Integral Equations, Calculus of Variations and their Applications
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Mod 06 Lecture Number 22
Classical Fredholm Theory Fredholm First Theorem-II

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Hello friends welcome to the today's lecture In this lecture we will discuss the Fredholm's first theorem we have discussed in say previous lectures so we will discuss some examples and Fredholm second theorem and third theorem in the case when Fredholm determinant is equal to zero It means that the lambda is a root of Fredholm determinant

So if you remember if you recall in previous classes we have discussed that the Fredholm equation

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Fredholm's First Theorem

The Fredholm equation

$$y(x) = f(x) + \lambda \int K(x,t)y(t)dt. \tag{33}$$

where the functions $f(x)$ and $k(x,t)$ are integrable, has a unique solution

$$y(x) = f(x) + \lambda \int \Gamma(x,t;\lambda)f(t)dt, \tag{34}$$

where the resolvent kernel,

$$\Gamma(x,t;\lambda) = D(x,t;\lambda)/D(\lambda), \tag{35}$$

with $D(\lambda) \neq 0$, is a meromorphic function of the complex variable λ, being the ratio of two entire functions defined by the series

y x equal to f of x plus lambda K x t y t D t and we assuming that f of x and K x are integrable has a unique solution equal to y x equal to f of x plus lambda gamma x t lambda f t d t where gamma x t lambda is known as resolvent kernel and it can be found as a ratio of two determinants D x t lambda divided D lambda Here D lambda is the Fredholm determinant and D x t lambda is the Fredholm minor corresponding to D lambda and here both D x t lambda and D lambda are m-tier functions and it is defined by we will see how it is defined And in this way this gamma x t lambda is a meromorphic function of the complex variable lambda and D x t lambda and D lambda

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$$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \cdots \int K \begin{pmatrix} x, & x_1, & \cdots, & x_m \\ t, & x_1, & \cdots, & x_m \end{pmatrix} dx_1 \cdots dx_m,$$
$$\tag{36}$$

and

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \cdots \int K \begin{pmatrix} x_1, & x_2, & \cdots, & x_m \\ x_1, & x_2, & \cdots, & x_m \end{pmatrix} dx_1 \cdots dx_m, \tag{37}$$

where

$$K \begin{pmatrix} x, & x_1, & \cdots, & x_m \\ t, & x_1, & \cdots, & x_m \end{pmatrix} = \begin{vmatrix} K(x,t) & K(x,x_1) & \cdots & K(x,x_m) \\ K(x_1,t) & K(x_1,x_1) & \cdots & K(x_1,x_m) \\ \vdots & \vdots & \vdots & \vdots \\ K(x_m,t) & K(x_m,x_1) & \cdots & K(x_m,x_m) \end{vmatrix} \tag{38}$$

is defined by the following infinite series D x t lambda is given by this and D lambda is this and where this symbol K x 1 to x m x 1 to x m is defined by equation number 38
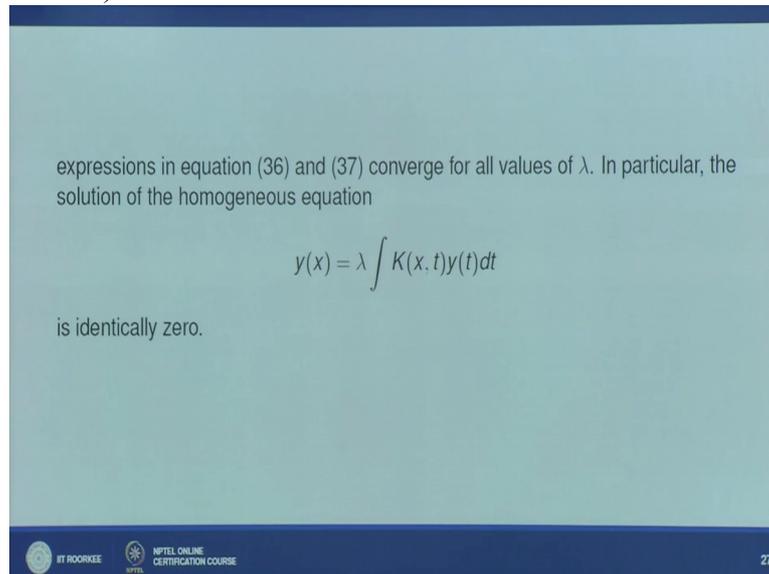
So and

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we can say that in particular if the

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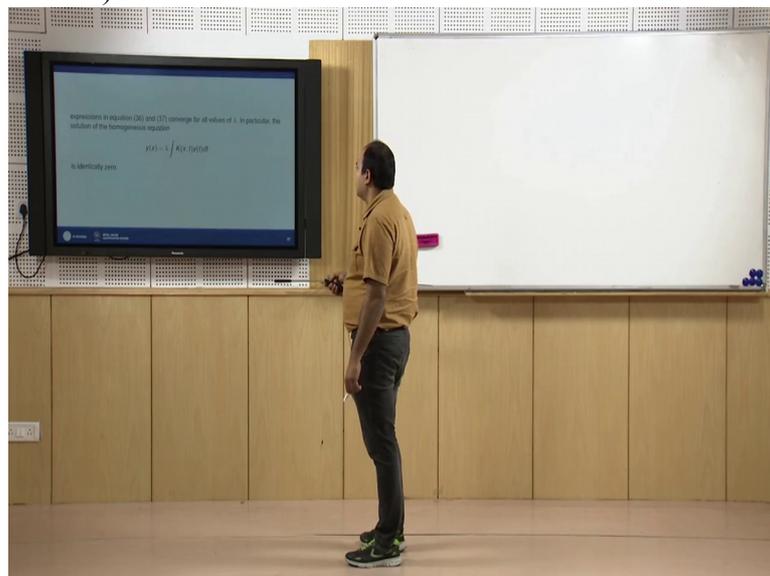
homogenous part that is f of x is simply zero then y x will have only a trivial solution

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The Fredholm equation

$$y(x) = f(x) + \lambda \int K(x,t)y(t)dt, \tag{33}$$

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$$y(x) = f(x) + \lambda \int \Gamma(x,t;\lambda)f(t)dt, \tag{34}$$

where the resolvent kernel,

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with $D(\lambda) \neq 0$, is a meromorphic function of the complex variable λ, being the ratio of two entire functions defined by the series

So it means that

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$$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \cdots \int K\begin{pmatrix} x, & x_1, & \cdots, & x_m \\ t, & x_1, & \cdots, & x_m \end{pmatrix} dx_1 \cdots dx_m, \tag{36}$$

and

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \cdots \int K\begin{pmatrix} x_1, & x_2, & \cdots, & x_m \\ x_1, & x_2, & \cdots, & x_m \end{pmatrix} dx_1 \cdots dx_m, \tag{37}$$

where

$$K\begin{pmatrix} x, & x_1, & \cdots, & x_m \\ t, & x_1, & \cdots, & x_m \end{pmatrix} = \begin{vmatrix} K(x,t) & K(x,x_1) & \cdots & K(x,x_m) \\ K(x_1,t) & K(x_1,x_1) & \cdots & K(x_1,x_m) \\ \vdots & \vdots & \vdots & \vdots \\ K(x_m,t) & K(x_m,x_1) & \cdots & K(x_m,x_m) \end{vmatrix} \tag{38}$$

if you know how to find D of x t lambda and D lambda then we can write down

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The Fredholm equation

$$y(x) = f(x) + \lambda \int K(x,t)y(t)dt, \qquad (33)$$

where the functions $f(x)$ and $k(x,t)$ are integrable, has a unique solution

$$y(x) = f(x) + \lambda \int \Gamma(x,t;\lambda)f(t)dt, \qquad (34)$$

where the resolvent kernel,

$$\Gamma(x,t;\lambda) = D(x,t;\lambda)/D(\lambda), \qquad (35)$$

with $D(\lambda) \neq 0$, is a meromorphic function of the complex variable λ, being the ratio of two entire functions defined by the series

our solution given in terms of the 34 And we have

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$$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \cdots \int K\left(\begin{array}{cccc} x, & x_1, & \cdots, & x_m \\ t, & x_1, & \cdots, & x_m \end{array} \right) dx_1 \cdots dx_m, \qquad (36)$$

and

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \cdots \int K\left(\begin{array}{cccc} x_1, & x_2, & \cdots, & x_m \\ x_1, & x_2, & \cdots, & x_m \end{array} \right) dx_1 \cdots dx_m, \qquad (37)$$

where

$$K\left(\begin{array}{cccc} x, & x_1, & \cdots, & x_m \\ t, & x_1, & \cdots, & x_m \end{array} \right) = \begin{vmatrix} K(x,t) & K(x,x_1) & \cdots & K(x,x_m) \\ K(x_1,t) & K(x_1,x_1) & \cdots & K(x_1,x_m) \\ \vdots & \vdots & \vdots & \vdots \\ K(x_m,t) & K(x_m,x_1) & \cdots & K(x_m,x_m) \end{vmatrix} \qquad (38)$$

seen in previous lectures that how to find out these expressions D x t lambda

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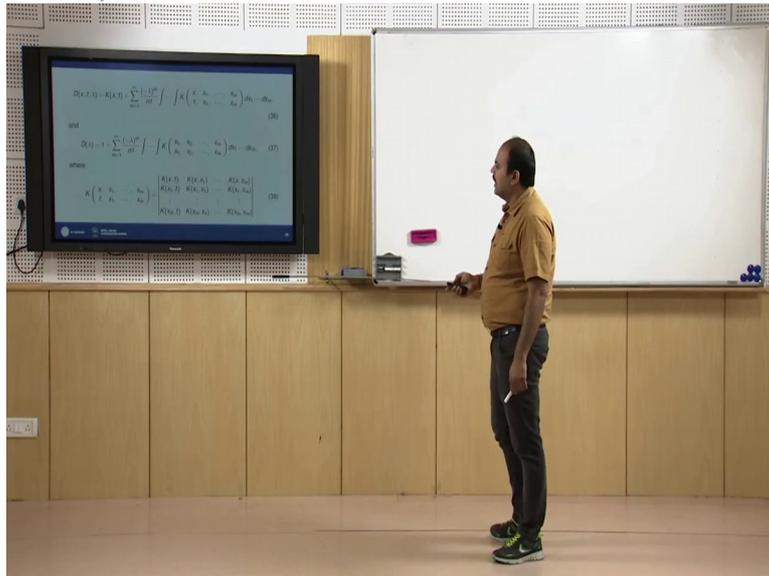
and D lambda

So in today's lecture we will first discuss uniqueness property of the solution and then we will see certain example based on the theory presented here So to look at the uniqueness

