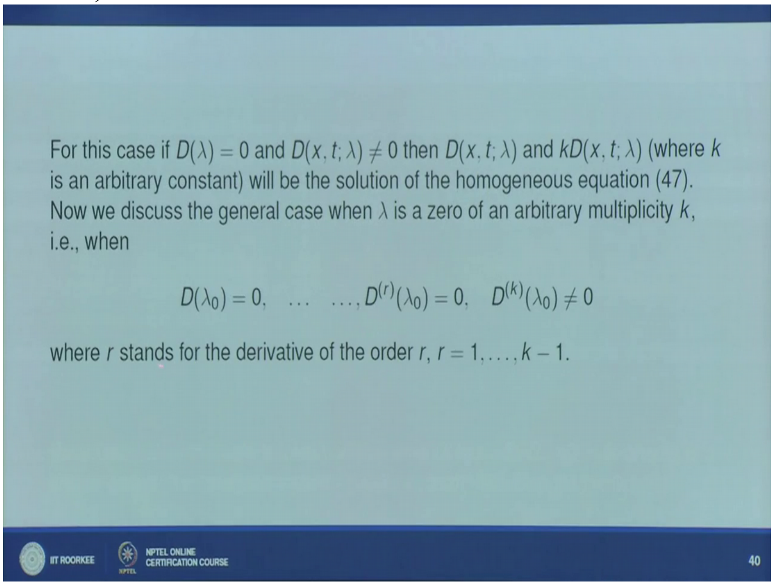

not only D x t lambda but any constant multiple of D x t lambda will be a solution of the homogenous equation, homogenous equation

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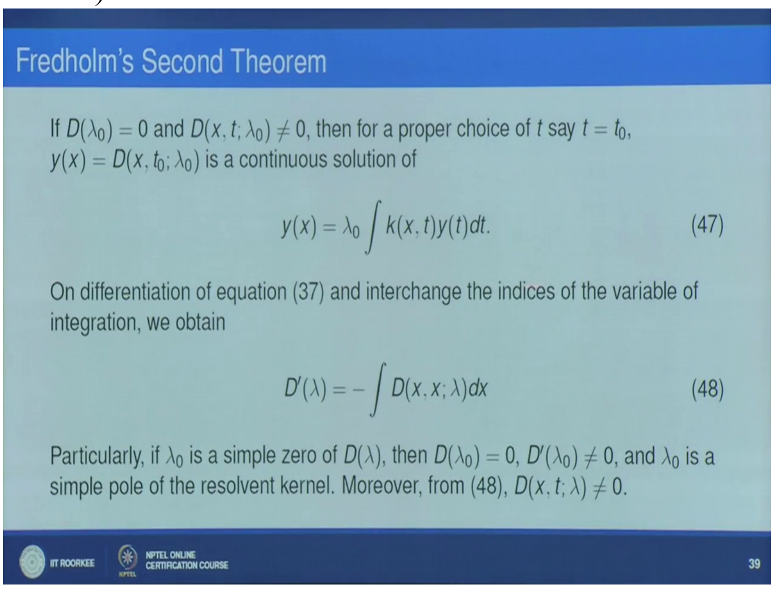

this equation.

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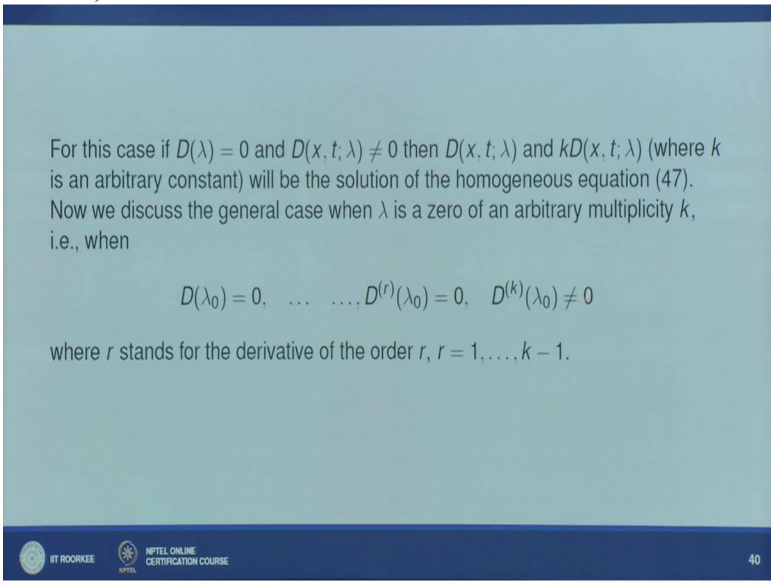

Now we try to consider, we generalize the case here. So here we have discussed the case when lambda naught is a

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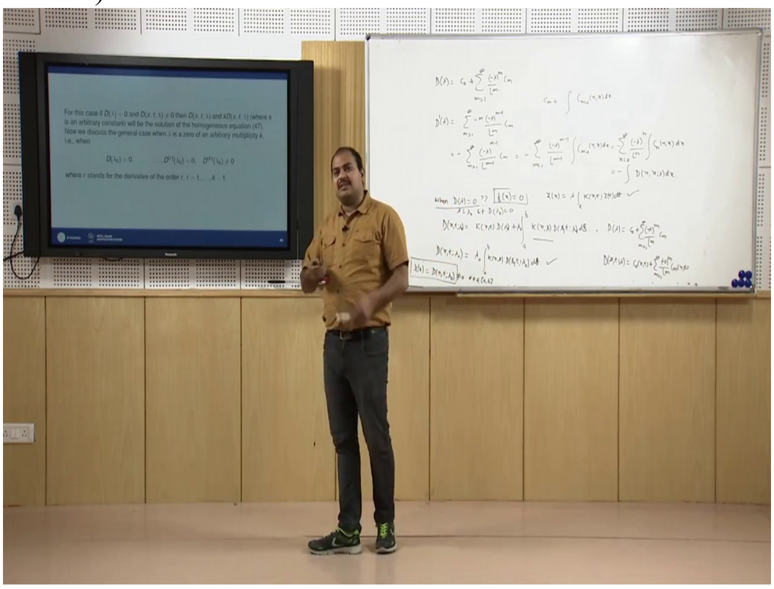

simple zero of D lambda. Now what happens when lambda naught is zero of D lambda with multiplicity say u K. So it means that D lambda naught is

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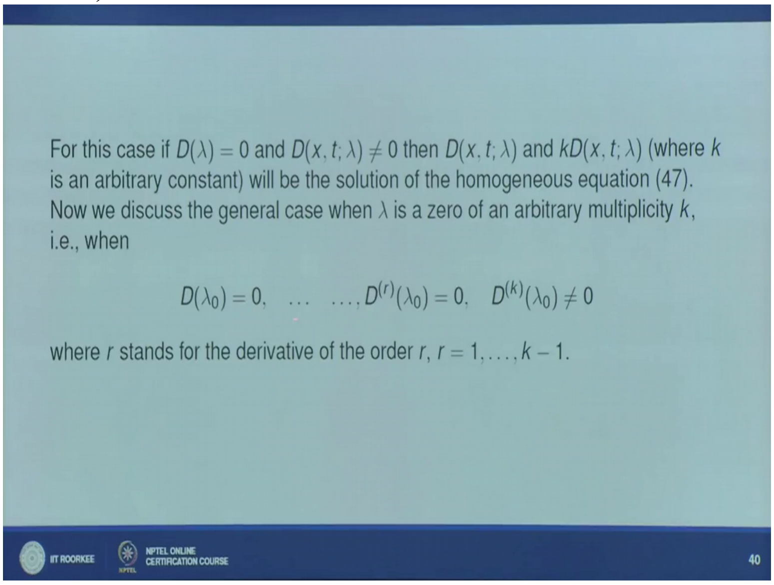

zero and up to D k minus 1 lambda naught is equal to zero but the k th derivative evaluated at lambda naught it is not equal to zero. So here we say that lambda equal to lambda naught, this is zero of D lambda of the order k, right and

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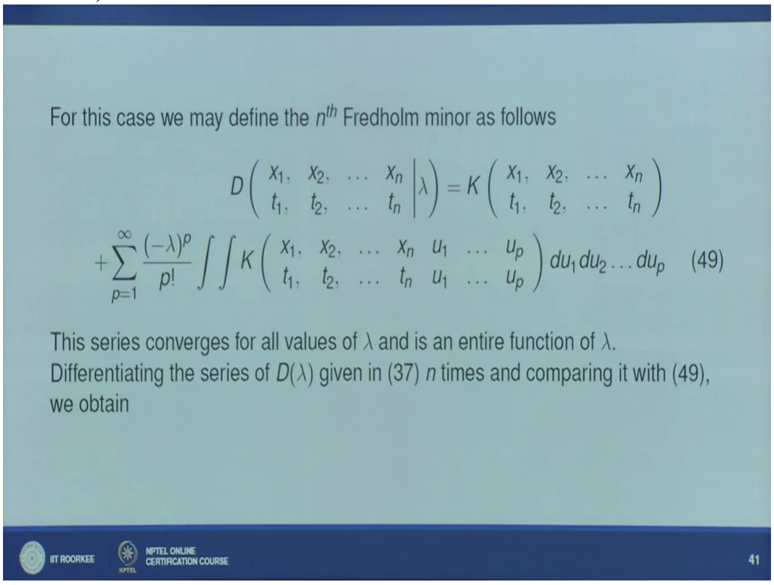

here we try to find out the solution

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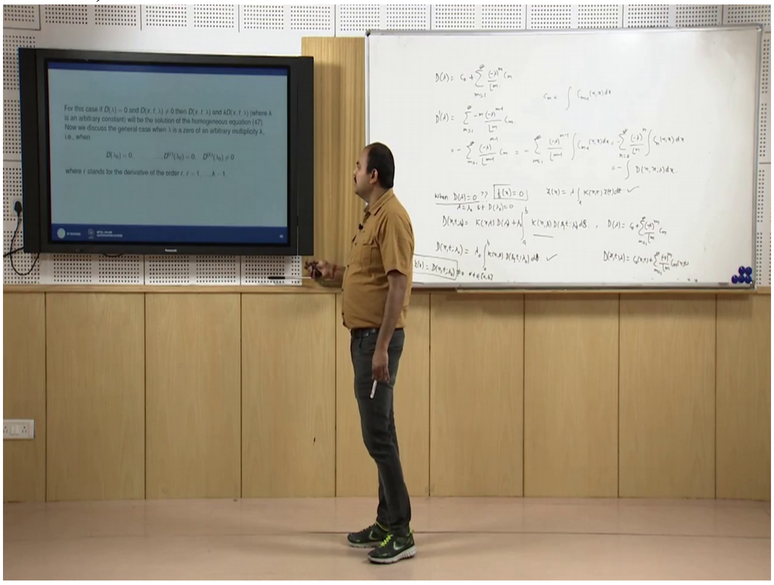

of our homogenous Fredholm equation given by 47.

So let me summarize here. When lambda naught is simple zero,

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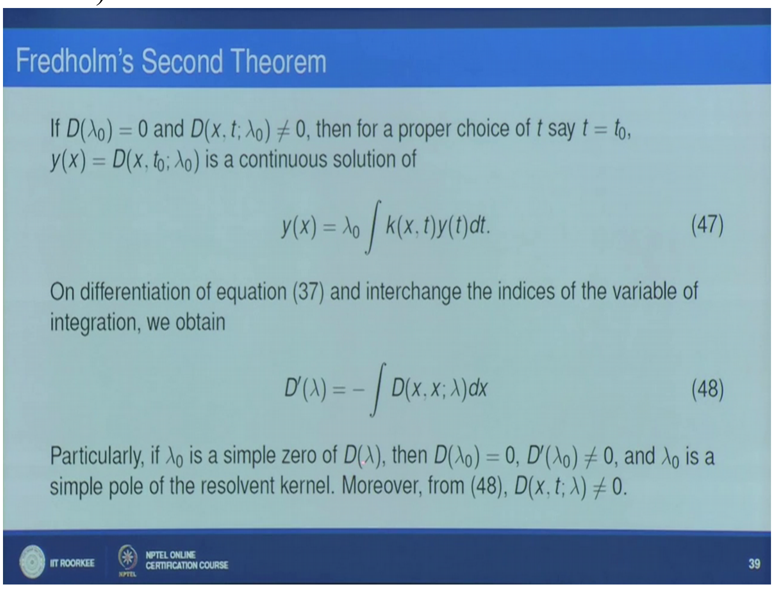

then D x t naught lambda naught is going to be a solution of this. Now we want to show that if lambda naught is a zero of D lambda with multiplicity k then in that case what should be extended or what should be the generalization of D x t naught lambda naught.

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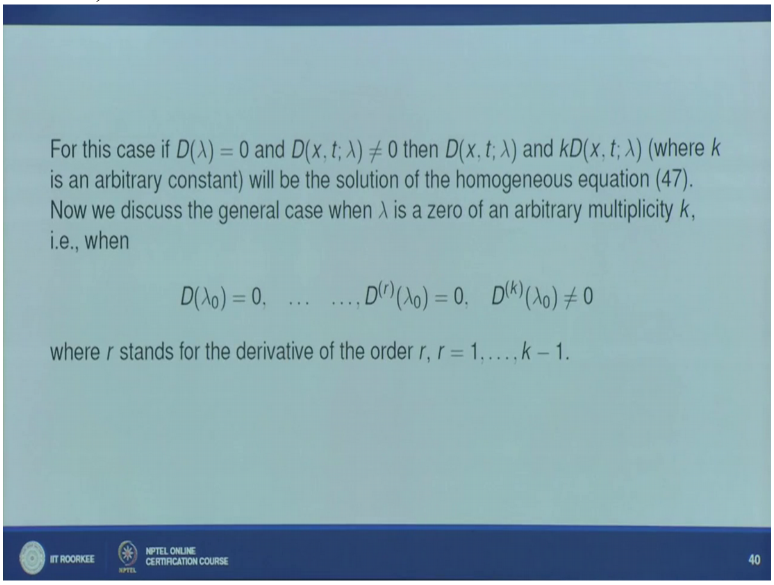

So here for this we define a

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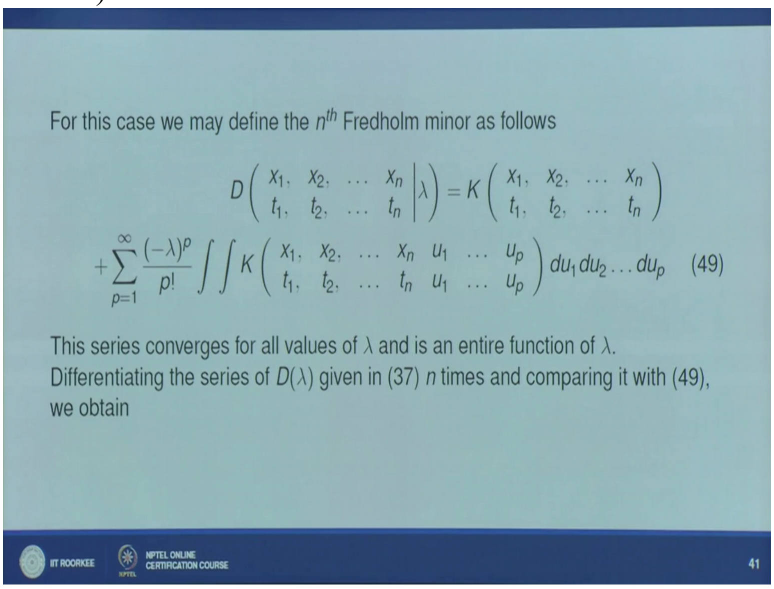

new term which is known as nth row minor and it is defined as d x 1 x 2 x n t 1 t 2 t n with respect to the given constant lambda and it is given by K x 1 t 1 x 2 t 2 up to x n t n plus p equal to 1 to infinity minus lambda to power p factorial p this is p integral K x 1 to x n t 1 to t n u 1 to u p u 1 to u p d u 1 d u 2 d u p. So if we can define nth Fredholm minor as defined as 49, then our claim is that this will be a kind of solution candidate for