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Uniqueness

To establish the uniqueness of solution of (1), Let $y_1(x)$ be a solution other than $y(x)$ of (1) provided that $D(\lambda) \neq 0$. Then
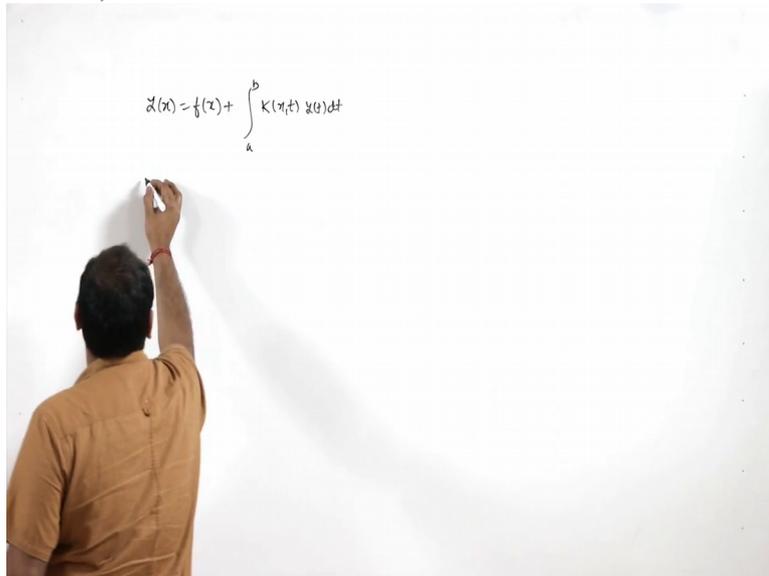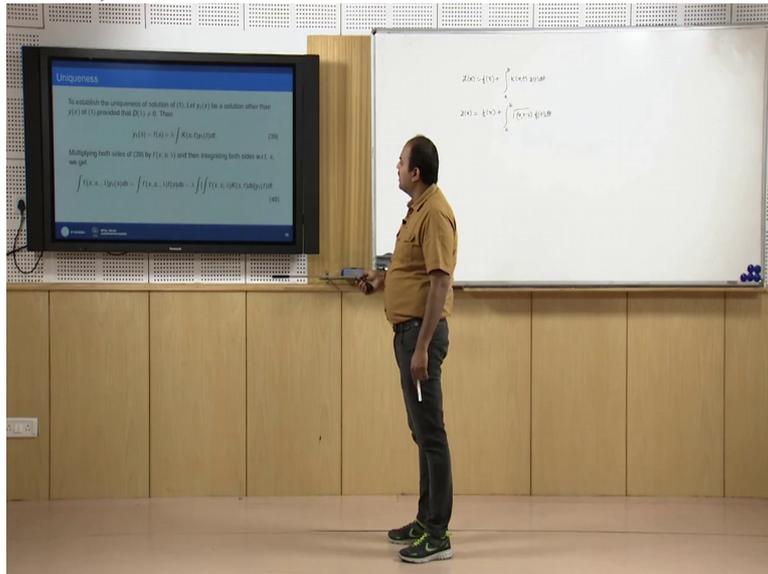
$$y_1(s) = f(s) + \lambda \int K(s,t)y_1(t)dt. \tag{39}$$

Multiplying both sides of (39) by $\Gamma(x,s;\lambda)$ and then integrating both sides w.r.t. s, we get

$$\int \Gamma(x,s,;\lambda)y_1(s)ds = \int \Gamma(x,s,;\lambda)f(s)ds + \lambda \int [\int \Gamma(x,s;\lambda)K(s,t)ds]y_1(t)dt. \tag{40}$$

we know that our solution of this Fredholm equation that is y of x equal to f of x plus you can say that it is a to b K of x t y t d t is the solution

$$y(x) = f(x) + \int_a^b K(x,t)\, y(t)\, dt$$

is given as y of x is equal to f of x plus a to b gamma x t lambda and f of t d t So we

$$y(x) = f(x) + \int_a^b K(x,t)\, y(t)\, dt$$

$$y(x) = f(x) + \int_a^b \overline{\Gamma(x,t;\lambda)}\, f(t)\, dt$$
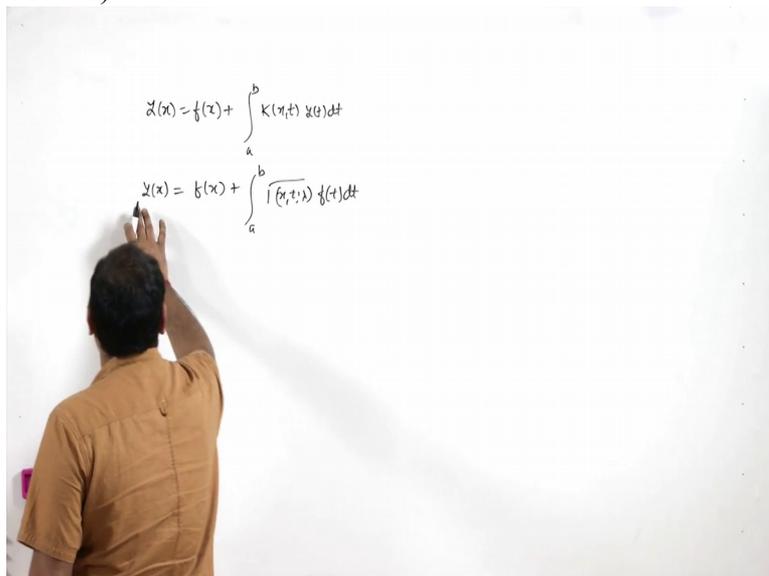
have shown that the solution of this can be written in terms of resolvent kernel and given by this equation Now if I assume that suppose

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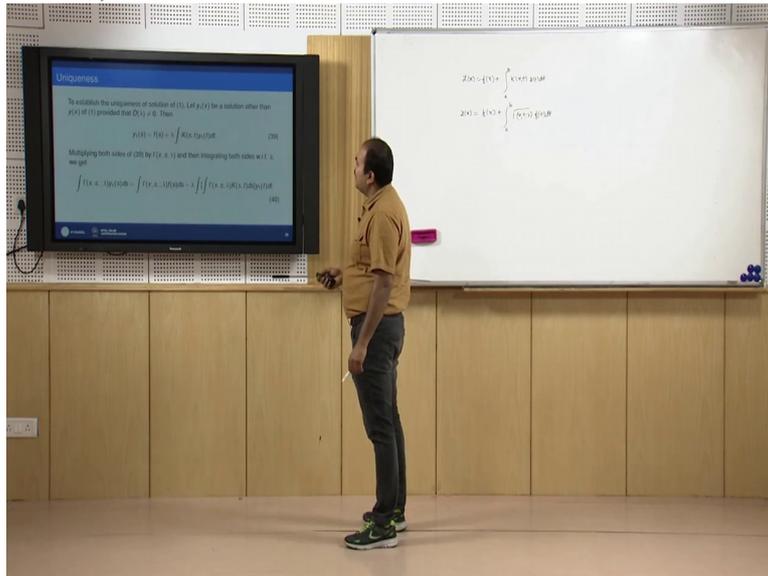
we have another solution that is say y 1 x which is also satisfying the equation number 39 means not only y is the solution suppose we have another solution called y 1 which also satisfies the same equation then we try to show that this y 1 is nothing but this y So this means that here we are assuming that y 1 is another solution of our Fredholm equation that is y 1 is equal to f of s plus lambda K s t y 1 t d t

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Uniqueness

To establish the uniqueness of solution of (1), Let $y_1(x)$ be a solution other than $y(x)$ of (1) provided that $D(\lambda) \neq 0$. Then

$$y_1(s) = f(s) + \lambda \int K(s,t) y_1(t) dt. \qquad (39)$$

Multiplying both sides of (39) by $\Gamma(x, s; \lambda)$ and then integrating both sides w.r.t. s, we get

$$\int \Gamma(x, s, ; \lambda) y_1(s) ds = \int \Gamma(x, s, ; \lambda) f(s) ds + \lambda \int [\int \Gamma(x, s; \lambda) K(s,t) ds] y_1(t) dt. \qquad (40)$$

So what we try to show here we try to show that this y 1 is nothing but y of s So here we just multiply

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the resolvent kernel gamma x s lambda and then you can integrate with respect to d of s

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Uniqueness

To establish the uniqueness of solution of (1), Let $y_1(x)$ be a solution other than $y(x)$ of (1) provided that $D(\lambda) \neq 0$. Then

$$y_1(s) = f(s) + \lambda \int K(s,t)y_1(t)dt. \qquad (39)$$

Multiplying both sides of (39) by $\Gamma(x,s;\lambda)$ and then integrating both sides w.r.t. s, we get

$$\int \Gamma(x,s;\lambda)y_1(s)ds = \int \Gamma(x,s;\lambda)f(s)ds + \lambda \int [\int \Gamma(x,s;\lambda)K(s,t)ds]y_1(t)dt. \qquad (40)$$

So we just multiply this equation number 39 by gamma x s lambda When you multiply here this f s gamma s x s lambda f of s d s plus lambda double integration gamma x s lambda K s t y K s t y 1 t 1 t Now what we try to do here once we multiply this then we have this gamma x s lambda your y 1 s d of s integration with respect to d of s equal to this gamma x s lambda and f of s d of s plus lambda You can put the limit all a to b and a to b here so gamma x s lambda and then here we have this part that K of x t K of x t and y 1 t then d t and then d s

So what we try to do here we try to show this can be simplified further So what we try to do here we just change the order of this integral
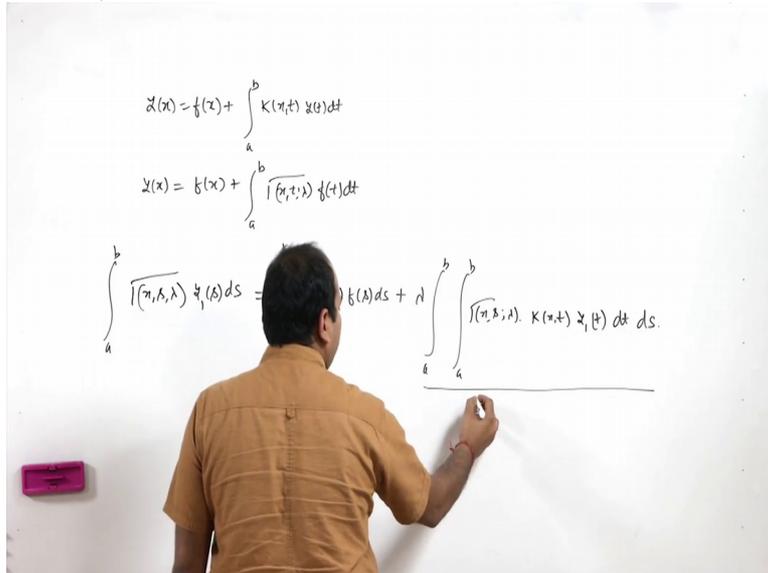
and we can write this as a to b and here a to b Now in place of inner integral with respect to t now let us once we change the order then inner integral is with respect to d s and outer one is d t When we do this then we can write it here as gamma x s lambda and K of x t here let me here correct this this is s of t so this is s of t So K of s of t now this y 1 t you can take it out from the integral so I can write it y 1 t like this So y 1 t K of s t d s and then d t And then lambda you can put it here right So this is equal to this

Now to find out the value of the inner integral we will use the relation
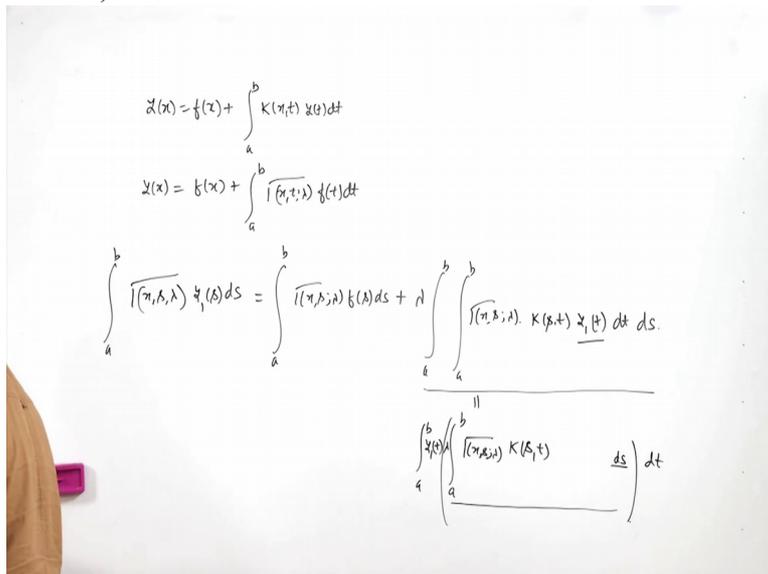
which is known as Fredholm relation satisfied by this resolvent kernel that is gamma x t lambda is equal to K of x t

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But we have

$$\Gamma(x, t; \lambda) = K(x, t) + \lambda \int \Gamma(x, s; \lambda) K(s, t) ds,$$

$$\lambda \int \Gamma(x, s; \lambda) K(s, t) ds = \Gamma(x, t; \lambda) - K(x, t). \qquad (41)$$

Using (41), (40) reduces to

$$\int \Gamma(x, s; \lambda) y_1(s) ds = \int \Gamma(x, s,; \lambda) f(s) ds + \int [\Gamma(x, t; \lambda) - K(x, t)] y_1(t) dt$$

or $\int \Gamma(x, t; \lambda) y_1(t) dt = \int \Gamma(x, t,; \lambda) f(t) dt + \int [\Gamma(x, t; \lambda) y_1(t) dt - \int K(x, t)] y_1(t) dt$
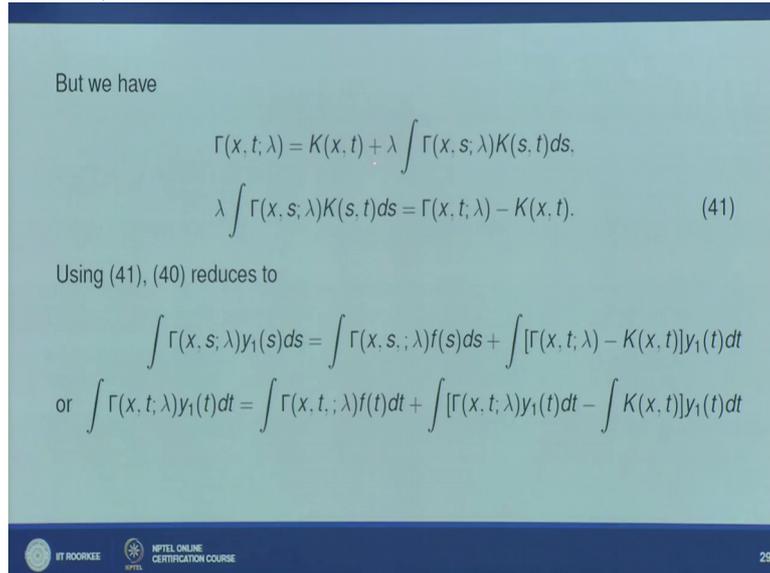
plus lambda gamma x s lambda K s t d s Now here this is the thing which we are searching for So this value can be written as gamma x t lambda minus K of x t So this integral can be replaced by the value gamma x t lambda minus K of x t So using this you can say that this is equal to a to b gamma x s lambda f of s d of s plus here it is a to b you have y 1 t and in place of this we are writing gamma x t lambda gamma x t lambda minus K of x t K of x t and this is with respect to d t Now when you simplify this

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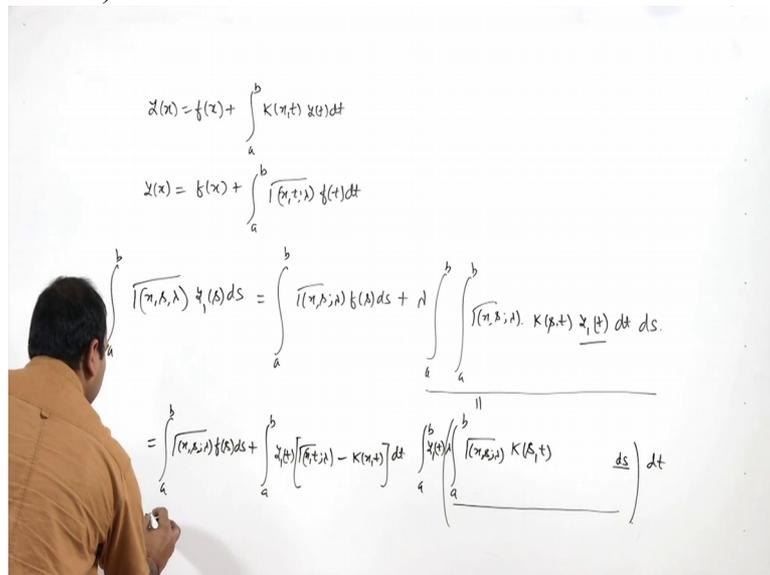
you will get a to b gamma x s lambda f of s d of s plus a to b y 1 t gamma x t lambda d t minus a to b y 1 t K of x t d t

Now if you look at this is the value

of what this is the value of this thing gamma a to b x s lambda y 1 s d s is equal to this thing Now if you compare then you can see that this value and the value presented here is same so these two will cancel each other and what you will get here is the value that this integral minus this integral
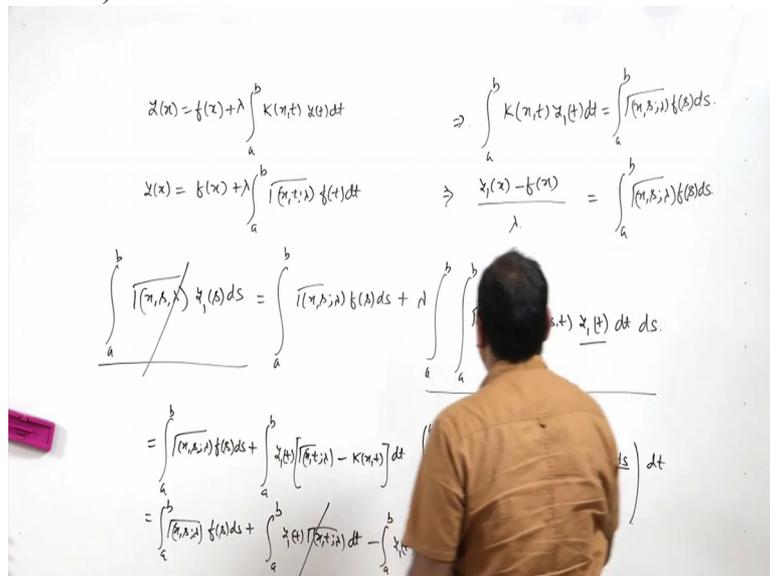
is equal to zero

Or I can write it here that this is what a to b and I can write it like K of x t y 1 t d of t is equal to a to b gamma x s lambda and f of s d of s And we already know

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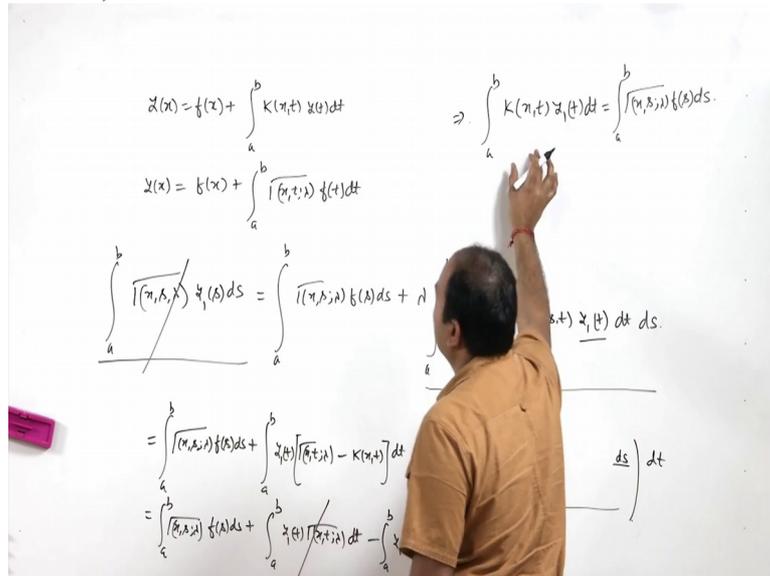
that y 1 satisfy this integral equation this then from here I can find out the value of a to b K x t y 1 t d t as from here I can say that this is nothing but y 1 x minus f of x divided by this lambda I think lambda is missing here lambda is you please correct there is a lambda here So if you correct this this is equal to a to b gamma x s lambda f of s d of s So this implies that

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if you simplify this further you will see that this implies that y 1 is equal to so here if you simplify this this is y 1 x is equal to f of x plus lambda times a to b gamma x of s lambda f of s d of s Now if you see then

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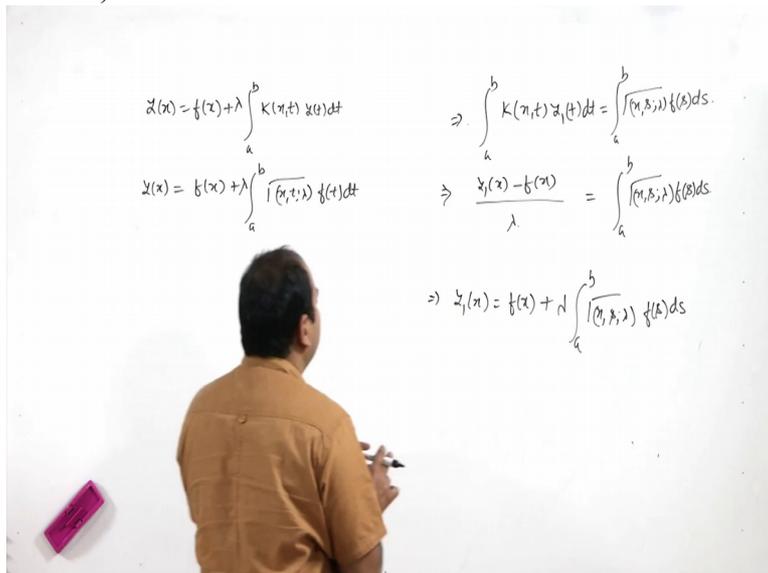
this value is given by y of x So you can say that this is nothing but y of x And this says that if

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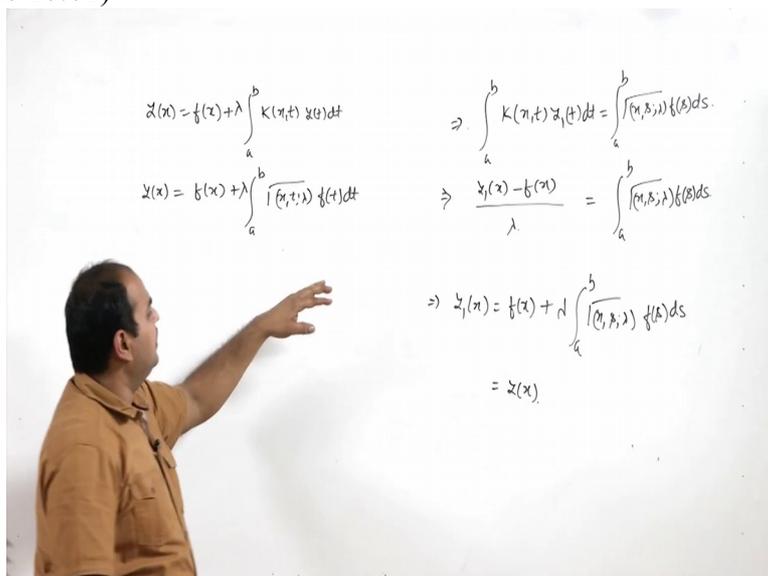
we have any solution of this integral equation it can be written in the form this So it means that this form is unique So we can assign the name y of x as this So if we have any other solution it has to be of this particular form So we can say that here this form is unique and our solution is also unique So it means that by this way whatever solution we are getting that is a unique solution Ok

So now once we have the uniqueness then we can start with the problem So let us consider the first example which is find the resolvent kernel for the

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integral equation y of x equal to f of x plus lambda zero to 1 x of s plus t y t d t Now once we have resolvent kernel with us then we can easily find out the solution of this equation So here if you look at here equation is what y of x equal to f of x plus lambda times here a is 0 and b is 1 x of x plus t y of t d of t