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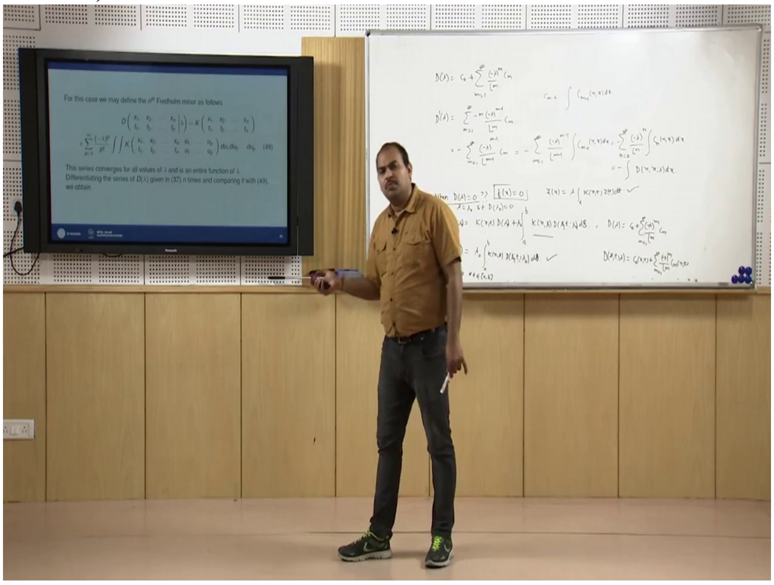

the homogenous integral equation. Now if you look at your equation number 49 in a clear manner, if we take

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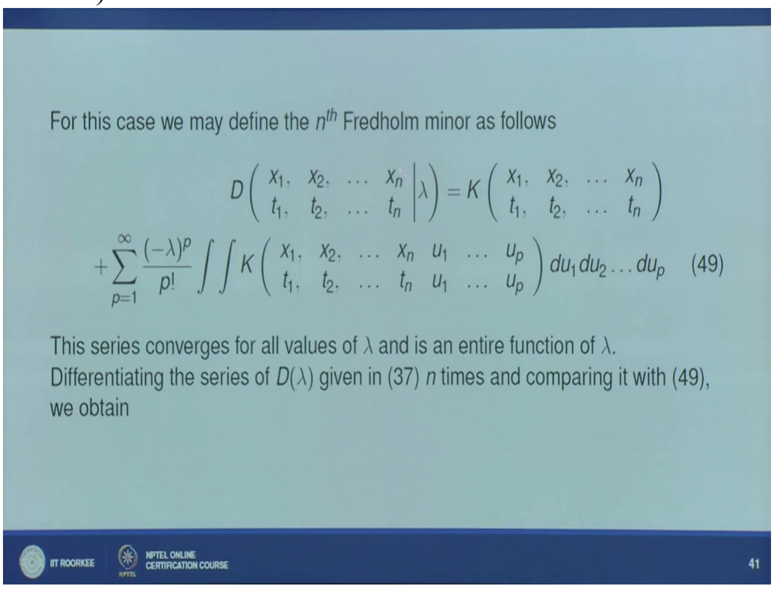

n equal to 1 so it means that we are considering the first Fredholm minor. So our claim is that this is nothing but your D x y lambda. If you look at equation number 49 and write down the expression for n equal to 1. So here if you look at this expression D of expression x 1 to x n and t 1 to t n with respect to lambda it is given as K x 1 to say x n t 1 to t n plus summation your p equal to 1 to infinity minus lambda to power p upon factorial p and this is p-fold integral and here we have K x 1 to x n t 1 to t n. And then u 1 to u p and u 1 to u p. And d u 1 to d u p. I hope we already

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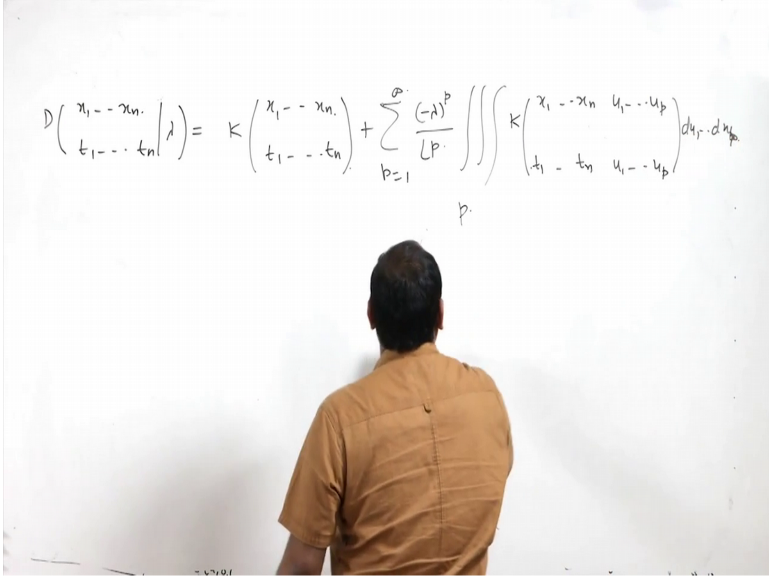

know this symbol. So using this symbol we can say that your d x y written like this.

Now here if we take n equal to 1, then we try to see that

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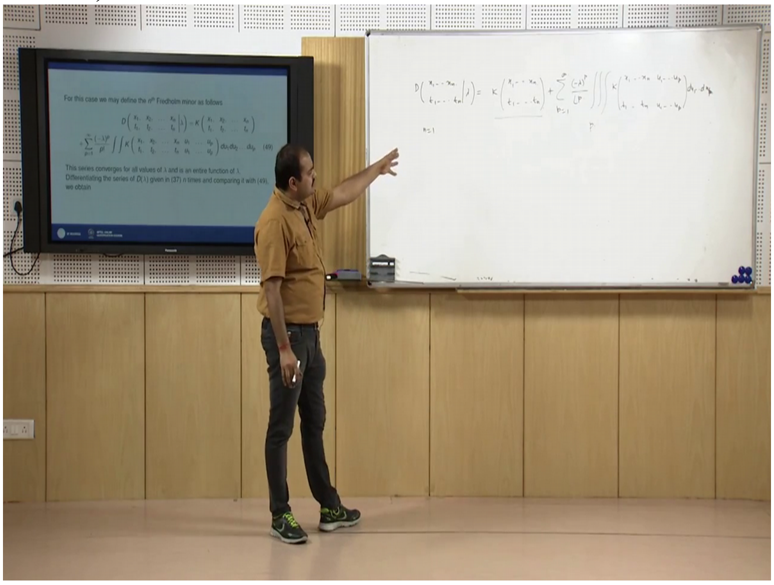

this is reduced to your Fredholm minor that is D x y lambda. So if you look at n equal to 1 then it is what, D x 1 t 1 given as lambda is equal to K x 1 t 1. Now if you look at the notation here, this is what, this is, notation is determinant here, x 1, so this, we already know this is K x 1 t 1 K x 1 t 2 and so on K x 1 t n and so on. So here it is K x n t 1 and K x n t n. So here using this notation you can say that K x 1 t 1 is nothing but K x t 1. So let me write it first and we will simplify this. t 1 plus summation p equal to 1 to infinity minus lambda to power p upon factorial p and this is p integral, and this is K x 1 and this is t 1 and then u 1 to u p and u 1 to u p and d u 1 to d of u p. So if you simplify

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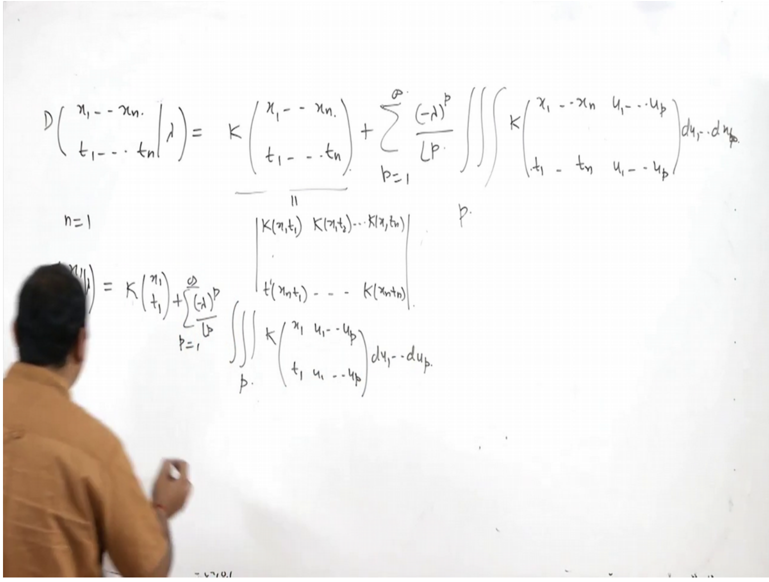

this is what, this I can write it, let me write this as x 1 t 1 lambda this is what K x 1 t 1 using this notation it is nothing but 1 cross 1 that is K x 1 t 1 plus summation p equal to 1 to infinity minus lambda to power p upon factorial p. And if you simplify this, you can say that this is nothing but c p of x of x 1 t 1.

So, and this I can write it as c naught x 1 comma t 1. So it means that, that this is the nth, Fredholm minor is a generalization

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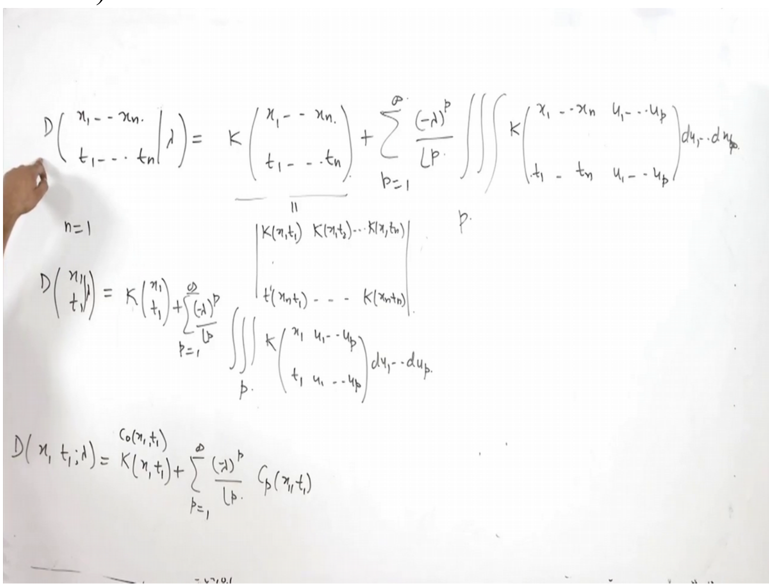

of this Fredholm minor for n equal to 1. So I can say if we want to write it, this is equal to this, Ok. So it means that the nth minor is a generalization of Fredholm minor. Or you can further simplify this as follows. If you want to write this, you can write this as D p x say y lambda or this is written as D of p is 1, p is n here, let me write it here. So I can say that

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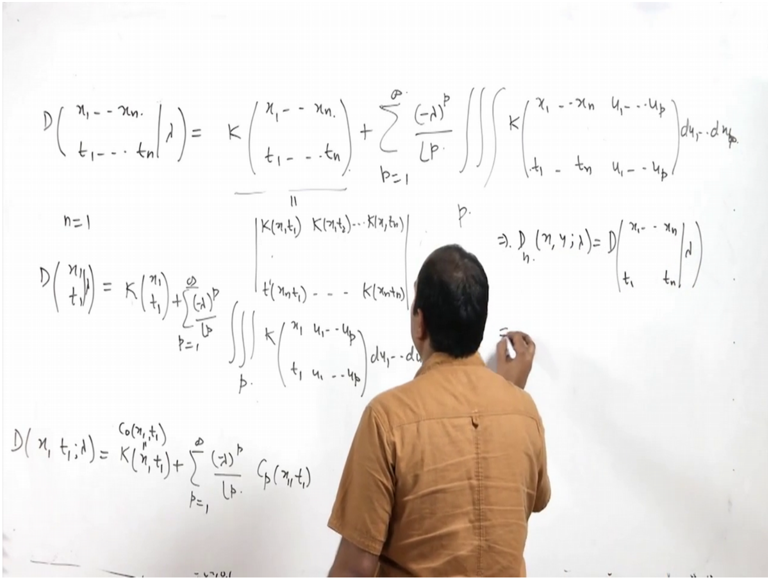

this is D n x y lambda. It is D x 1 to x n t 1 to t n lambda. Let me write it here.