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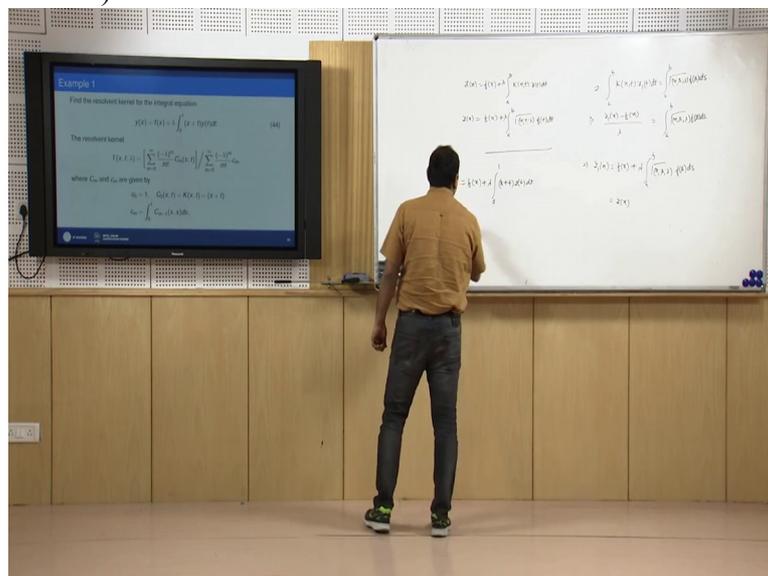
So here what is given is K of x t is equal to x plus t and a is equal to 0 and b equal to 1 So what we need to

do here we need to if we want to find out the solution we need to find out the resolvent kernel

So let us first find out the resolvent kernel And resolvent kernel is given by gamma x t lambda is equal to D of x t lambda divided by D of lambda So here if you recall then D of x t lambda is defined as this c naught x of t plus summation minus 1 to power m lambda to power m divided by factorial m c m x of t and D lambda is defined as c naught small c naught plus summation this is m is equal to 1 to infinity So m equal to 1 to infinity minus 1 to power m lambda to power m divided by factorial m small c m And how to find out

this small c naught and c m we have already seen that these are interconnected and these can be found by the following relations So here your small c naught small c naught is always 1 and C naught of x of t is given by K of x t and your small
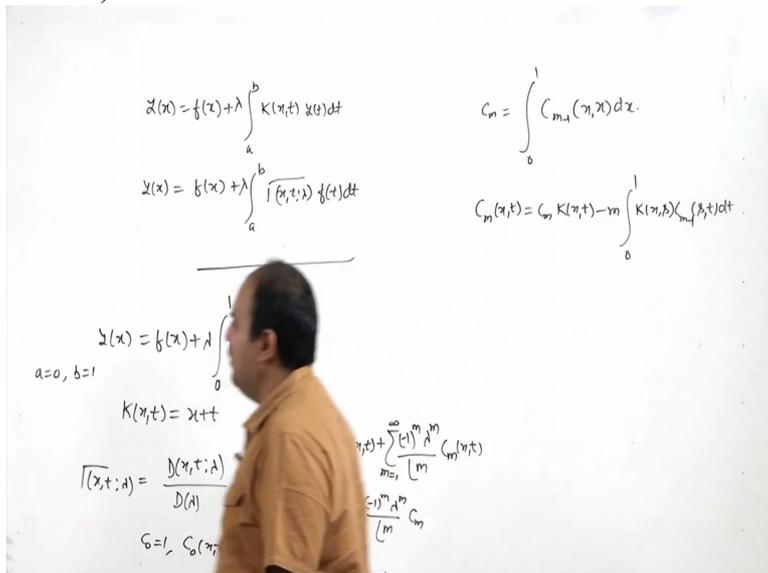
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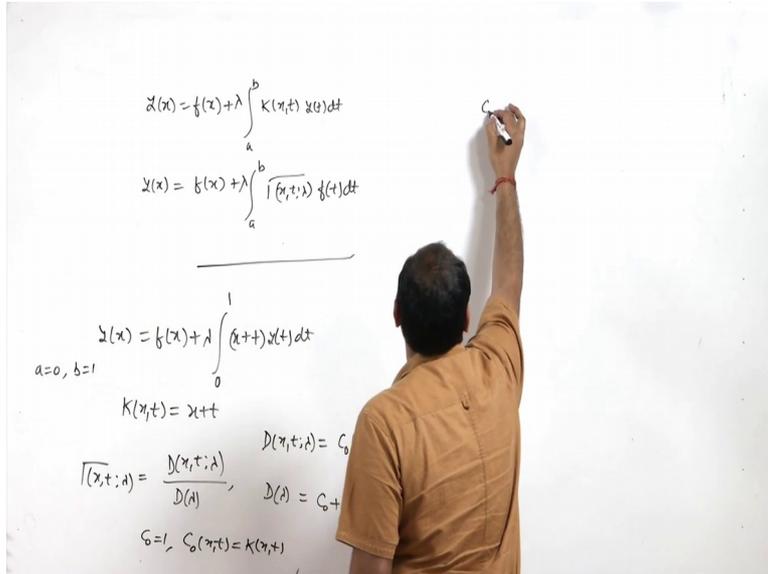
c m is given by integral c m minus 1 x comma x D of x whatever be the limit limit is right now it is zero to 1 here

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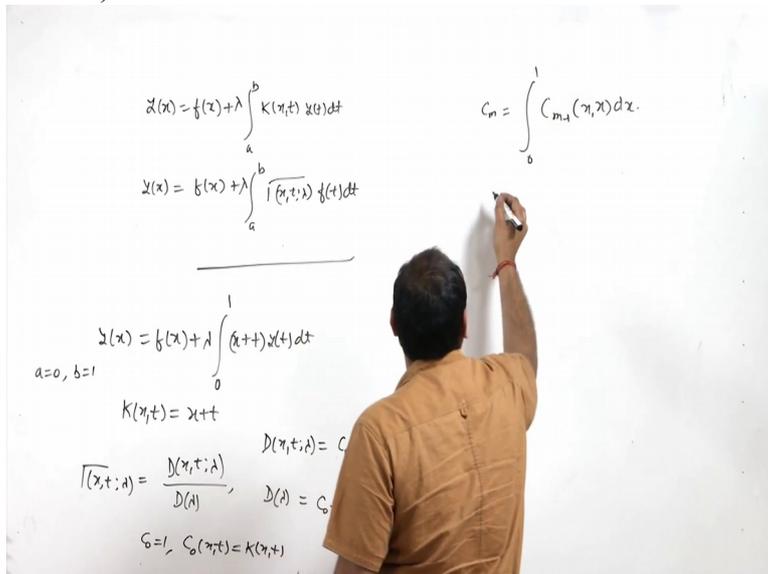
and then this capital C m x t is given as small c m K of x t minus m zero to 1 this K of x s c m minus 1 s of t d t Ok So with the help

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of these relations you can find out all these c m c i x t and small c is and you can put the values of small c m and capital C m to find out D x t lambda and D lambda and then you can find out gamma x t lambda using the series expansion of D x t lambda and D lambda here So here to find out c naught is already given as 1 c naught x t as

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x plus t and c m you can find out by the relation zero to 1 c m minus 1 x of x comma x d of x So first thing is that small c naught and capital c naught is given to you Then with the help of capital C naught x t you can find out the value

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small c 1 and small c 1 is nothing but zero to 1 and then c naught x comma x d of x that is 0 to 1 and this is what this is x plus t here so it is 2 of x d of x so it is nothing but x square by 2 and 0 to 1 so that is equal to 1 So small c 1

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you can find it like this So small c 1 is given by

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$$C_m = c_m K(x, t) - m \int_0^1 K(x, s) C_{m-1}(s, t)ds. \tag{45}$$

Thus

$$c_1 = \int_0^1 2x\,dx = 1.$$

$$C_1(x, t) = (x + t) - \int_0^1 (x + s)(s + t)ds = \frac{1}{2}(x + t) - xt - \frac{1}{3},$$

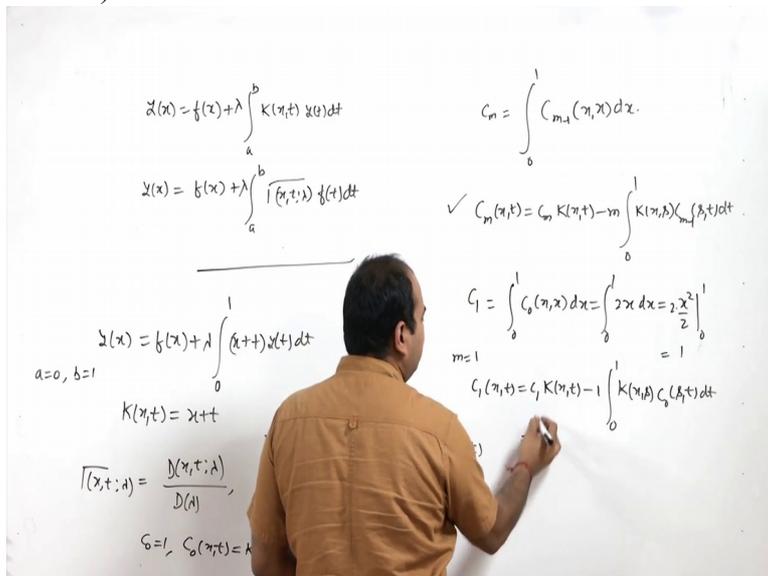$$c_2 = \int_0^1 \left(x - x^2 - \frac{1}{3}\right)dx = -\frac{1}{6}$$

$$C_2(x, t) = -\frac{1}{6}(x + t) - 2\int_0^1 (x + s)\left[\frac{1}{2}(s + t) - st - \frac{1}{3}\right]ds = 0.$$

this one And then once we have this small c 1 then we can use the recurrence relation to find out capital C m x of t. So we can use this formula for m equal to 1 So m equal to 1 it is c 1 x of t is equal to c 1 K of x of t minus 1 zero to 1 K of x comma s and this is c zero s of t d t. Now everything you know. Small c 1 you know

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small c 1 is just calculated and it is equal to 1 K x t is known to us so it is x plus t minus 0 to 1 K of x is x plus s and then c naught x t is nothing but K of s t So it is given as s plus t d of s So this is small mistake here d of s So it is d of s Ok
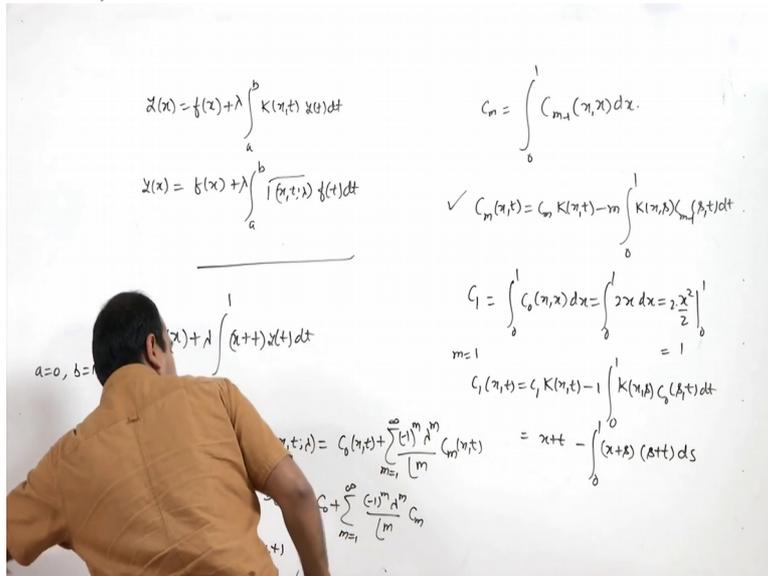
Ok so you can calculate this and it is equal to x plus t here minus now we need to integrate with respect to this thing s so we can say it is zero to 1 and we can simplify this We can write s as constant term you can take it out So this I can write it here This is x s plus x of t plus s square plus s t and d of s So you can write
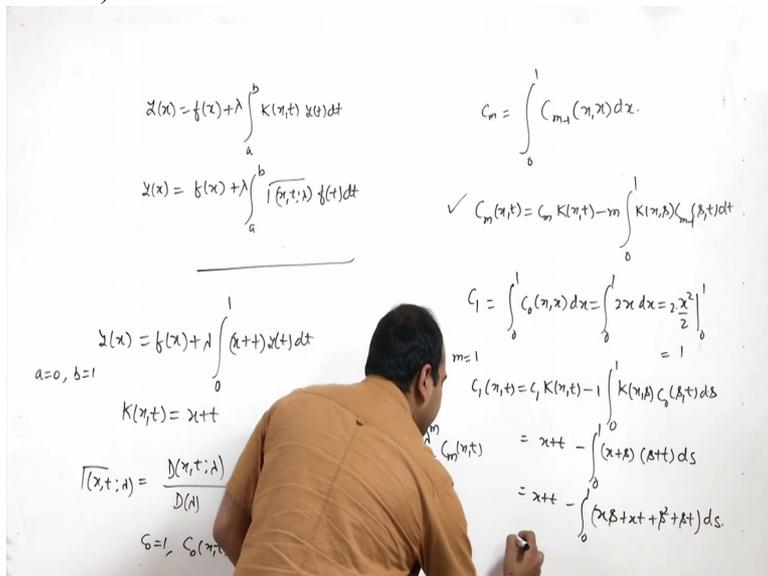
it here this as x plus t and you can simplify this is not very difficult integral and you can write it your thing here So here if we combine the this coefficient of x so you can write it here this s square by 2 x plus t zero to 1 minus x of t s zero to 1 plus s cube by 3 and there is a minus sign outside so it is minus here zero to 1 And if you simplify

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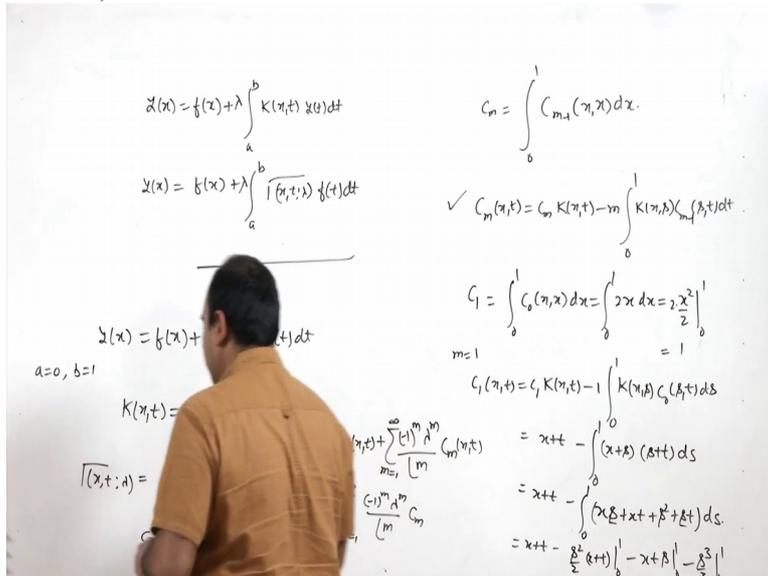
you will get the value of c 1 x of t and it is if you simplify your value of c 1 x t is given by this

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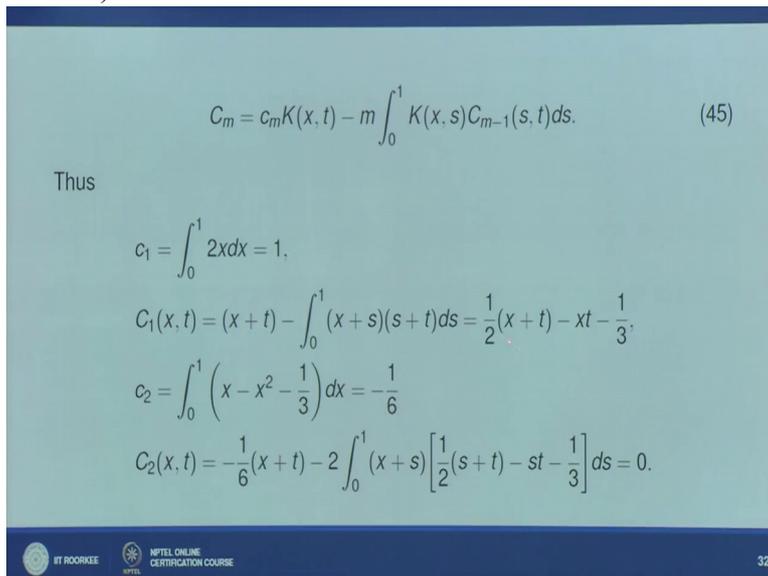

$$C_m = c_m K(x, t) - m \int_0^1 K(x, s) C_{m-1}(s, t) ds. \qquad (45)$$

Thus

$$c_1 = \int_0^1 2x dx = 1.$$

$$C_1(x, t) = (x + t) - \int_0^1 (x + s)(s + t) ds = \frac{1}{2}(x + t) - xt - \frac{1}{3}.$$

$$c_2 = \int_0^1 \left(x - x^2 - \frac{1}{3} \right) dx = -\frac{1}{6}$$

$$C_2(x, t) = -\frac{1}{6}(x + t) - 2 \int_0^1 (x + s) \left[\frac{1}{2}(s + t) - st - \frac{1}{3} \right] ds = 0.$$

that is 1 by 2 x plus t minus x t minus 1 by 3 So you can simplify you can get that Ok

So here if you put the value of s equal to 1 you can say it is half x plus t and this is what it is minus of x of t and minus 1 by 3 So c 1 x t you can get it from this And once we have this capital C 1 x t you can find out small c 2 zero to 1 c 1 x x d of x And you can calculate and it is given by minus 1 by 6 and again you can use the recurrence relation given in equation number 45 for m equal to 2 then we can say that c 2 x of t is equal to c 1 small c 2 so small c 2 is just calculated that is minus 1 by 6 so minus 1 by 6 K of x t that is x plus t minus m that

is 2 zero to 1 K of x s means x plus s and c 1 s t so c 1 s t you can get it from this equation and if you calculate this is coming out to be zero Now as soon as you were getting some c 2 some c i x t as zero then with the help of this recurrence relation and the relation

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small c m as zero to 1 c m minus 1 x s d s you can see that as soon as your c i is zero then your small c i plus 1 is going to be zero and here

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with the help of equation number 45 you can see that C i capital C i plus 1 is also coming out to be zero So in this case as soon as any one of your capital C i x t is zero then this infinite series is truncated into finite series and all others say higher order say c i x c i plus 1 x t is nothing but zero So in this case

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Since $C_2(x, t)$ vanishes, from (45) we find that $C_3 = C_4 = \ldots$ and $c_3 = c_4 = \ldots$ also vanish. Therefore resolvent kernel

$$\Gamma(x, t; \lambda) = \frac{(x + t) - [1/2(x + t) - xt - 1/3]\lambda}{1 - \lambda - (\lambda^2/12)}.$$

as soon as any one of your capital C i x t is zero then this infinite series is truncated into finite series and all others say higher order say c i x c i plus 1 x t is nothing but zero In this case since c 2 x t is zero this implies that capital C 3 capital C 4 and so on all are simply zero Similarly small c 3 c 4 all are simply vanish and it is equal to zero And in this case your gamma x t lambda is given by d x t lambda which is now reduced to only C naught x t minus lambda times C1 x t This is the value of C 1 x t divided by D lambda Now D lambda is again c naught minus lambda c 1 small c 1 is 1 minus lambda square and c 1 is small c 1 is c 2

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$$C_m = c_m K(x, t) - m \int_0^1 K(x, s) C_{m-1}(s, t) ds. \tag{45}$$
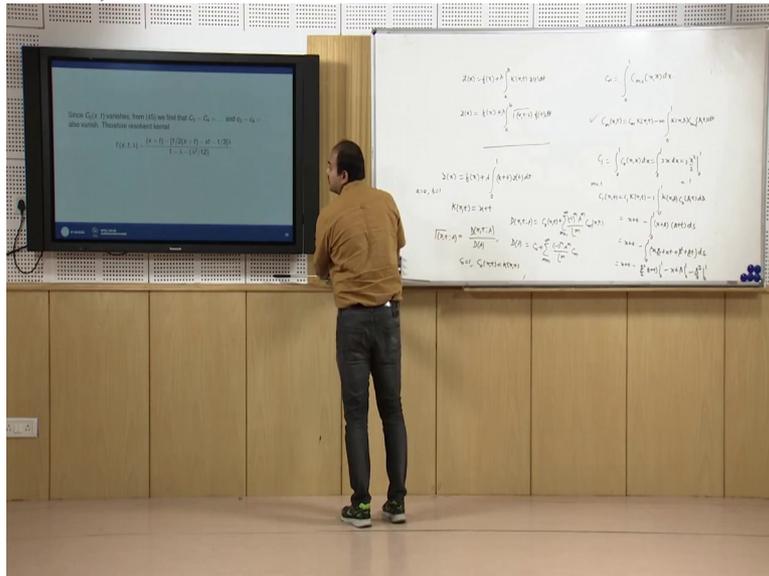
Thus

$$c_1 = \int_0^1 2x\, dx = 1.$$

$$C_1(x, t) = (x + t) - \int_0^1 (x + s)(s + t) ds = \frac{1}{2}(x + t) - xt - \frac{1}{3}.$$

$$c_2 = \int_0^1 \left(x - x^2 - \frac{1}{3}\right) dx = -\frac{1}{6}$$

$$C_2(x, t) = -\frac{1}{6}(x + t) - 2 \int_0^1 (x + s)\left[\frac{1}{2}(s + t) - st - \frac{1}{3}\right] ds = 0.$$

you can find out c 2 is minus 1 by c So you can write it gamma x t lambda this

gamma x t lambda as this Let me write it here It may not be clear here So let me write it

So here your small c naught is 1 small c 1 is 1 here small c 2 is minus 1 by 6 and capital C naught x of t is equal to K of x t that is x plus t and C 1 x t is coming out to be half of x plus t minus x of t minus 1 by 3 here So here with the help of
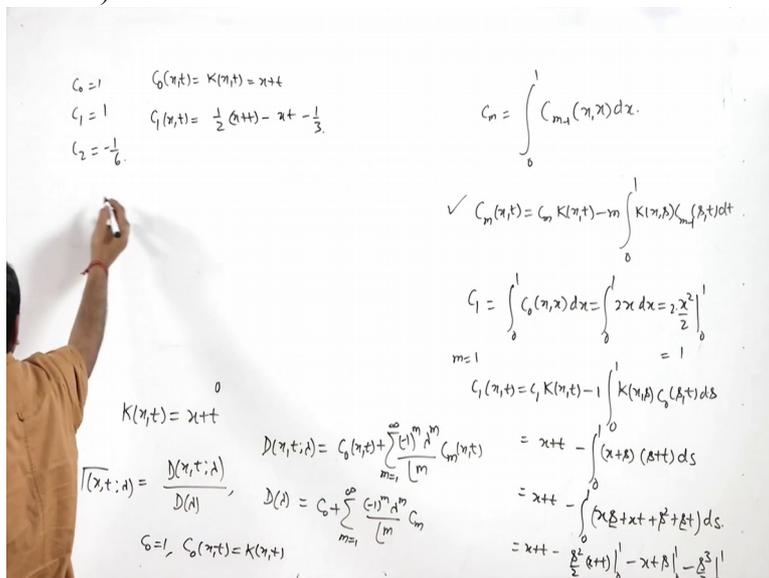
this you can calculate D lambda as c naught minus c 1 lambda plus lambda square upon 2 c 2 And rest are all zeroes right So

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this can be written as 1 minus lambda and this you can keep it minus lambda square by 12 c 2 is this

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Ok so D lambda is given by this And D x t lambda is equal to that is x plus t minus lambda times c 1 x t that is 1 by 2 x plus t minus x of t minus 1 by 3 So D x t lambda is given by this D lambda is given by this And you can find out gamma x t lambda as a ratio of these two and it is given by this thing gamma x t lambda as the ratio of these two and it is given by this thing gamma x t lambda and once you have your gamma x t lambda then