This I can write it here as B naught x, say B naught x y plus summation p equal to 1 to infinity, you can write it B p. This x y, Ok x y, right. So here B naught x y is defined

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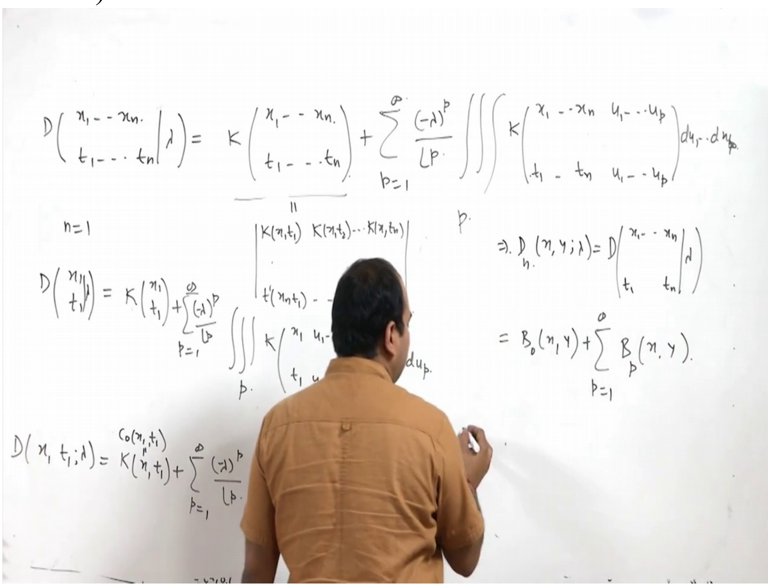

as what, B naught x y is given by this thing. So you can say that x is nothing but your x 1 to x n and y is your t 1 to t n. And B naught x y is defined as this, so this you can call it B naught x y and this you can it B p of x y. So if you define it like this, this is nothing but

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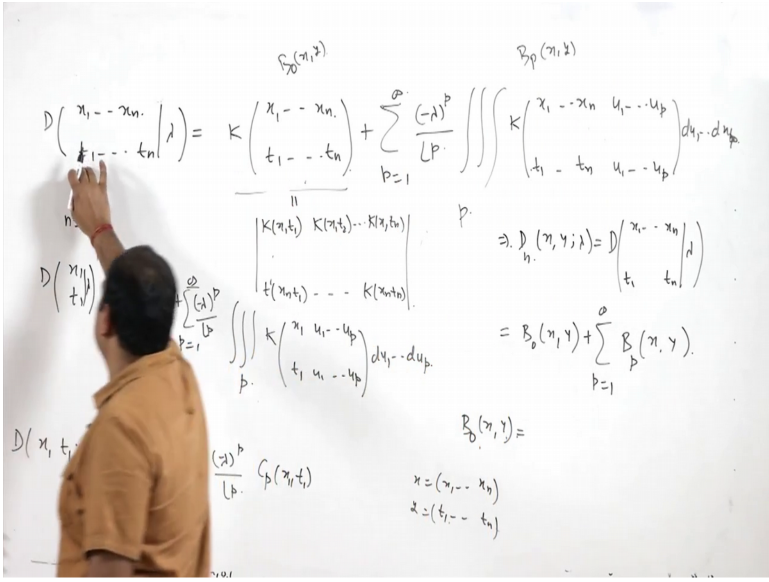

generalization of the first Fredholm minor, Ok. So we try to show that this is going to give you the solution of the homogenous Fredholm equation. So here let us relate this nth Fredholm minor

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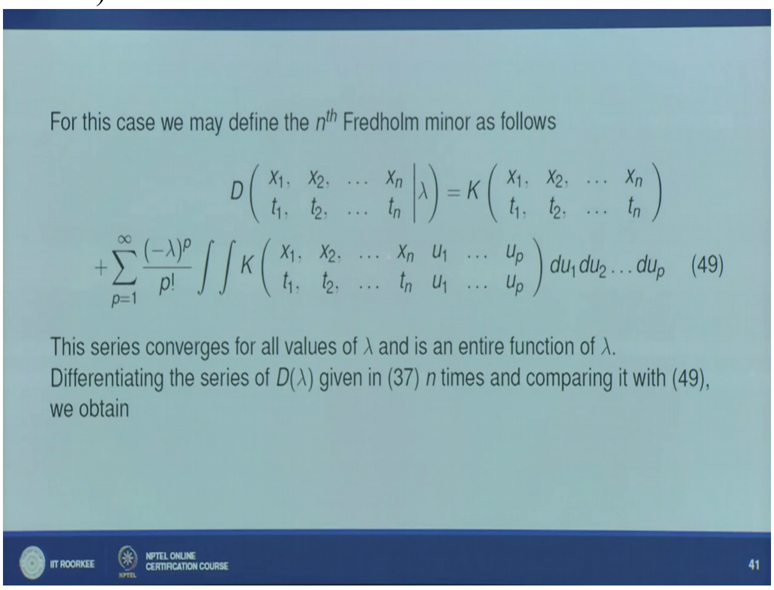

with the derivative of Fredholm determinant that is D lambda. So if we look at, we have already this kind of relation given as

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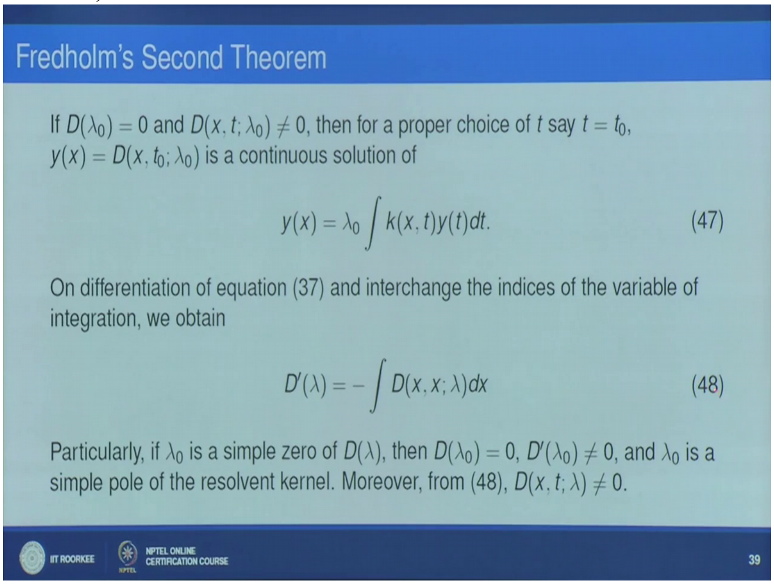

48 so when n equal to 1, we have a relation that D dash lambda is equal to minus D x comma x lambda. Now in the case when n is more than 1, then do we have any relation of this 48 kind? Then here we will not do it but when you differentiate D of lambda that is Fredholm determinant with n th times you will see that it is coming to be minus 1 to power n,

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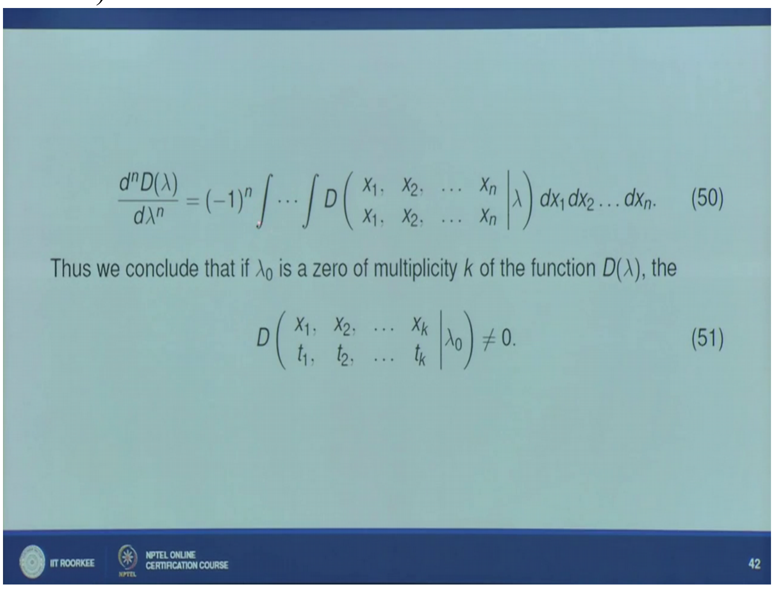

this is n times n-fold integral D of x 1 to x n, x 1 to x n lambda d x 1 to d x n.

So it means that we do have the similar kind of relation as we have seen that D dash lambda is equal to minus of integral D x comma x lambda. Similarly this equation number 55 shows that when n is more than 1, then also we have a similar kind of relation. And we say that if lambda naught is a zero of multiplicity k of the function D lambda then means what that D K lambda naught is non zero. So it means that with the help of this equation number 50, we can say that b kth minus kth Fredholm minor that is d x 1 to x k, x 1 to x k, t 1 to t k, this lambda naught is not equal to zero. This means that if this is a kind of relation between the zeroes of D lambda and the solution candidate, that is this. So it means that if lambda naught is a zero of multiplicity of k of this function D lambda then kth minor is not equal to zero. But we may have that there is a k minus 1th minor

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or say, k minus 2th minor may also not be equal to zero. So it means that we can always choose a number q which is less than k such that the qth minor of Fredholm minor, the kth minor of D lambda is not equal to zero, Ok. So with the help of that we can define the notion, that is index, Ok. So we will define that later on. But before that we will try to see that this nth minor, nth Fredholm minor will satisfy similar kind of integral equation. So if we know that this D x y lambda is equal to K of x t D lambda, I think I have to write t here, that D x t lambda is equal to K x t plus D lambda plus lambda a to b K of x of s D of s t lambda d of s. So it means that

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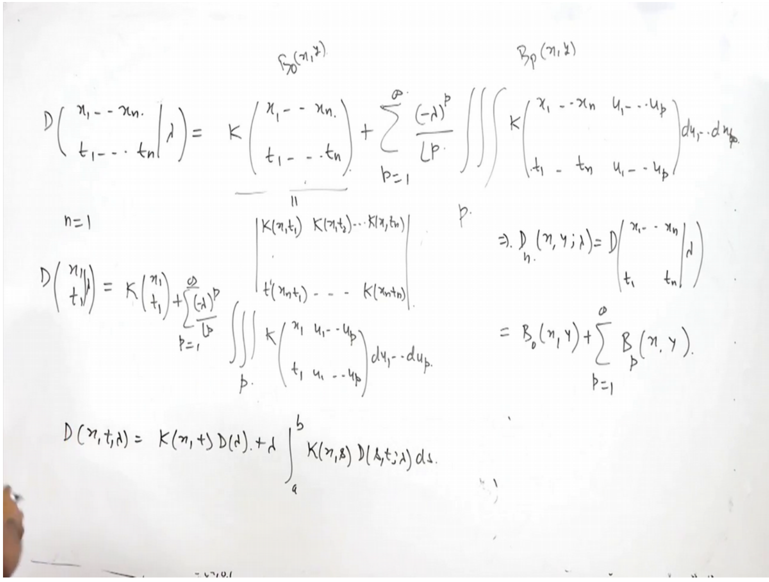

we already know that this Fredholm minor satisfies this relation, we want to know the generalization that is n th minor of, nth Fredholm minor will also satisfy similar kind of equation or not, so for that what try to do here,we look at the similar kind of expression.

So here if you have

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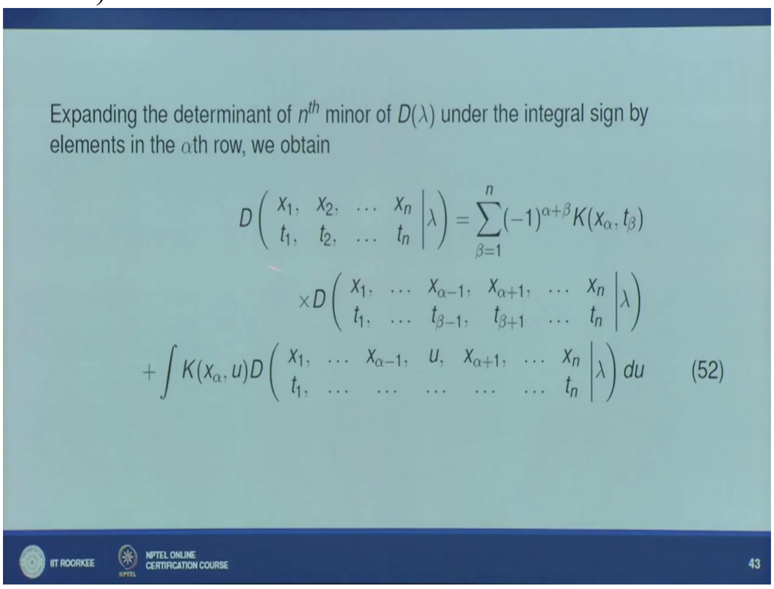

this expression nth minor of, nth Fredholm minor of D lambda and if we expand this determinant with the help of say, along the alpha th row and when we simplify we can get this kind of relation 52. This simplification involves a lot of say, calculation thing.

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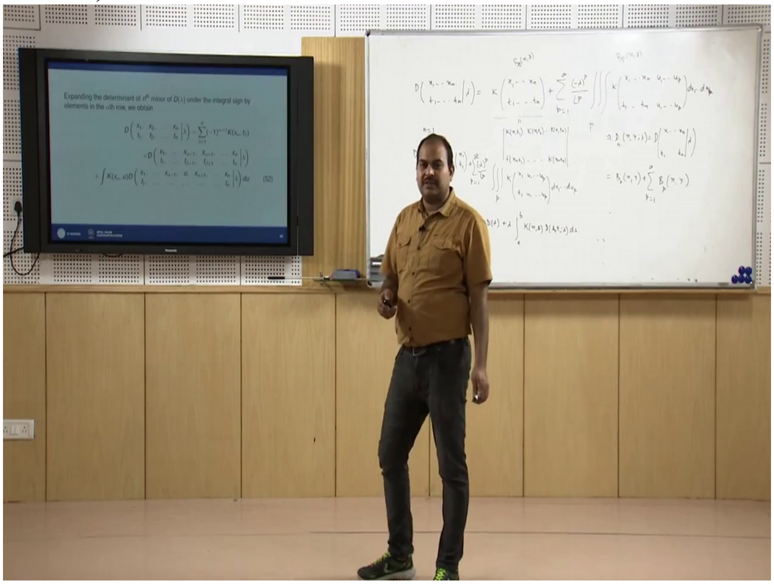

So I am not presenting here the calculation part. Only thing I am giving here is that this D, this nth minor of D lambda will satisfy this integral equation 52 and this when you are doing the expansion of your determinant

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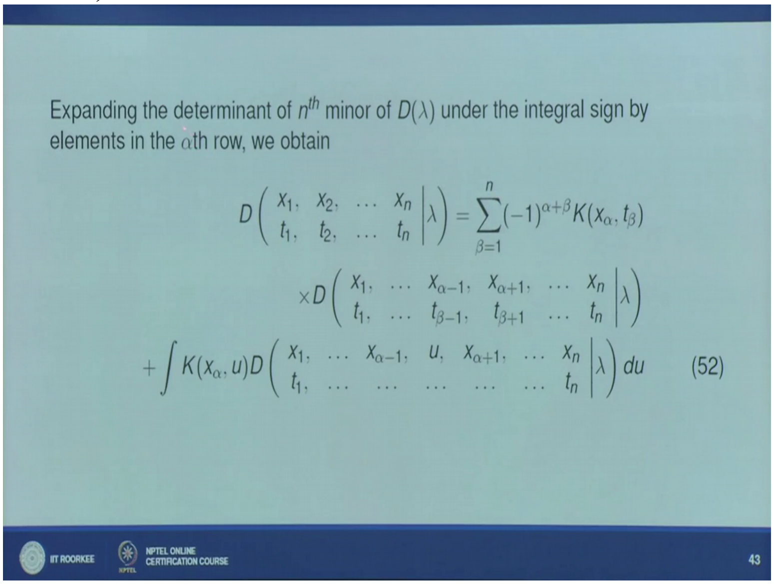

along the alpha th row. If you do the same expression with the beta th

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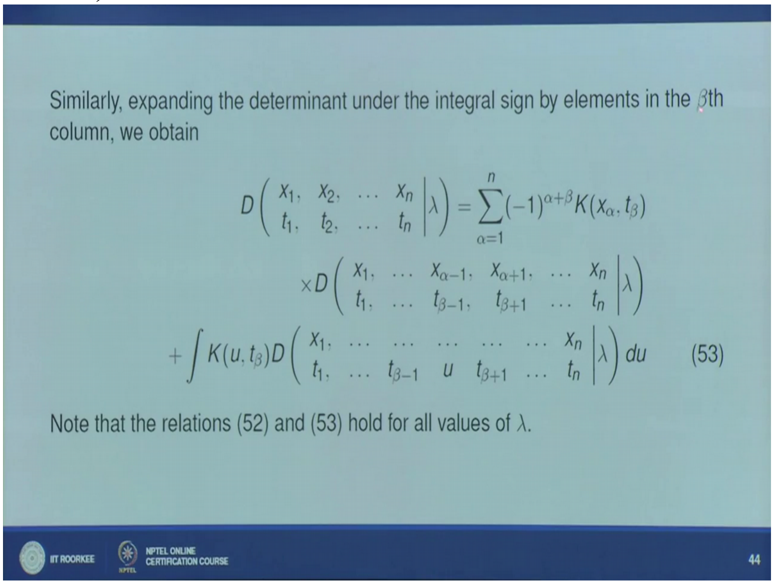

column, then we also have the similar kind of integral equation. So here if you look at this equation number 52 and the equation number 53 is kind of integral equation similar to the integral equation satisfied by