(Refer Slide Time 21:25)

Since $C_2(x, t)$ vanishes, from (45) we find that $C_3 = C_4 = \ldots$ and $c_3 = c_4 = \ldots$ also vanish. Therefore resolvent kernel

$$\Gamma(x, t; \lambda) = \frac{(x + t) - [1/2(x + t) - xt - 1/3]\lambda}{1 - \lambda - (\lambda^2/12)}.$$

solution you can write it by y of x equal to f of x plus zero to 1 here in this case it is a zero and b is equal to 1 and gamma x of s lambda f of s d s right So

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resolvent kernel is given to us and you can find out the solution of this problem So now let us move to next problem and here also equation is given as y of x equal to 1 plus lambda zero to pi sin of x plus t y t

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Example 2

Solve the integral equation

$$y(x) = 1 + \lambda \int_0^\pi \sin(x + t) y(t) dt. \tag{46}$$

Here

$$c_0 = 1, \quad C_0(x, t) = K(x, t) = \sin(x + t)$$

$$c_1 = \int_0^\pi C_0(x, x) dx = \int_0^\pi \sin(x + x) dx = 0$$

$$C_1(x, t) = c_1 K(x, t) - \int_0^\pi K(x, s) C_0(s, t) ds$$

$$= 0 - \int_0^\pi \sin(x + s) \sin(s + t) ds = -\frac{1}{2}\pi \cos(x - t).$$

d t So here we want to solve this integral equation f x is given as 1 lambda as 1 lambda is given as lambda only and a is zero and b is equal to pi and K of x t is given as sin of x plus t So as I point as we pointed out this small c naught is always 1 and small capital C naught x t is K of x t which is already given as sin x plus t here Now let us start calculating the coefficient for D lambda and D x t lambda

So c 1 is equal to zero to pi c naught x comma x D of x And when you put the value here then it is sin of x plus x d of x and it is nothing but it is coming out to be zero Similarly now moving on to find out the capital C 1 x t Then from the recurrence relation it is c 1 small c 1 K x t minus m m is 1 here zero to pi K of x s c of x t d of s So here C 1 is coming out to be zero K of x t so zero times this sin of x t is nothing but zero minus zero to pi sin of x plus s sin of x plus t d of s Here you can use the formula sin a sin b and you can simplify this and you can get a relation like this So

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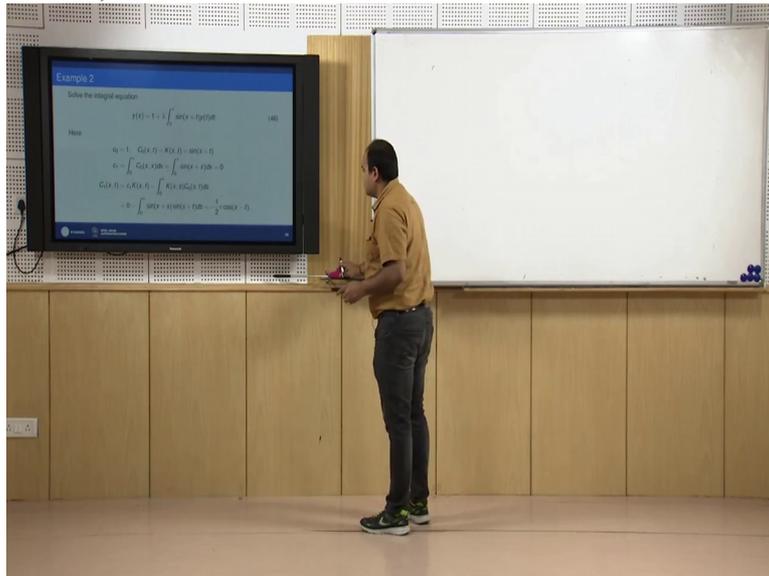
c 1 x t is basically what So c 1 x of t c 1 is zero into K of x t So that is coming out to be zero minus 1 zero to pi here and sin of x plus s and sin of s plus t d of s is that Ok So

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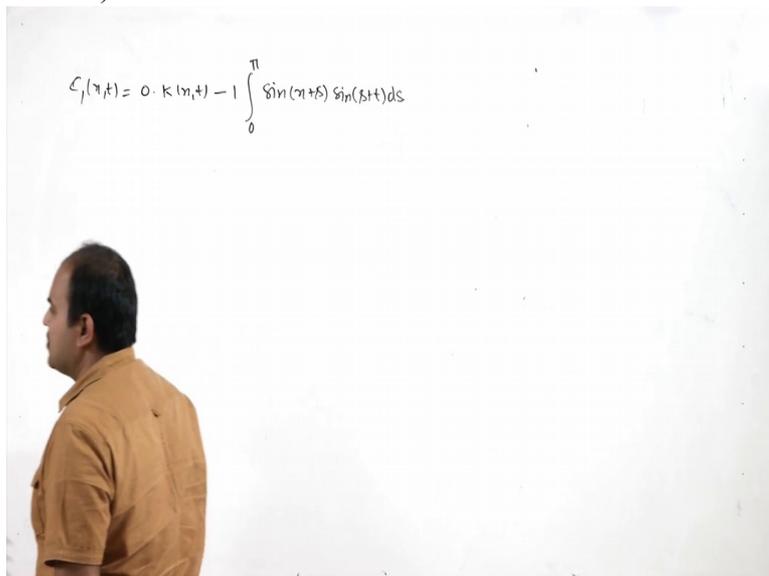
this is nothing but zero this is zero simply minus zero to pi and then you can write it you multiply by 2 and divide by 2 and this you can write it cos of a minus b so that is you can write it is this is a and this is b So a minus b means x minus t and so plus and here we have cos of x plus s plus s plus t and yeah d of s is it Ok

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$$C_1(x,t) = 0 \cdot K(x,t) - 1 \int_0^\pi \underbrace{\sin(x+s)}_{A} \underbrace{\sin(s+t)}_{B} ds$$

$$= -\frac{1}{2} \int_0^\pi \Big[\cos(x-t) + \cos(x+s+s+t) \Big] ds$$
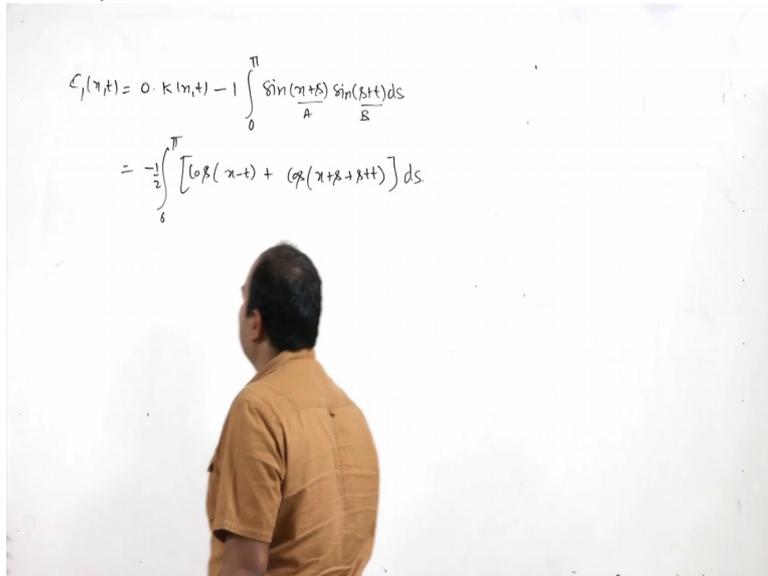
So this you can simplify and you can check that this is coming out to be minus 1 by 2

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Example 2

Solve the integral equation

$$y(x) = 1 + \lambda \int_0^\pi \sin(x+t) y(t) dt. \tag{46}$$

Here

$$c_0 = 1, \quad C_0(x,t) = K(x,t) = \sin(x+t)$$

$$c_1 = \int_0^\pi C_0(x,x) dx = \int_0^\pi \sin(x+x) dx = 0$$

$$C_1(x,t) = c_1 K(x,t) - \int_0^\pi K(x,s) C_0(s,t) ds$$

$$= 0 - \int_0^\pi \sin(x+s) \sin(s+t) ds = -\frac{1}{2}\pi \cos(x-t).$$

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pi cos of x minus t So you have c 1 of x t as minus 1 by 2 pi of cos of x minus t cos x minus t So C 1 x t is and we already have

$$C_1(x,t) = 0 \cdot k(x,t) - 1 \int_0^\pi \underbrace{\sin(x+s)}_{A} \ \underbrace{\sin(s+t)}_{B} ds$$

$$= -\frac{1}{2} \int_0^\pi \left[\cos(x-t) + \cos(x+s+s+t) \right] ds$$

$$C_1(x,t) = -\frac{1}{2} \pi \cos(x-t)$$

c naught is equal to 1 c 1 x of c naught x of t as sin of x plus t Now just now we have calculated

$$C_1(x,t) = 0 \cdot k(x,t) - 1 \int_0^\pi \underbrace{\sin(x+s)}_{A} \ \underbrace{\sin(s+t)}_{B} ds$$

$$= -\frac{1}{2} \int_0^\pi \left[\cos(x-t) + \cos(x+s+s+t) \right] ds$$

$$C_1(x,t) = -\frac{1}{2} \pi \cos(x-t)$$

$$C_0 = 1 \quad C_0(x,t) = \sin(x+t)$$

C 1 x of t that is minus 1 by 2 pi cos of x minus t here and small
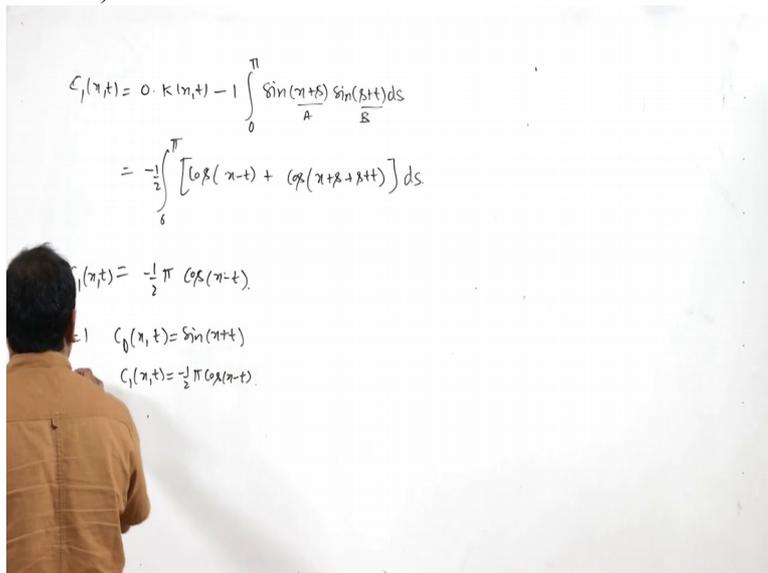
c 1 we have calculated as zero then with this help with this c 1 you can find out the value c 2 and it is coming out to be minus 1 upon 2 pi zero to pi here and this is c 1 x x d of x and this minus pi by 2 zero to pi and this is what cos of x minus x is what Ok so this c 1 x comma x is already containing this minus pi by 2 I am just writing the formula here