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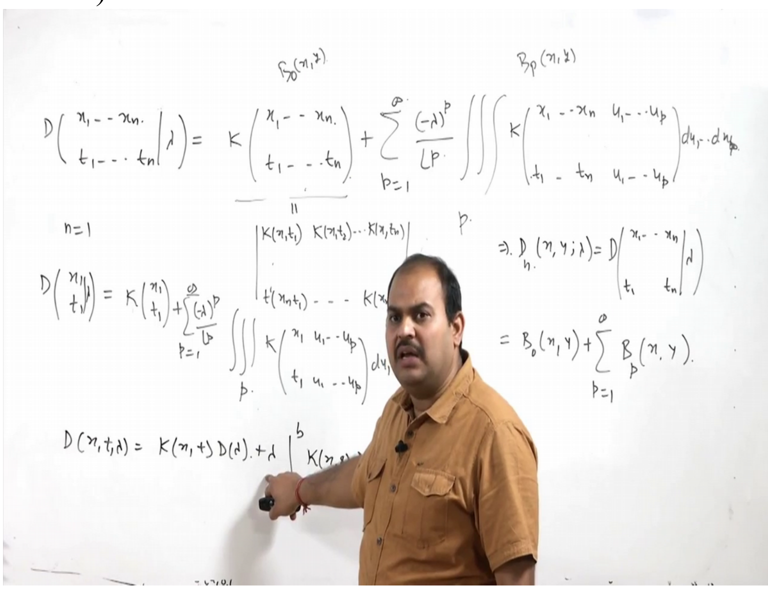

D x t lambda. So we try to see that this equation number

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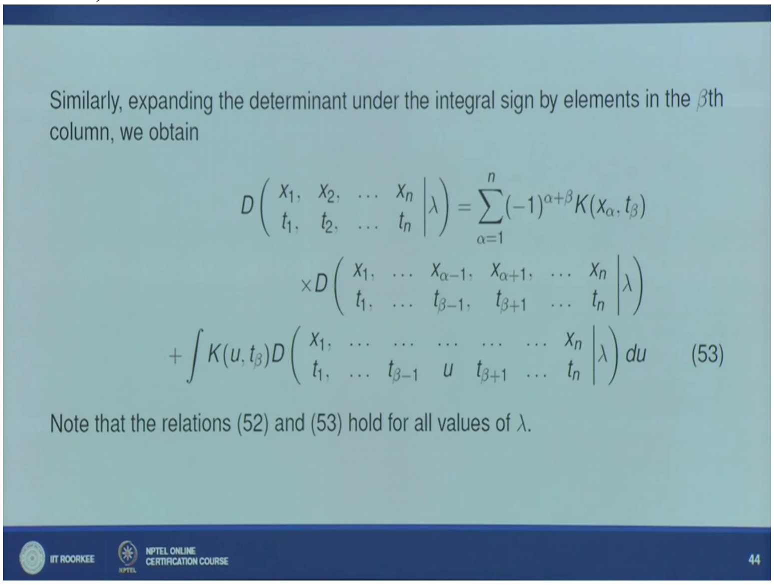

52 and 53 will give you, say some method to find out the solution when your D lambda is not equal to, D lambda is equal to zero. So here as we know that this expression, then D x t lambda

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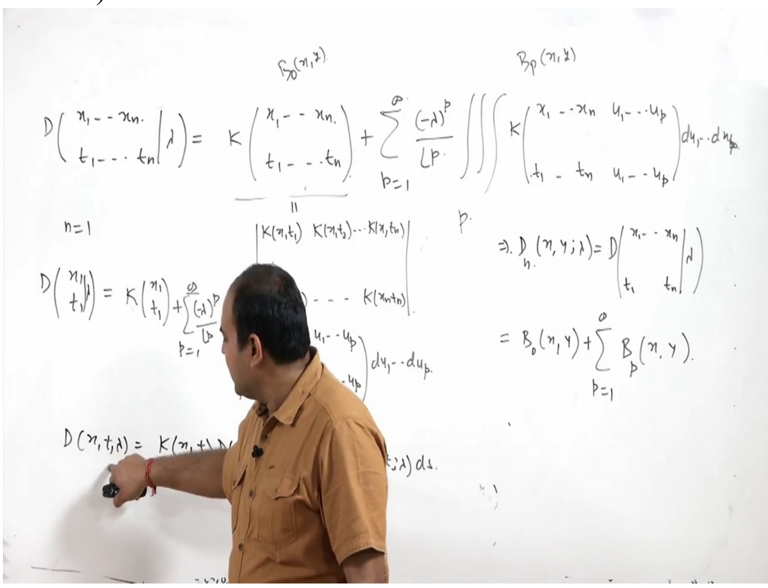

is a, basically you can find out this expression is valid for all values of lambda. In the same way you can say that your 52 and 53,

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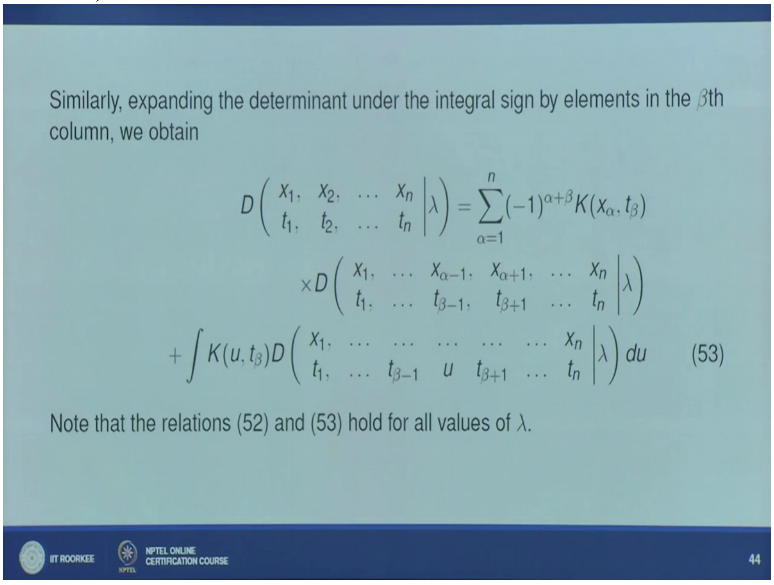

these two expression also hold for all values of lambda. So not only those values for which D lambda is not equal to zero. It is valid for all values of lambda.

So with the help of 52 and 53 we try to now discuss the case when D lambda naught, lambda naught is zero of D lambda. So now we can say that 52 and 53, we tried to get this, that we can find the solution

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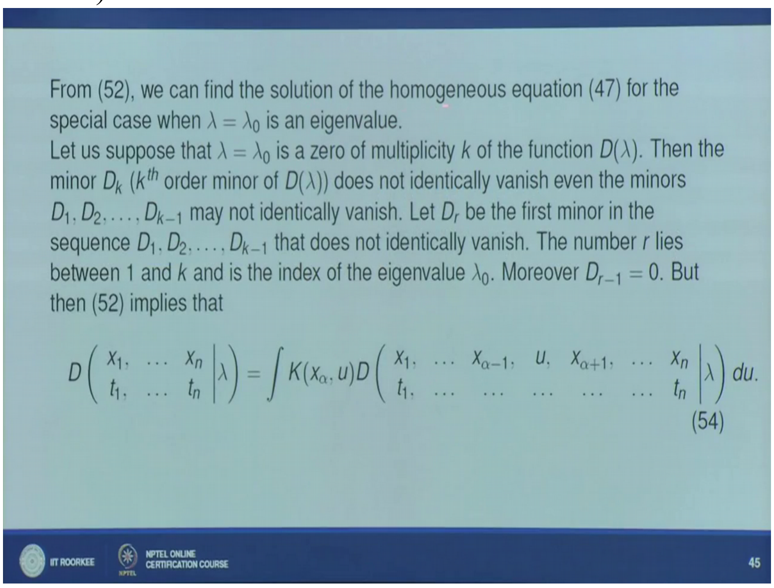

of the homogenous equation for the special case when lambda equal to lambda naught is an eigenvalue. So let us assume that this lambda equal to lambda naught is a zero of multiplicity k of the function D lambda. So it means that D lambda, D dash lambda

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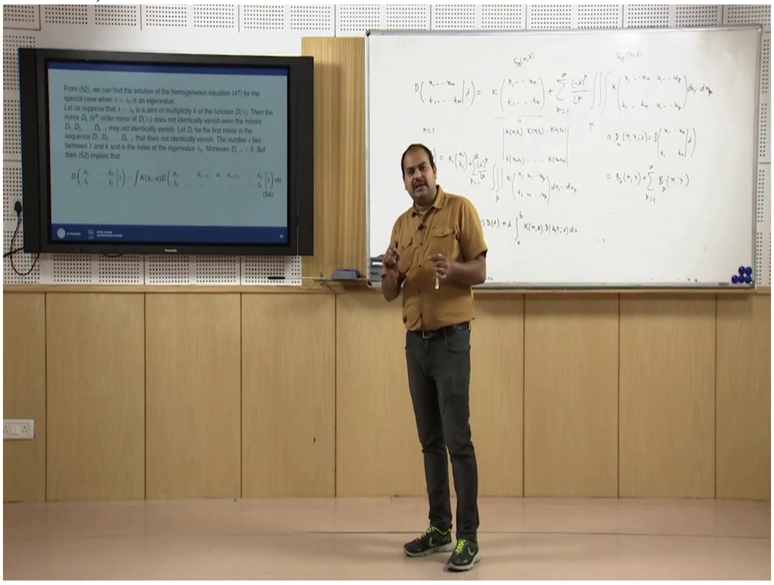

up to D to power k minus 1 lambda naught is equal to zero, let me write it here so that, so here we know that lambda equal to lambda naught is a zero of multiplicity k. So it means that D lambda naught is equal to zero and equal to D dash lambda naught is equal to zero and so on up to D k minus 1 lambda naught is equal to zero. And your

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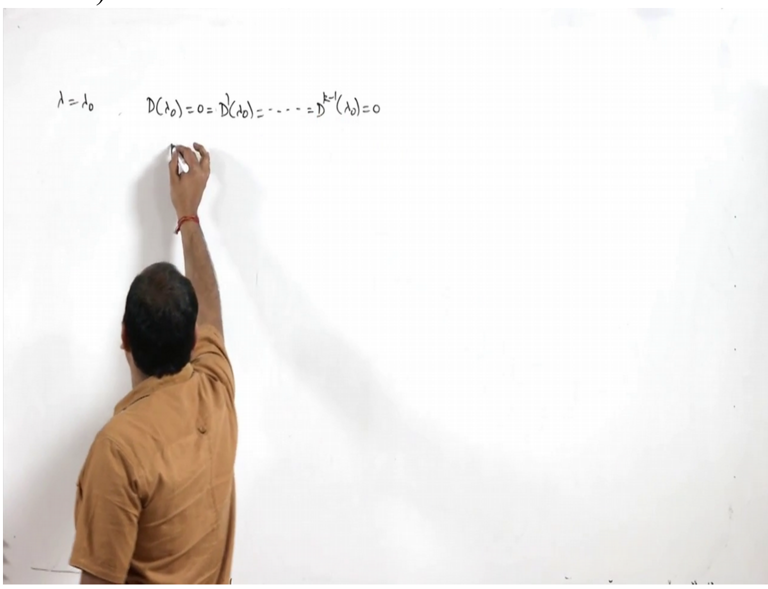

D k lambda naught is not equal to zero, Ok.

Now we know that, from this relation that is minus 1 to power k D k lambda naught

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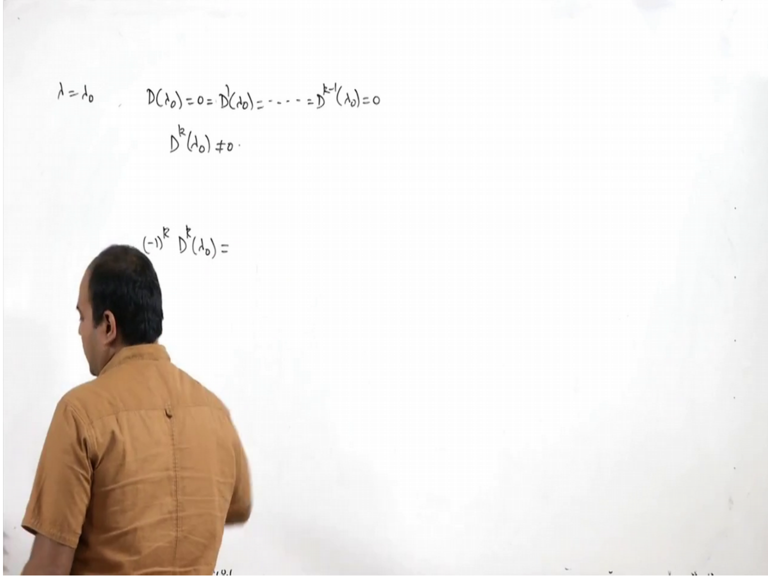

is equal to your relation, if you remember

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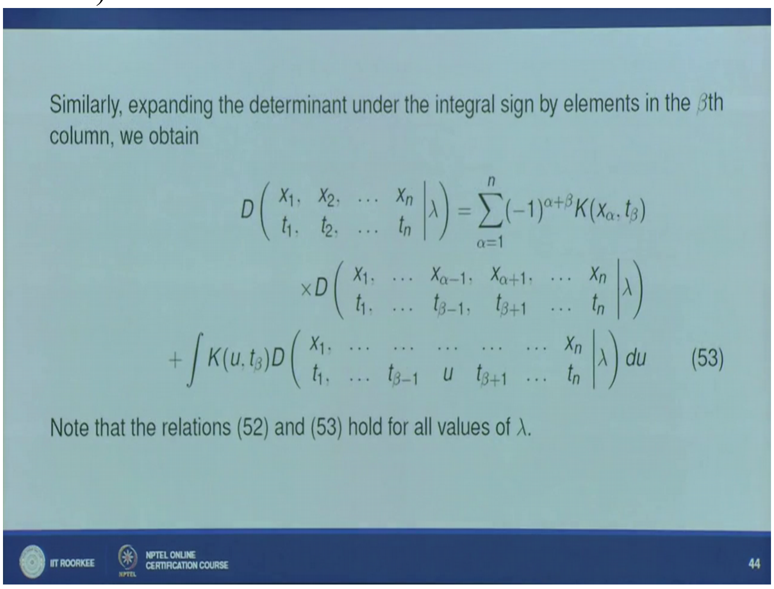

equation number 50 here, this relation that D k, d and D lambda, differentiation of D lambda with with respect to lambda nth time , this 50. So this means that if D k lambda is non zero, the corresponding kth minor is also non-zero, right? So it means that, this implies that the k th minor of D lambda is also not equal to zero. But

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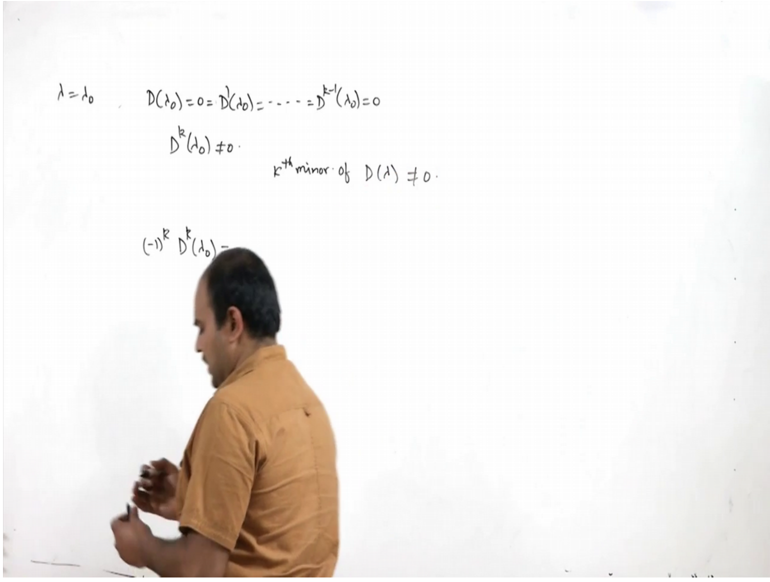

it may happen that this K minus 1th minor or say anything lesser than that is also equal to, is also not equal to zero. So what we try to do here, we try to find out say,

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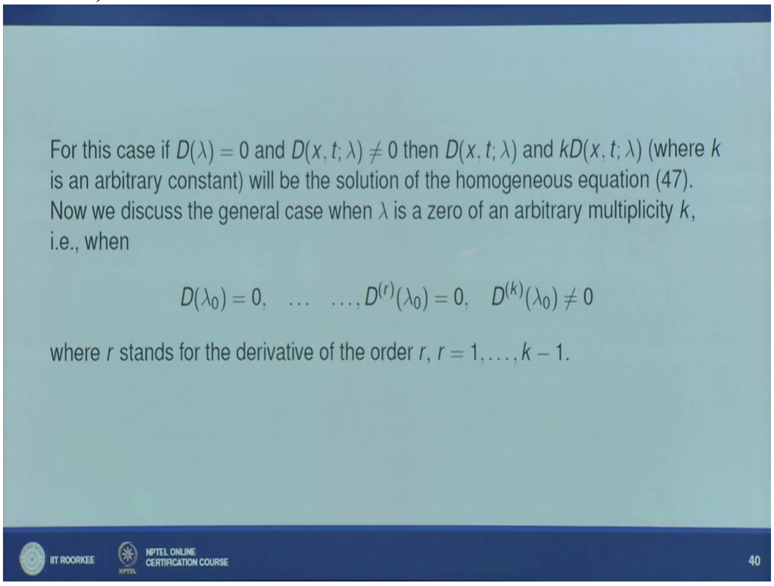

the value of r for which, just a minute, yeah, here, here

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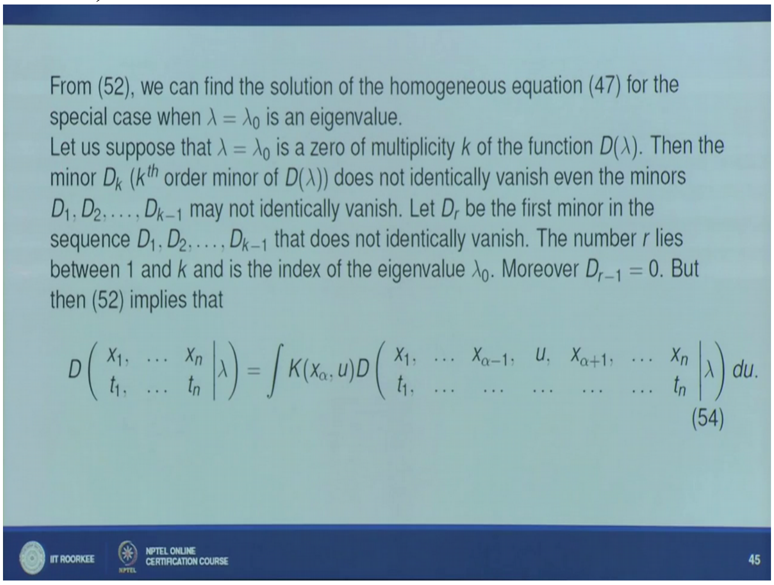

we already know that lambda equal to lambda naught is a zero of multiplicity k of the function D lambda then the minor D k does not identically vanish, even the minus D 1 to D k minus may not identically vanish. So let us find out, say r such that D r be the first minor in the sequence that does not identically vanish. So it means that, that before that it means that D r minus 1 is equal to zero. So that means that kth minor is not vanishing. It may happen that this k minus 1th member minor will also not vanish. So what we can

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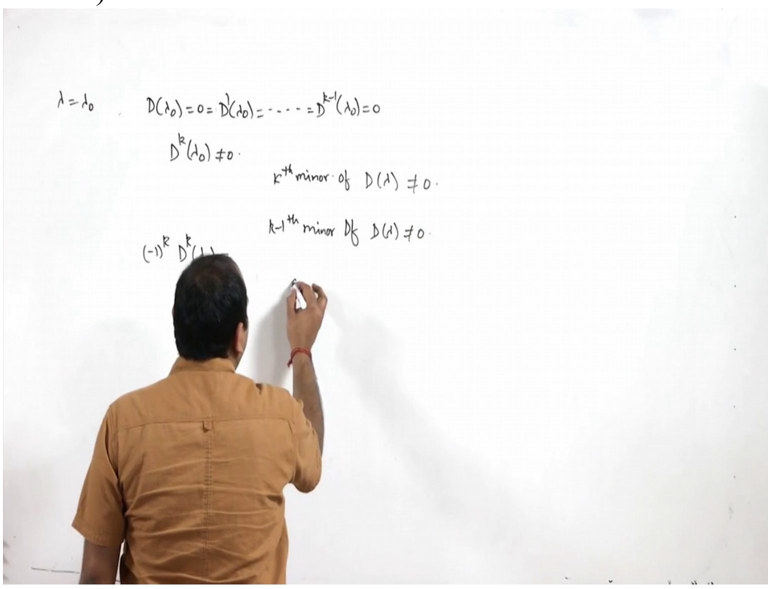

choose, say r which is between 1 to say, this k such that rth minor of D lambda does not vanish. But r minus 1th minor of D lambda is equal to zero. Is that Ok?

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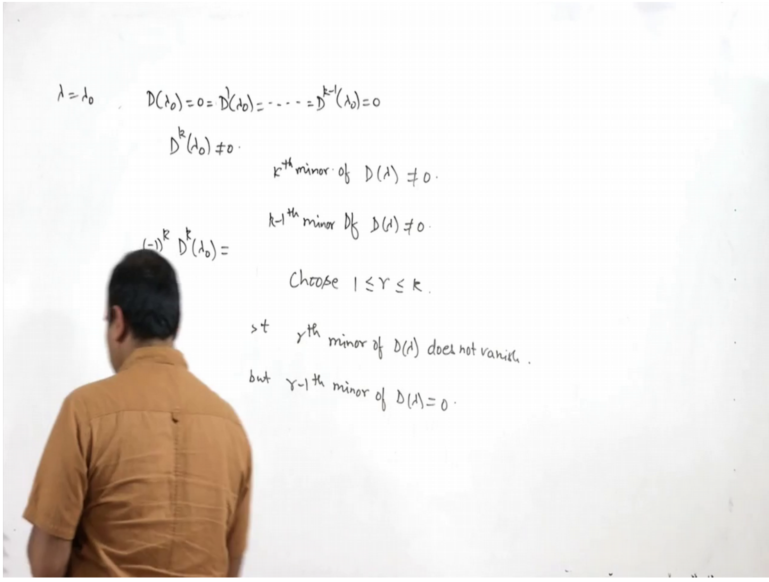

So it means that if we do this, such a number, this r, such a number is called index of lambda naught. So if r is the index of lambda naught that is rth minor of D lambda does not vanish but r minus 1th minor of D lambda is equal to zero, in fact any minor whose order is less than r is going to vanish.

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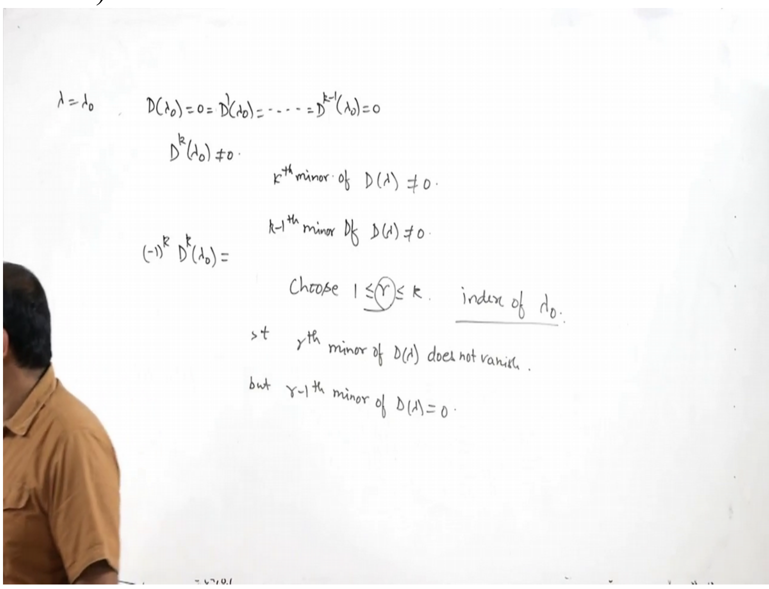

In that case equation number 52 reduces to this integral

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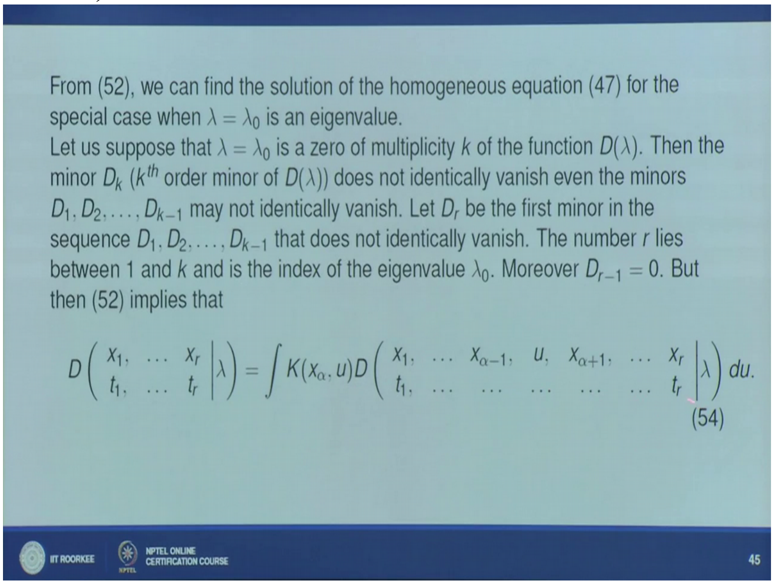

equation D x 1 to x r t 1 to r lambda equal to integral K x alpha u d x 1 to x alpha minus 1. Please remember that this K x alpha u means at alpha th position there is a u. So alpha th position here is replaced by this u d u. So if I, so we already know that r is the index of the number lambda naught then this expression is not going to be, so rth minor is not going to vanish. So it means that this is a non-zero quantity. So it means that this D x 1 to x r t 1 to t r, this r th minor satisfies the homogenous Fredholm integral equation. So here we can say

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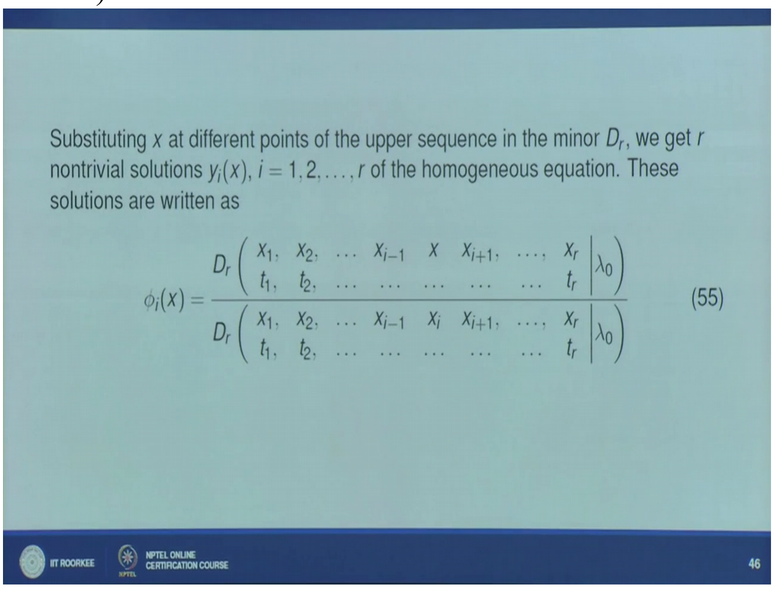

that if we substitute x at the different points of the upper sequence in the minor D r we can get our non trivial solution y j x which is defined like this. So here what we are doing here, we are replacing the ith upper, ith position of upper sequence by x and divided by D of r here and we can say that phi of x so this ratio which is, this ratio is the function of x and we can say that this phi of x is the solution of the Fredholm integral equation which

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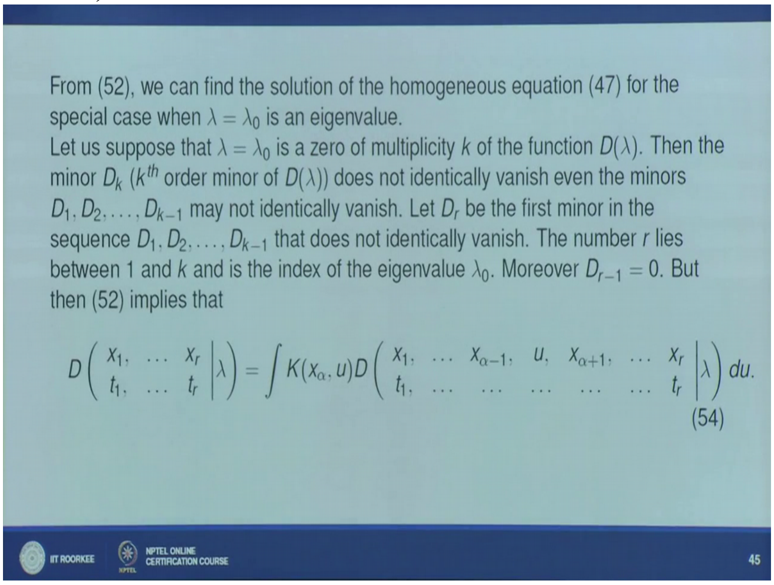

is homogenous here.