So cos of x minus x means it is simply so this is nothing but minus pi square by 2 So small c 2 is
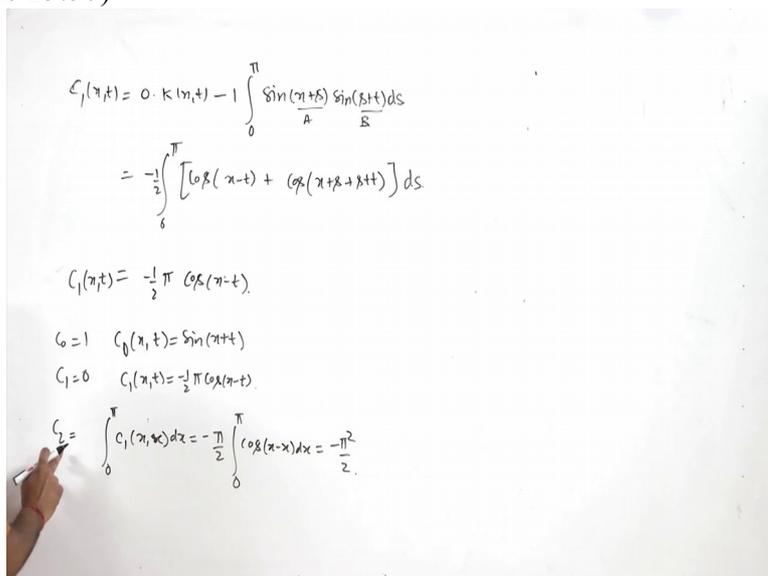
known to us now It is pi square by 2 Then capital C 2 x of t you can get it small c 2 t of x t minus m is 2 here zero to pi and this is K of x of s c 1 c 1 s of t s of t d of s

$$C_1(x,t) = 0 \cdot K(x,t) - 1 \int_0^{\pi} \underbrace{\sin(x+s)}_{A} \underbrace{\sin(s+t)}_{B} ds$$

$$= -\frac{1}{2} \int_0^{\pi} \left[\cos(x-t) + \cos(x+s+s+t) \right] ds$$

$$C_1(x,t) = -\frac{1}{2} \pi \cos(x-t)$$

$C_0 = 1 \quad C_0(x,t) = \sin(x+t)$

$C_1 = 0 \quad C_1(x,t) = -\frac{1}{2}\pi\cos(x-t)$

$$C_2 = \int_0^{\pi} C_1(x,x)dx = -\frac{\pi}{2} \int_0^{\pi} \cos(x-x)dx = -\frac{\pi^2}{2}$$

$$C_2(x,t) = C_2 K(x,t) - 2 \int_0^{\pi} K(x,s) C_1(s,t) ds.$$

So now c 2 is known to us c 1 is known to us Now we can find out and c 2 small c 2 is minus pi square by 2 and sin of x plus t minus 2 times zero to pi and this is sin of x plus s and c 1 s t is coming out to be here it is minus 1 by 2 pi So this you can take it out So this is cos of s minus t d of s

$$C_1(x,t) = 0 \cdot K(x,t) - 1 \int_0^{\pi} \underbrace{\sin(x+s)}_{A} \underbrace{\sin(s+t)}_{B} ds$$

$$= -\frac{1}{2} \int_0^{\pi} \left[\cos(x-t) + \cos(x+s+s+t) \right] ds$$

$$C_1(x,t) = -\frac{1}{2} \pi \cos(x-t)$$

$C_0 = 1 \quad C_0(x,t) = \sin(x+t)$

$C_1 = 0 \quad C_1(x,t) = -\frac{1}{2}\pi\cos(x-t)$

$$C_2 = \int_0^{\pi} C_1(x,x)dx = -\frac{\pi}{2} \int_0^{\pi} \cos(x-x)dx = -\frac{\pi^2}{2}$$

$$C_2(x,t) = C_2 K(x,t) - 2 \int_0^{\pi} K(x,s) C_1(s,t) ds.$$

$$= -\frac{\pi^2}{2} \sin(x+t) - 2 \int_0^{\pi} \sin(x+s) \left(\frac{-1}{2}\pi\right) \cos(s-t) ds.$$

So you just look at here c 2 small c 2

$$c_2 = \int_0^\pi C_1(x, x)\,dx = -\int_0^\pi \frac{1}{2}\pi \cos(x - x)\,dx = -\frac{1}{2}\pi^2,$$

$$C_2(x, t) = c_2 K(x, t) - 2\int_0^\pi K(x, s)C_1(s, t)\,ds$$

$$= -\frac{1}{2}\pi^2 \sin(x + t) + 2\int_0^\pi \frac{1}{2}\pi \sin(x + s)\cos(s - t)\,ds = 0.$$

Since $C_2(x, t)$ vanishes, from (45) we find that $C_3 = C_4 = \ldots$ and $c_3 = c_4 = \ldots$ also vanish. Therefore

$$\Gamma(x, t; \lambda) = \left[\sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} C_m(x, t)\right] \bigg/ \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} c_m$$

$$= \frac{\sin(x + t) + \frac{1}{2}\pi\lambda \cos(x - t)}{1 - \frac{1}{4}\pi^2\lambda^2}.$$

is just now we have calculated as minus pi square by 2 and c 2 x t we are going to calculate here and it is minus 1 by two pi square sin x plus t plus 2 times zero to pi 1 by two pi sin of x plus s cos of s minus t d of s Now here you can use the formula sin a cos b and when you simplify you will see that it is coming out to be zero Please you please find out and show that it is equal to
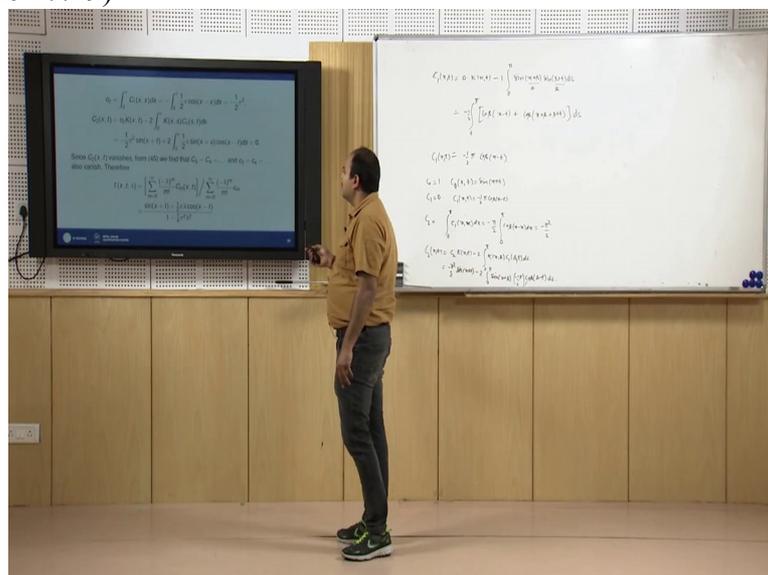
zero And as we have seen that this C 2 x t is zero capital C 2 x t is zero then

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$$c_2 = \int_0^\pi C_1(x,x)\,dx = -\int_0^\pi \frac{1}{2}\pi\cos(x-x)\,dx = -\frac{1}{2}\pi^2,$$

$$C_2(x,t) = c_2 K(x,t) - 2\int_0^\pi K(x,s)C_1(s,t)\,ds$$

$$= -\frac{1}{2}\pi^2 \sin(x+t) + 2\int_0^\pi \frac{1}{2}\pi\sin(x+s)\cos(s-t)\,ds = 0.$$

Since $C_2(x,t)$ vanishes, from (45) we find that $C_3 = C_4 = \ldots$ and $c_3 = c_4 = \ldots$ also vanish. Therefore

$$\Gamma(x,t;\lambda) = \left[\sum_{m=0}^\infty \frac{(-\lambda)^m}{m!}C_m(x,t)\right]\Big/\sum_{m=0}^\infty \frac{(-\lambda)^m}{m!}c_m$$

$$= \frac{\sin(x+t) + \frac{1}{2}\pi\lambda\cos(x-t)}{1 - \frac{1}{4}\pi^2\lambda^2}.$$

with this formula you can say that small c 3 is zero and when small c 3 is zero capital C 2 is zero You can check from the recurrence relation that your capital C 3 is also zero and hence we can say that all the all the c i plus 1 x of t where i is 2 all are simply vanishing Similarly small c 3 c 4 all are simply vanished and you can say that this expression this infinite series is converted into a finite sum Similarly the denominator also this infinite series is truncated into a finite sum

So you can write it here this as C naught x of t C naught x t is sin of x plus t plus lambda times minus lambda times C 1 x t So C 1 x t is

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Example 2

Solve the integral equation

$$y(x) = 1 + \lambda \int_0^\pi \sin(x+t)y(t)\,dt. \qquad (46)$$

Here

$$c_0 = 1, \quad C_0(x,t) = K(x,t) = \sin(x+t)$$

$$c_1 = \int_0^\pi C_0(x,x)\,dx = \int_0^\pi \sin(x+x)\,dx = 0$$

$$C_1(x,t) = c_1 K(x,t) - \int_0^\pi K(x,s)C_0(s,t)\,ds$$

$$= 0 - \int_0^\pi \sin(x+s)\sin(s+t)\,ds = -\frac{1}{2}\pi\cos(x-t).$$

minus 1 by 2 pi cos of x minus t So using this value you can say that

$$c_2 = \int_0^\pi C_1(x,x)\,dx = -\int_0^\pi \frac{1}{2}\pi\cos(x-x)\,dx = -\frac{1}{2}\pi^2,$$

$$C_2(x,t) = c_2 K(x,t) - 2\int_0^\pi K(x,s)C_1(s,t)\,ds$$

$$= -\frac{1}{2}\pi^2\sin(x+t) + 2\int_0^\pi \frac{1}{2}\pi\sin(x+s)\cos(s-t)\,ds = 0.$$

Since $C_2(x,t)$ vanishes, from (45) we find that $C_3 = C_4 = \ldots$ and $c_3 = c_4 = \ldots$ also vanish. Therefore

$$\Gamma(x,t;\lambda) = \left[\sum_{m=0}^\infty \frac{(-\lambda)^m}{m!}C_m(x,t)\right]\Big/\sum_{m=0}^\infty \frac{(-\lambda)^m}{m!}c_m$$

$$= \frac{\sin(x+t) + \frac{1}{2}\pi\lambda\cos(x-t)}{1 - \frac{1}{4}\pi^2\lambda^2}.$$

it is 1 by 2 pi lambda cos of x minus t And similarly C 2 capital C 2 x t and C 3 x t all are zero so there is no other term here and similarly in denominator also this is C naught C 1 is coming out to be zero minus lambda square and this is nothing but your value C 2 So C 2 you can write it here So 1 so D lambda is basically what Let me write it here
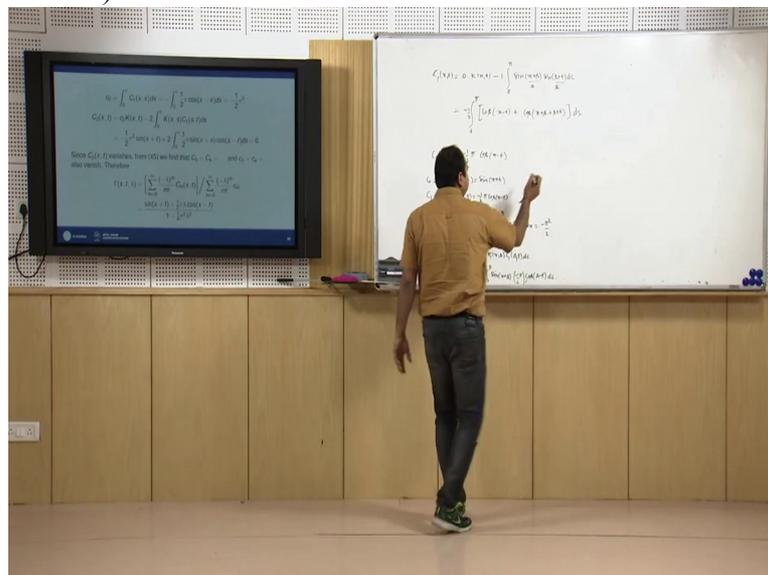
D lambda is equal to c naught minus lambda C 1 plus lambda square upon factorial 2 C 2 so and all others are simply zero So you can put