So here in this way we can get r number of linearly independent solution of this. Please look at here, the expression given in terms of 55, when here if I represent phi by x by that expression, then if I look at f i of x alpha, right so it means that at

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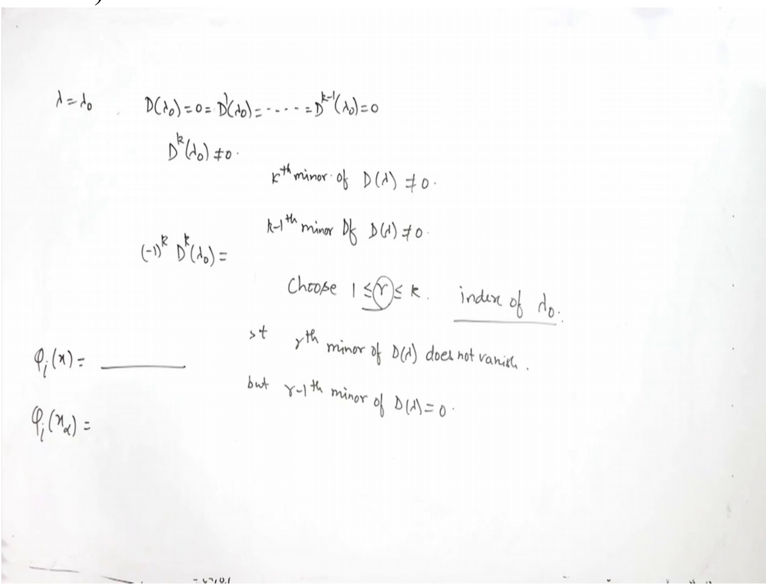

alpha th position if we take the x alpha so it means that if I take here, if we take, phi i x alpha

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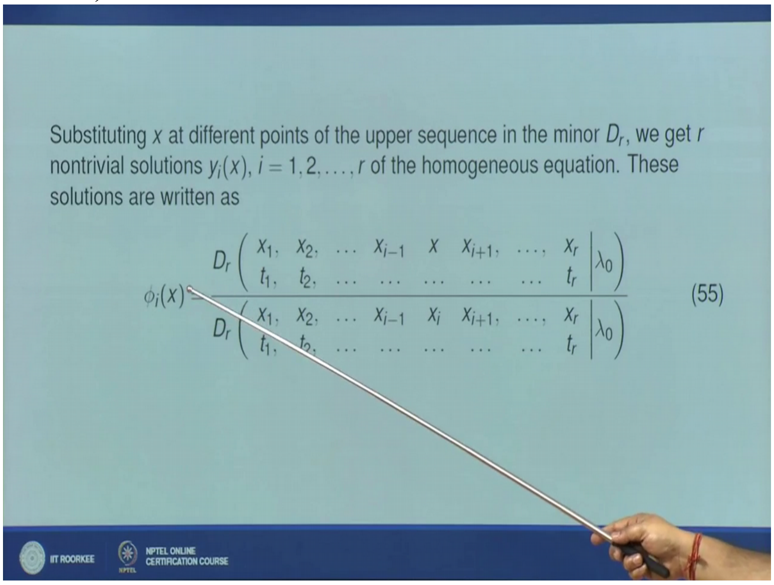

means, at ith position if we take x of alpha then this numerator is going to vanish. Because that represent that your ith row is identical to alpha th row here. So it means that if we take x as x alpha and alpha is not equal to i, then in that case numerator is going to vanish because of, in this determinant there are two rows which are identical to each other. But if I take the x as x i, it means here if we take x as x i, it means that phi i x i is, then this is, this expression is same as the expression given in denominator. So it means that in this case when x is replaced by x i then phi i x i is nothing but your y. So I can say that phi i x alpha is equal to zero if alpha naught equal to i and it is equal to 1 when alpha is equal to i. So this

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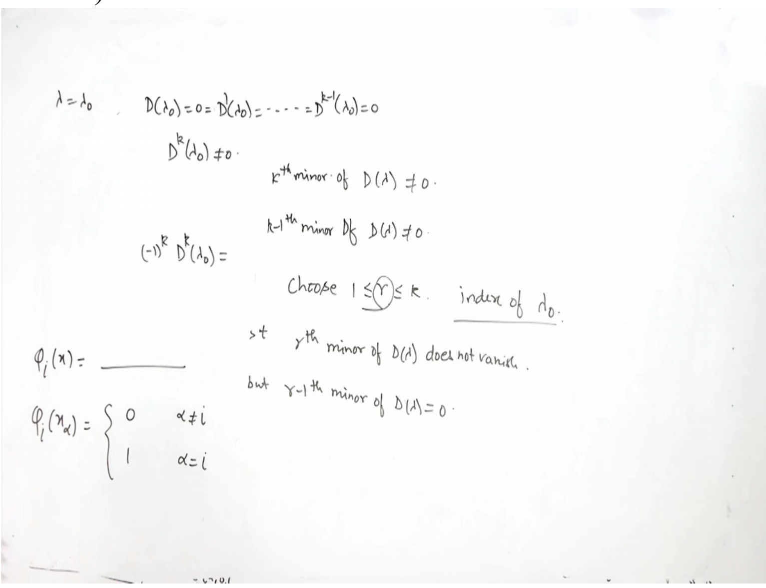

means that this phi i x given by this is having this particular property

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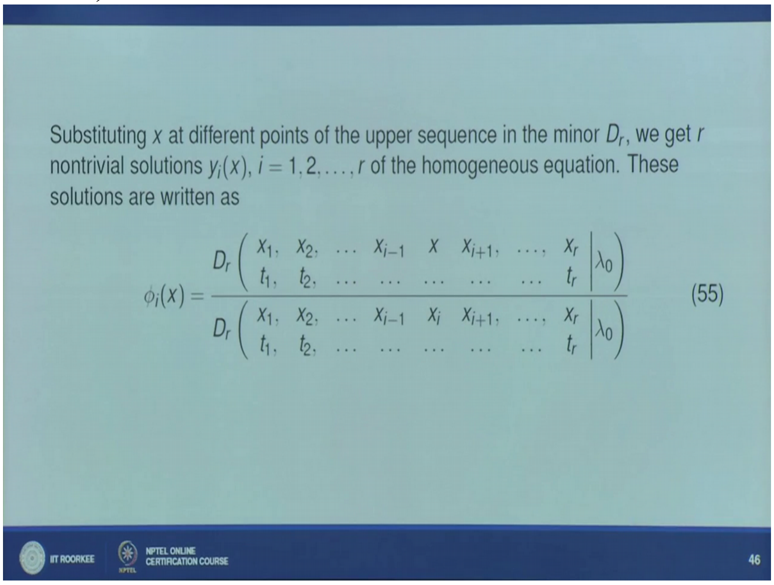

that phi i x alpha is equal to zero when alpha naught equal to y and equal to 1 when alpha equal to y. And with the help of this property you can easily check that this solution phi i x for i is equal to say 1 to say r, these are linearly independent solutions. In fact you can easily check that summation c i phi i x is equal to zero, e equal to 1 to r and then you can replace x equal to say, x j and you can check that

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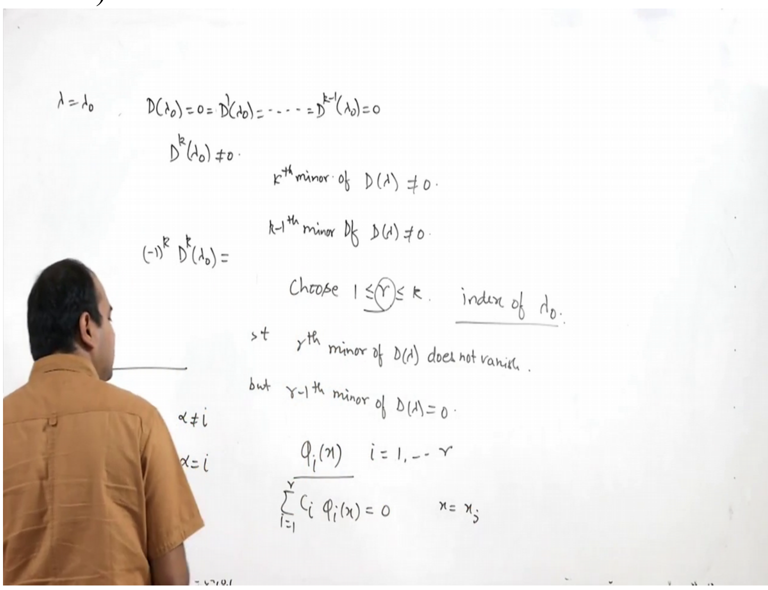

all the corresponding coefficients c ks are equal to zero. So in this way we can say that all these solutions given by

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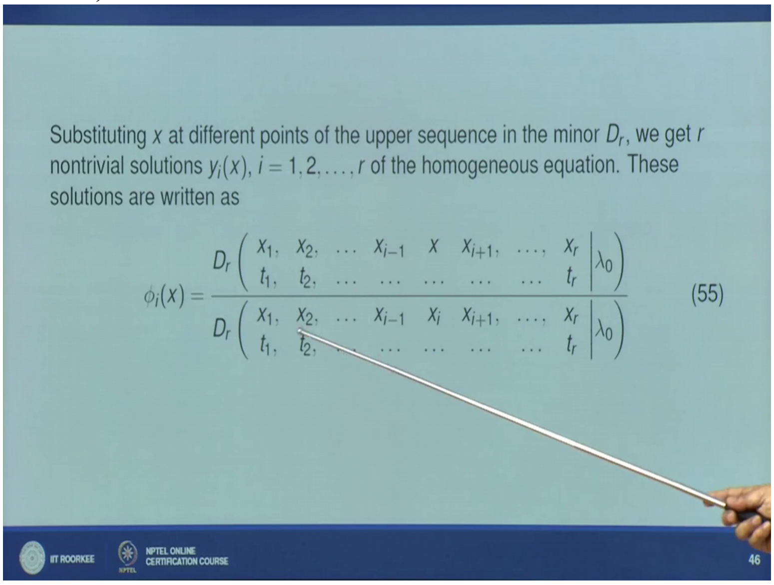

this phi i x are linearly independent.

Please remember that r is the index of D lambda corresponding to lambda equal to lambda naught. So here we can summarize the discussion in the theorem which is known as Fredholm's second theorem. It says that if lambda naught is zero

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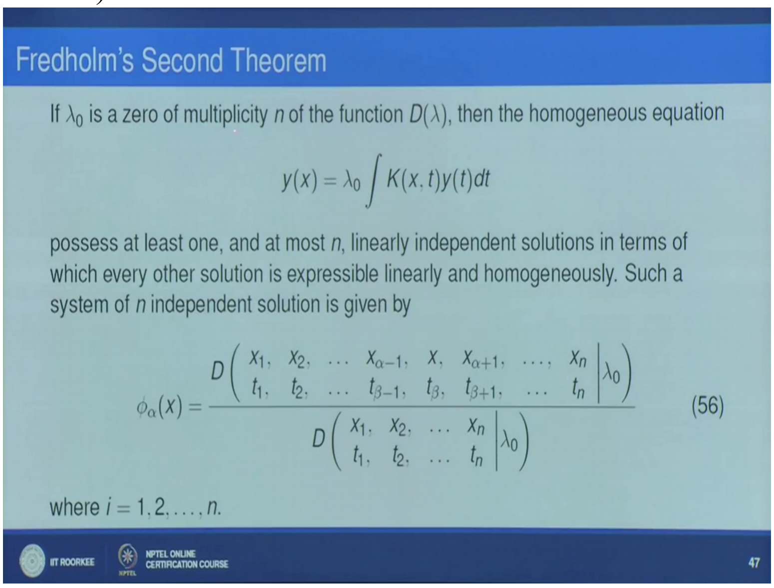

of multiplicity n of the function D lambda then the homogenous equation y x equal to lambda naught K x t y t d t possesses at least 1 and at most n linearly independent solutions in terms of which every other solution is expressible, linearly and homogenously. And this such, this system of n independent solutions is given by the expression here, phi alpha x equal to d x 1 to x l for minus 1at alpha th position, it is x, x alpha l to x n and this t 1 to t beta minus 1, t beta and t m. And in denominator it is the nth minor of, nth Fredholm minor of D lambda. Is that Ok?

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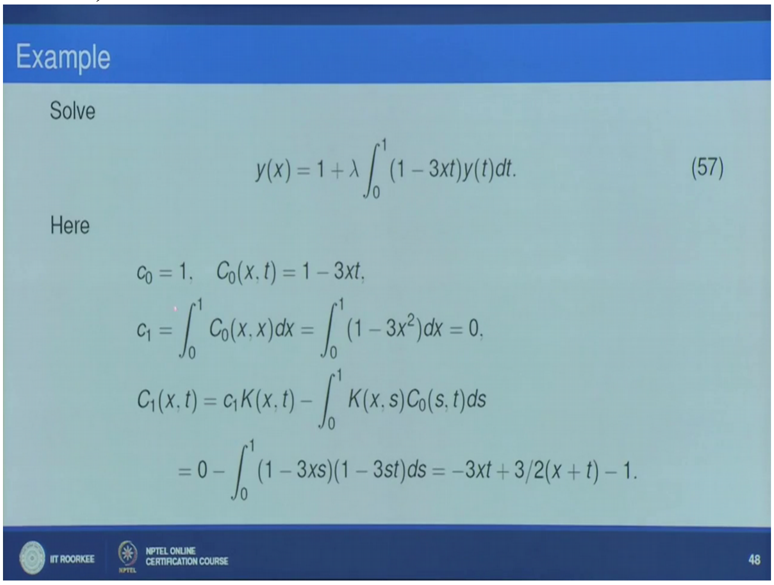

And then let us consider one

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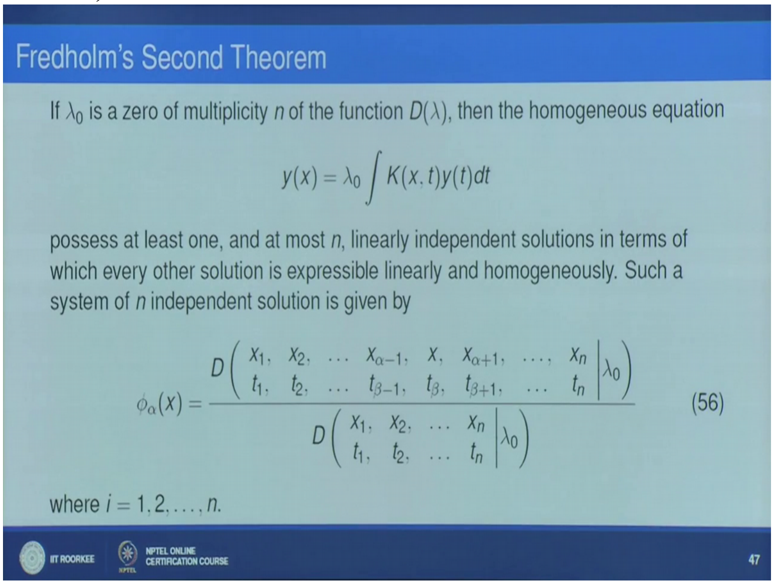

example based on Fredholm second theorem.

So here example is y of x equal to lambda zero to 1, 1 minus t x t y t d t. If you remember we have already discussed this example with the help of Fredholm first theorem. So here we considering the homogenous equation, now as we have done in

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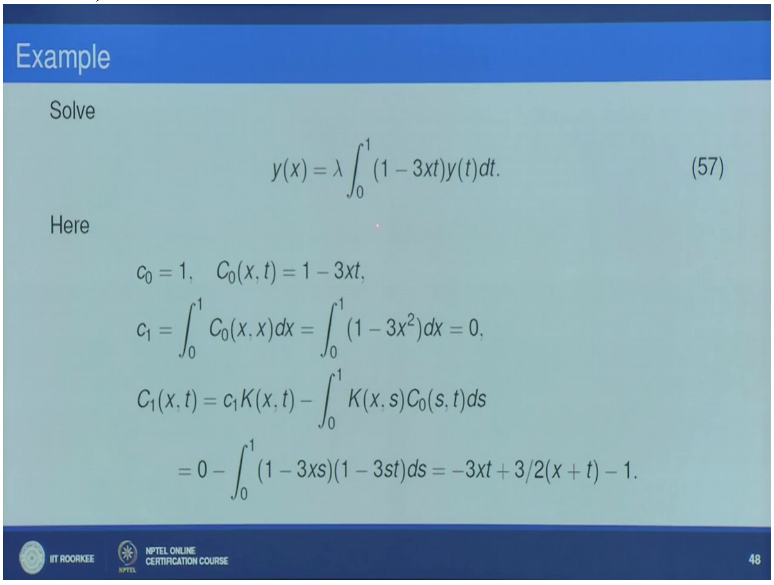

the previous lecture, we can calculate all the c is and you can see that your

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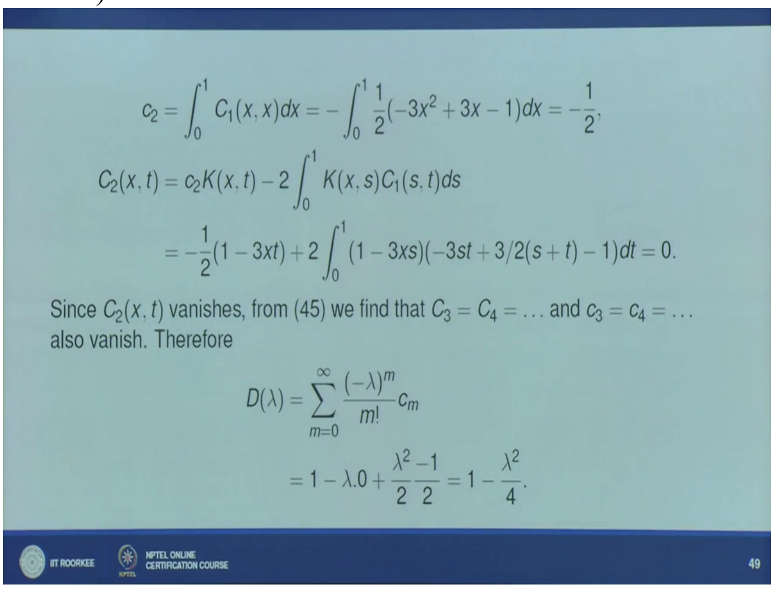

D lambda is given by m equal to zero to infinity minus lambda to power m factorial m small c m and all these c is are already calculated in previous lecture. Similarly we can write it here and you can say that D lambda is nothing but 1 minus lambda square by 4. And

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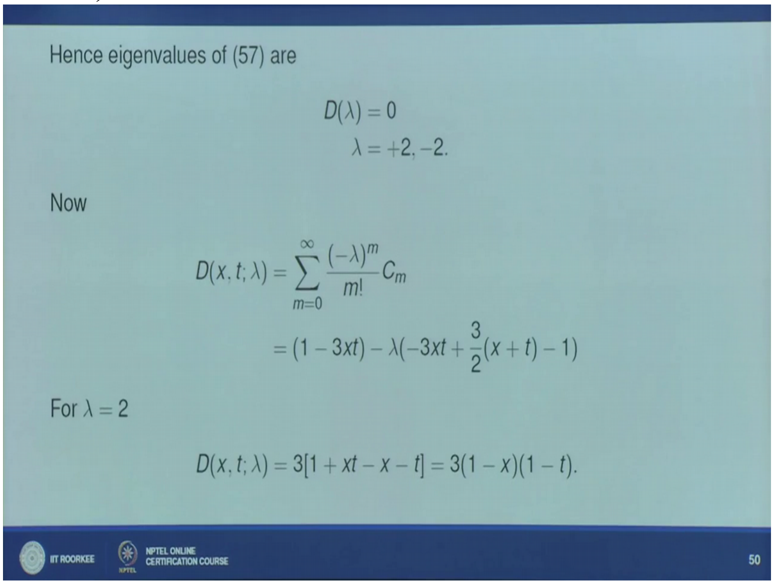

if you look at D lambda is equal to 1 minus lambda square by 4. So if I take lambda in a way such that D lambda is equal to zero. So let us take lambda equal to plus 2 or minus 2. So we can

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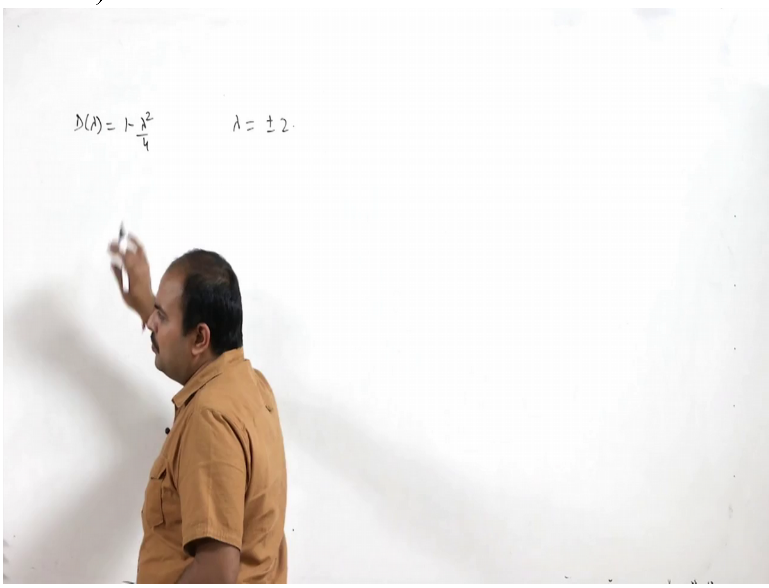

say that lambda equal to plus minus 2 is known as the eigenvalues of the equation, integral equation and in this case your D x t lambda which we have already calculated, let me write it here, D x t lambda is given as 1 minus 3 x t plus minus lambda times minus 3 x t plus 3 by 2 x plus t minus 1 and when you simplify this is

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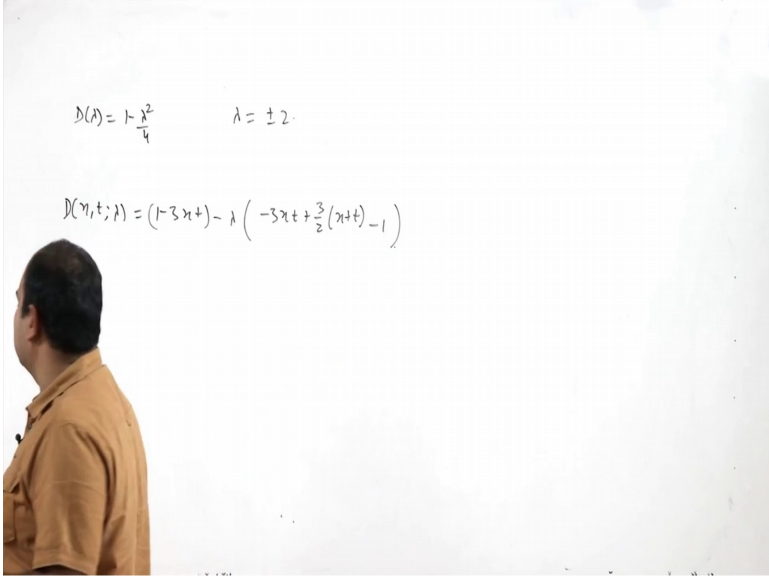

for lambda equal to 1, lambda equal to 2 d x t 2 is given as 3 of 1 minus x 1 minus t. So it means that

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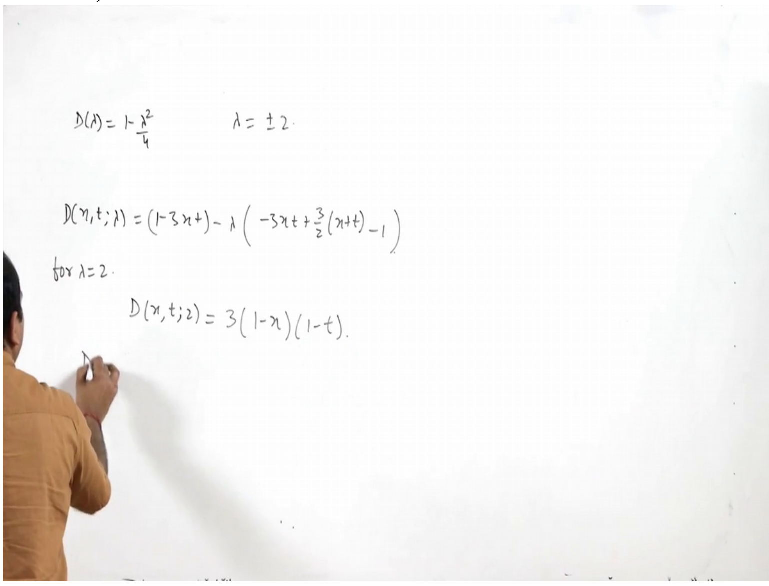

here your d of 2 is equal to zero but d dash lambda is not equal to zero at lambda equal to 2, that you can easily verify here. Then in this case we have already seen

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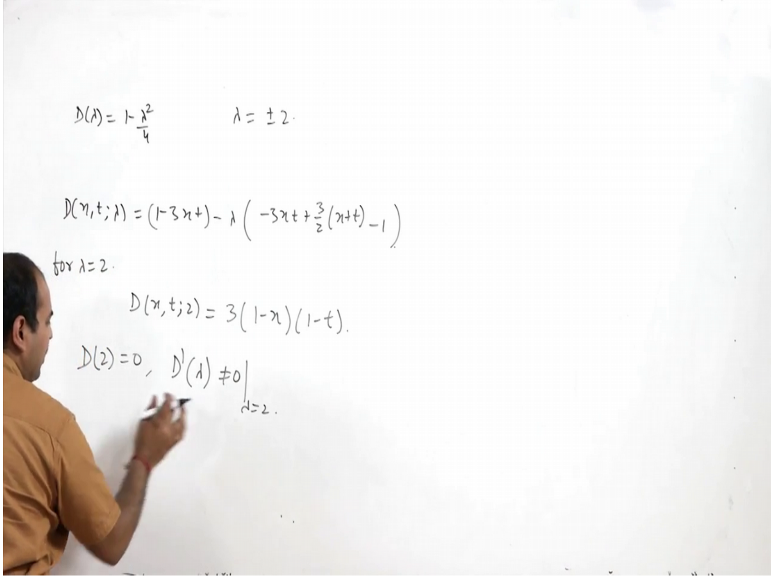

that in this case, your solution y of x is given as D of x, say some particular value of t, let us say t naught and 2, that is going to be solution of this. So here we

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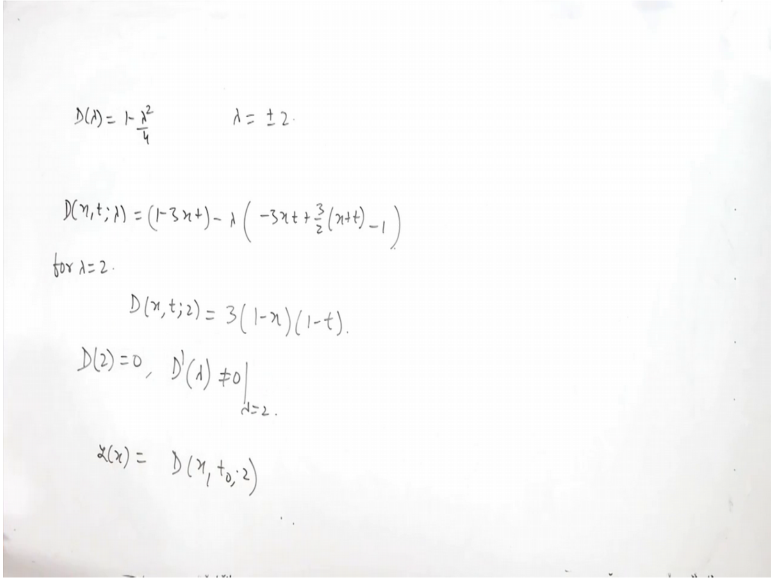

just try to see that for lambda is equal to 2, D x t lambda is reduced to 3 1 minus x l minus t here

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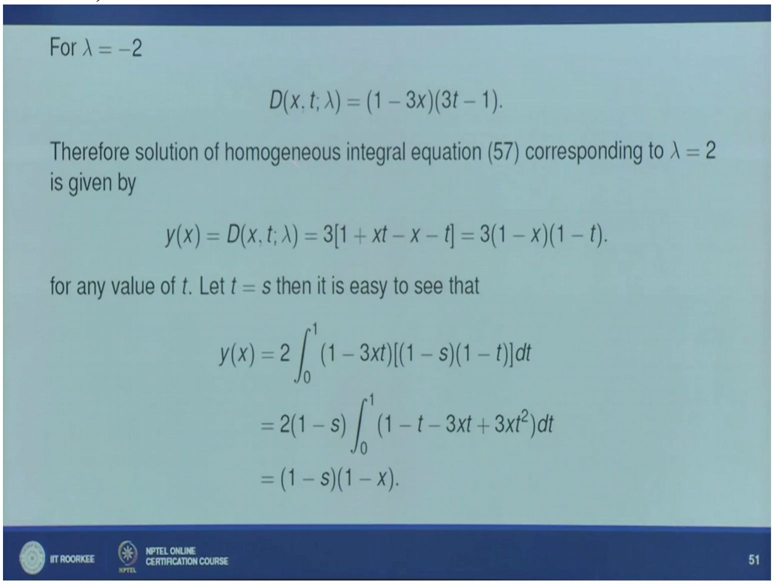

and for lambda equal to 2, -2 you can say that D x t lambda is given by 1 minus 3 s into 3 minus 1 So we can say that in this case if we apply the result which is Fredholm's second theorem that

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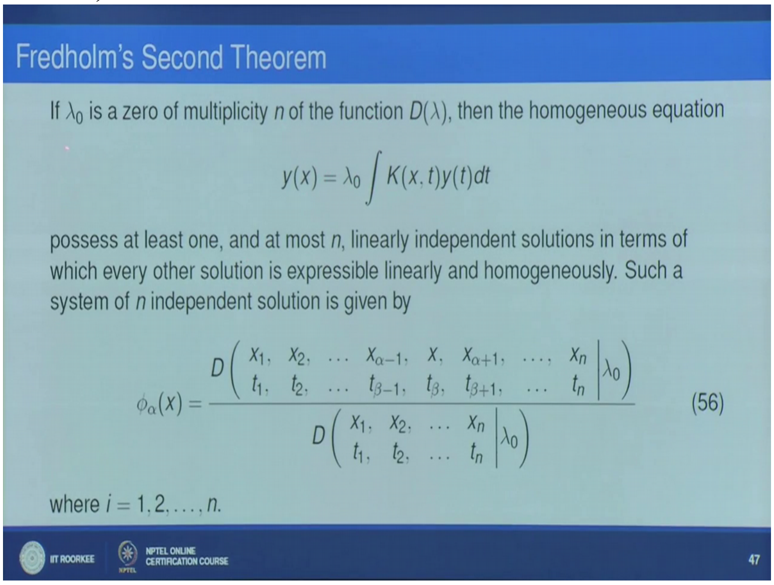

if lambda naught is a zero of multiplicity n so here n is equal to 1. In that case we have at least 1 and at most 1. So we can say that at least 1

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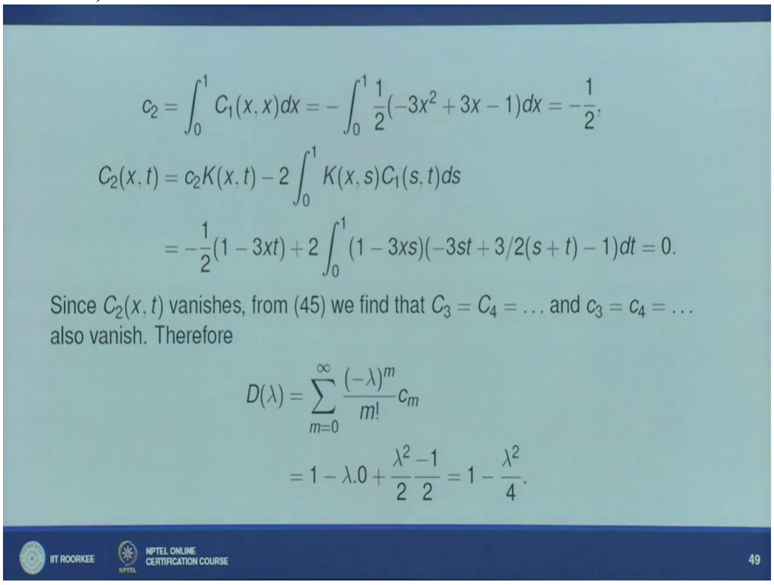

solution is

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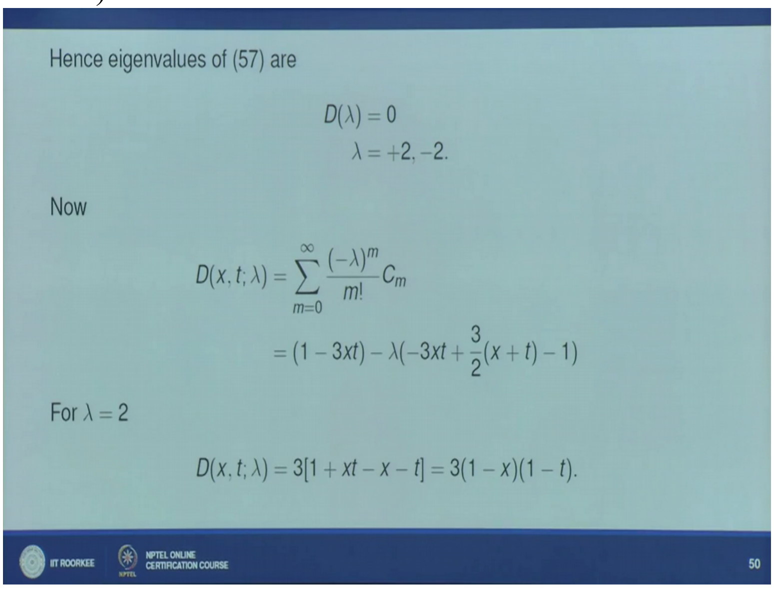

given and that solution you can find out by nth minor of your Fredholm, of D lambda so here first minor of D lambda is given by D x t lambda. Ok

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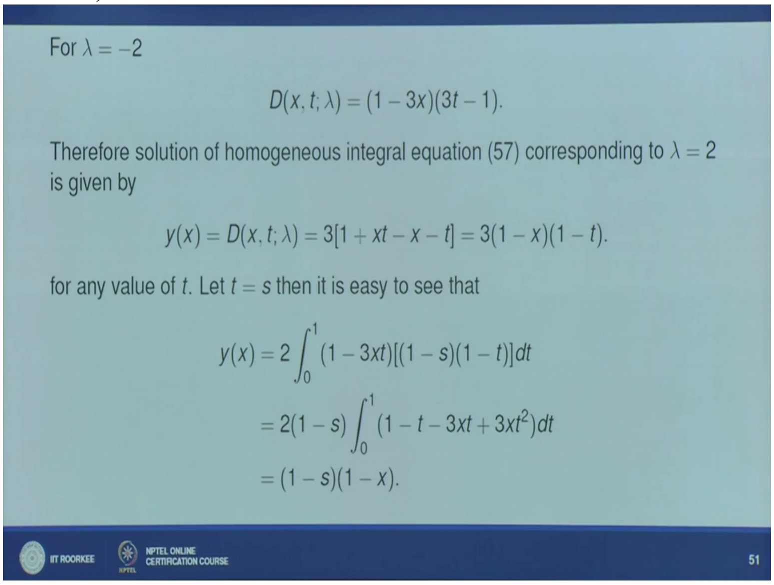

so here we can say that corresponding to lambda equal to 2, the solution y of x is given by D of x t lambda. So here lambda is equal to 2 and you can take any value of t. So when you put lambda equal to 2, it is reduced to 3 into 1 minus x 1 minus 2. So it means, our claim is that for any value of t, this expression 3 into 1 minus x 1 minus t is a solution of the homogenous integral equation.

So that we can verify here. So we can take let us say, t equal to s. You can take any value of t here. So let t equal to

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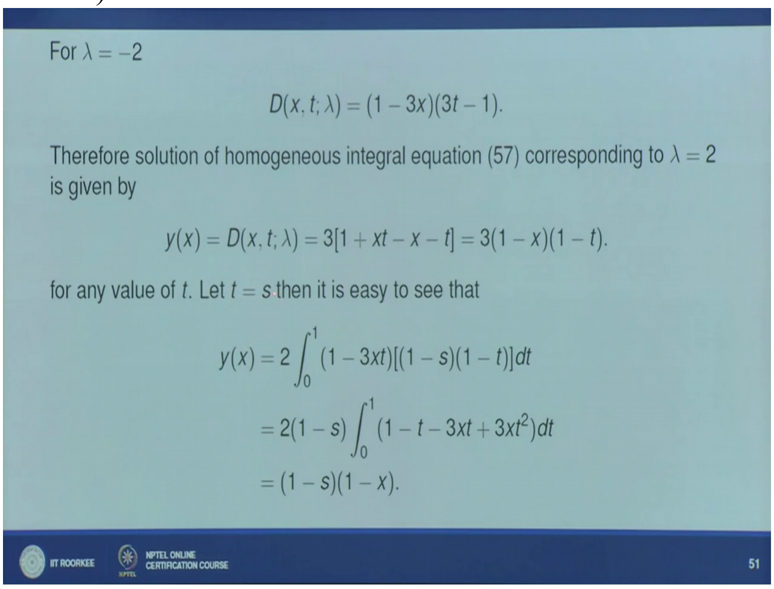

s and we can verify then this expression D x t lambda is going to be a solution of this. If you calculate it is coming out to be 1 minus s 1 minus x. It means that this expression gives you a solution of your homogenous integral equation for lambda equal to 2. Similarly you can do for lambda equal to minus 2. So you can check that for lambda equal to minus 2, this expression D x t lambda equal to 1 minus 3 x t 3 minus 1 is a solution of the homogenous integral equation for any values of t between this 0 to 1, Ok.

So let us try to find out the solution

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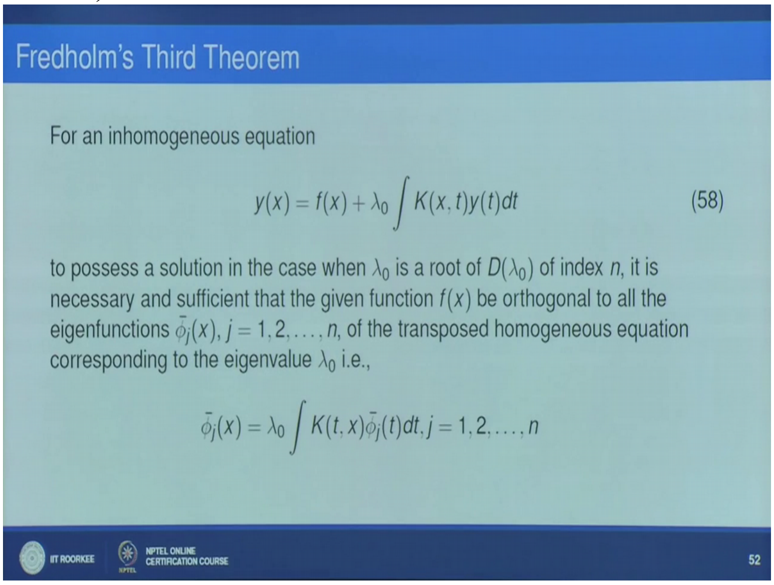

for non-homogenous equation. So that is the content of Fredholm's third theorem which says that this inhomogenous equation or non-homogenous equation y x equal to f x plus lambda naught K a to b K x t y t d t will have a solution in a case when lambda naught is a zero of, is a root of D lambda naught of index n. And the necessary and sufficient condition for this is that the function f of x is orthogonal to all the eigen function phi j bar x of the transpose homogenous equation. What is the transpose homogenous equation? That your phi j bar x is the solution of this equation lambda naught a to b K t x phi j bar t d t. So if you remember the homogenous part, the homogenous part is lambda naught K x t y t d t. So the transpose equation is this, lambda naught K t x phi j bar t d t. So if we can find out the solution of this transpose homogenous equation with the help of Fredholm's second theorem, and f x is orthogonal to all those phi j bar x then your solution is given by this, y of x

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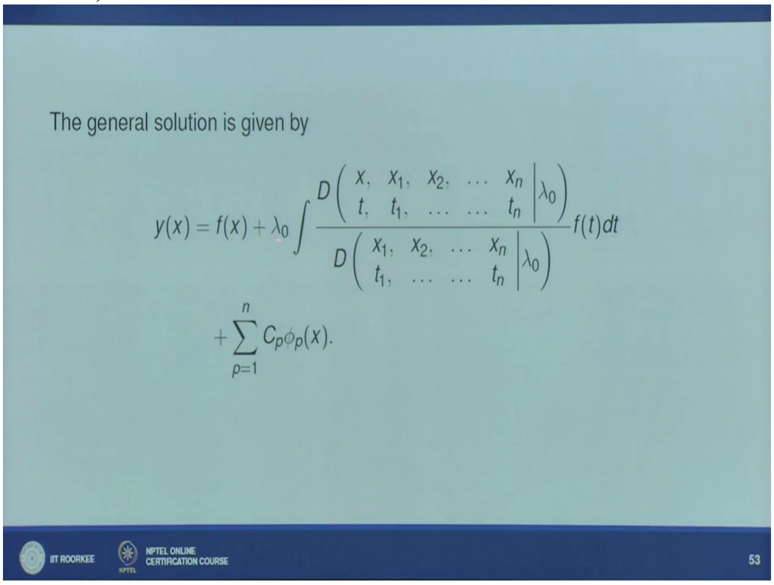

equal to f of x plus lambda naught and if you remember we have, this is nothing but phi j x here. So plus p equal to 1 to n c p phi p x. So here this we can denote as say some h x y lambda naught, this is kind of resolvent kernel in this particular case. And this phi p is the solution of the homogenous integral

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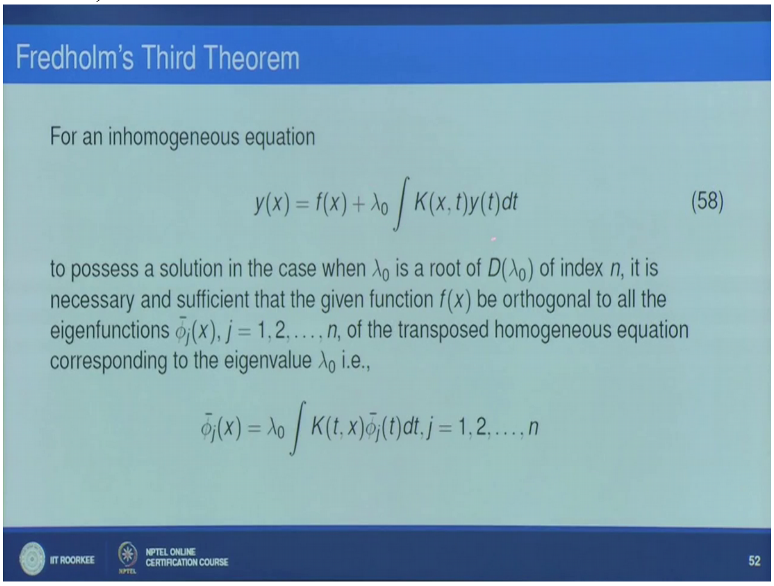

equation. So it means that in this case, if f x satisfies this condition that it is orthogonal to this, and phi j bar satisfies the associated transpose equation, then solution

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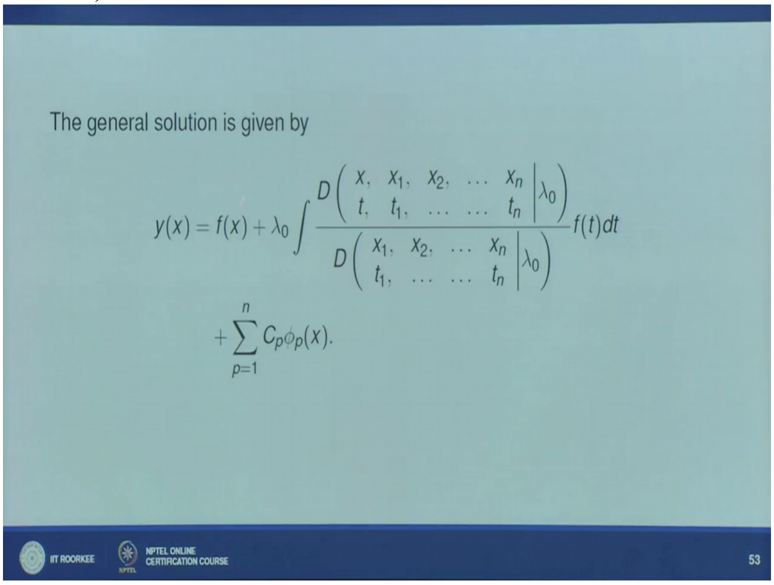

is given by this. And so we can find with the help of Fredholm second theorems which we have discussed earlier. This is the simple generalization of Fredholm alternative theorem. So here we will stop.

And so we can summarize it like this. That Fredholm first theorem is giving you the solution method when D lambda naught is not equal to zero. And second theorem gives you the solution of Fredholm integral, Fredholm integral equation which is homogenous in the case when D lambda naught is equal to zero or you can say that lambda naught is a root of D lambda of multiplicity say m. And in Fredholm third theorem we try to find out the solution of non-homogenous integral equation. And thank you for listening us. We will meet in a next lecture. Thank you.