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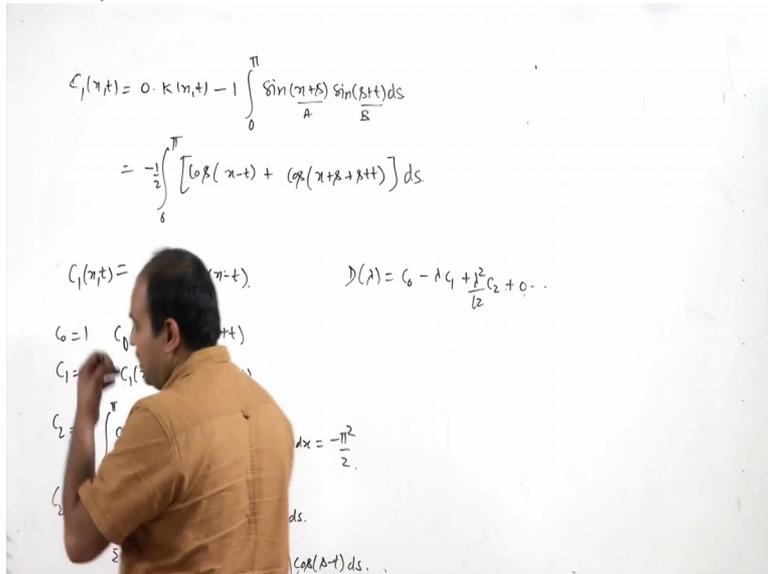
all the value of C naught C 1 and C 2 and you can see that D lambda is coming out to be 1 minus 1 by 4 pi square lambda square So here also your gamma x t lambda is given by say

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$$c_2 = \int_0^\pi C_1(x, x)dx = -\int_0^\pi \frac{1}{2}\pi \cos(x - x)dx = -\frac{1}{2}\pi^2,$$

$$C_2(x, t) = c_2 K(x, t) - 2\int_0^\pi K(x, s)C_1(s, t)ds$$

$$= -\frac{1}{2}\pi^2 \sin(x + t) + 2\int_0^\pi \frac{1}{2}\pi \sin(x + s)\cos(s - t)ds = 0.$$

Since $C_2(x, t)$ vanishes, from (45) we find that $C_3 = C_4 = \ldots$ and $c_3 = c_4 = \ldots$ also vanish. Therefore

$$\Gamma(x, t; \lambda) = \left[\sum_{m=0}^\infty \frac{(-\lambda)^m}{m!}C_m(x, t)\right] \Big/ \sum_{m=0}^\infty \frac{(-\lambda)^m}{m!}c_m$$
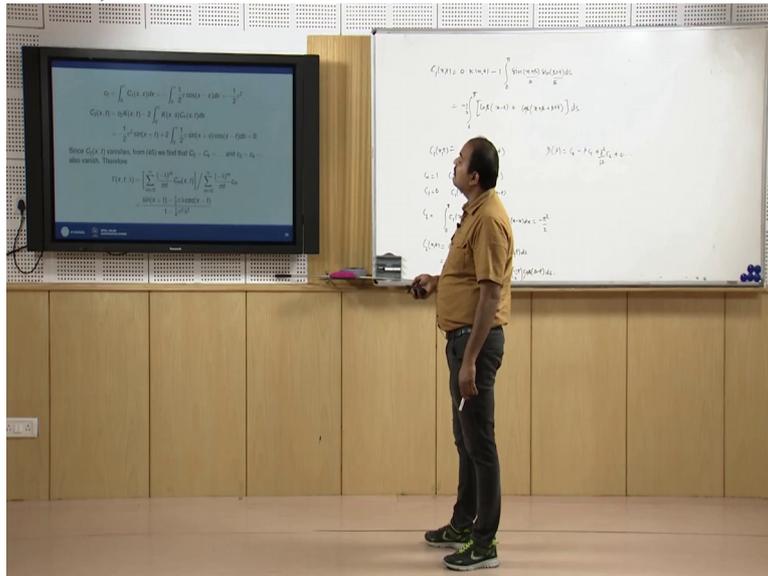
$$= \frac{\sin(x + t) + \frac{1}{2}\pi\lambda\cos(x - t)}{1 - \frac{1}{4}\pi^2\lambda^2}.$$

this this kind of ratio of two finite series And once we have your resolvent kernel calculated

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then you can find out the solution by putting gamma x t lambda

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$$c_2 = \int_0^\pi C_1(x,x)dx = -\int_0^\pi \frac{1}{2}\pi\cos(x-x)dx = -\frac{1}{2}\pi^2,$$

$$C_2(x,t) = c_2 K(x,t) - 2\int_0^\pi K(x,s)C_1(s,t)ds$$

$$= -\frac{1}{2}\pi^2\sin(x+t) + 2\int_0^\pi \frac{1}{2}\pi\sin(x+s)\cos(s-t)ds = 0.$$

Since $C_2(x,t)$ vanishes, from (45) we find that $C_3 = C_4 = \ldots$ and $c_3 = c_4 = \ldots$ also vanish. Therefore

$$\Gamma(x,t;\lambda) = \left[\sum_{m=0}^\infty \frac{(-\lambda)^m}{m!}C_m(x,t)\right] \Big/ \sum_{m=0}^\infty \frac{(-\lambda)^m}{m!}c_m$$

$$= \frac{\sin(x+t) + \frac{1}{2}\pi\lambda\cos(x-t)}{1 - \frac{1}{4}\pi^2\lambda^2}.$$

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into the equation that is

you can write down your solution as y of x equal to f of x plus lambda a to b gamma x t lambda f of s d of s So for this particular
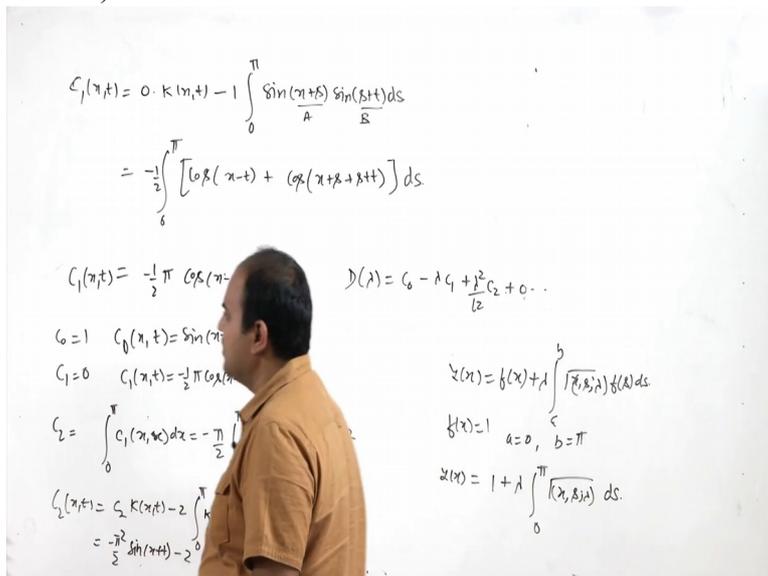
problem your f of s is simply 1 and a is equal to zero and b is equal to pi So you write your solution

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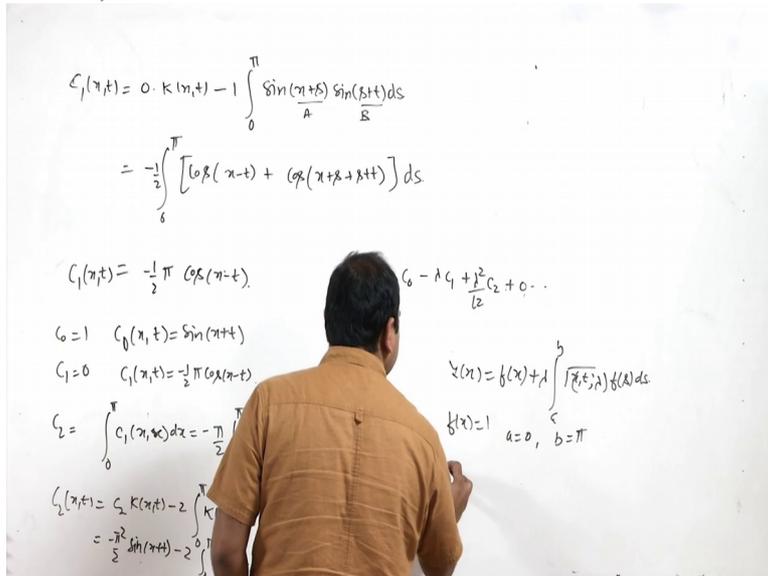
as y of x is equal to 1 plus lambda zero to pi and gamma x sorry it is s here So it is not t it is s here So here you can write s so lambda f s is 1 so here So gamma x s lambda just now

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we have calculated like this gamma x t lambda is given by this So you can

$$c_2 = \int_0^\pi C_1(x,x)\,dx = -\int_0^\pi \frac{1}{2}\pi\cos(x-x)\,dx = -\frac{1}{2}\pi^2,$$

$$C_2(x,t) = c_2 K(x,t) - 2\int_0^\pi K(x,s)C_1(s,t)\,ds$$

$$= -\frac{1}{2}\pi^2\sin(x+t) + 2\int_0^\pi \frac{1}{2}\pi\sin(x+s)\cos(s-t)\,ds = 0.$$

Since $C_2(x,t)$ vanishes, from (45) we find that $C_3 = C_4 = \dots$ and $c_3 = c_4 = \dots$ also vanish. Therefore

$$\Gamma(x,t;\lambda) = \left[\sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!}C_m(x,t)\right] \Big/ \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!}c_m$$

$$= \frac{\sin(x+t) + \frac{1}{2}\pi\lambda\cos(x-t)}{1 - \frac{1}{4}\pi^2\lambda^2}.$$

calculate gamma x s lambda from this So you can find out the solution like this And there is one more example example number 3 And here also we need to find out the solution of integral equation that is

Example 3

Solve the integral equation

$$y(x) = 1 + \int_0^1 (1-3xt)y(t)\,dt.$$

Here

$$c_0 = 1, \quad C_0(x,t) = K(x,t) = 1 - 3xt$$

$$c_1 = \int_0^1 C_0(x,x)\,dx = \int_0^1 (1-3x^2)\,dx = 0$$
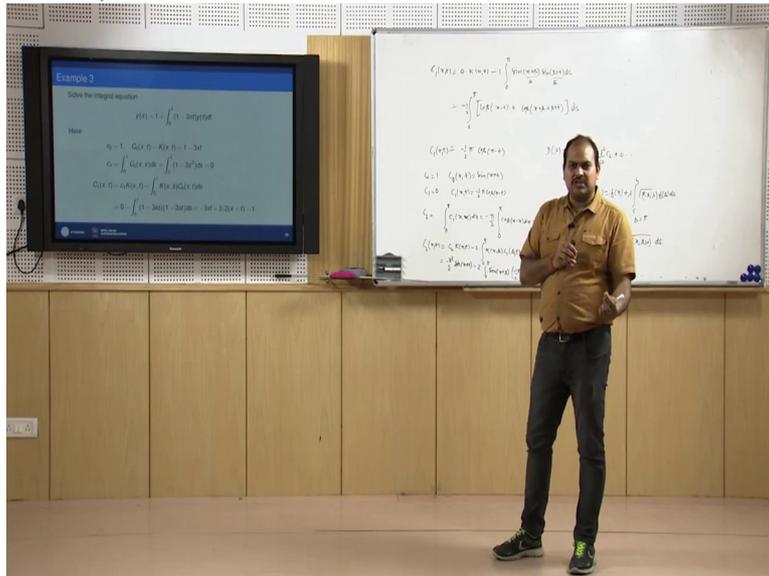
$$C_1(x,t) = c_1 K(x,t) - \int_0^1 K(x,s)C_0(s,t)\,ds$$

$$= 0 - \int_0^1 (1-3xs)(1-3st)\,ds = -3xt + 3/2(x+t) - 1.$$

y of x is equal to 1 plus zero to 1 1 minus c x t y t d t So here f of x is simply 1 lambda is equal to 1 And a is zero and b is equal to 1 And K x t is given by 1 minus D x t So as we have done in previous problems here also we find alternatively small
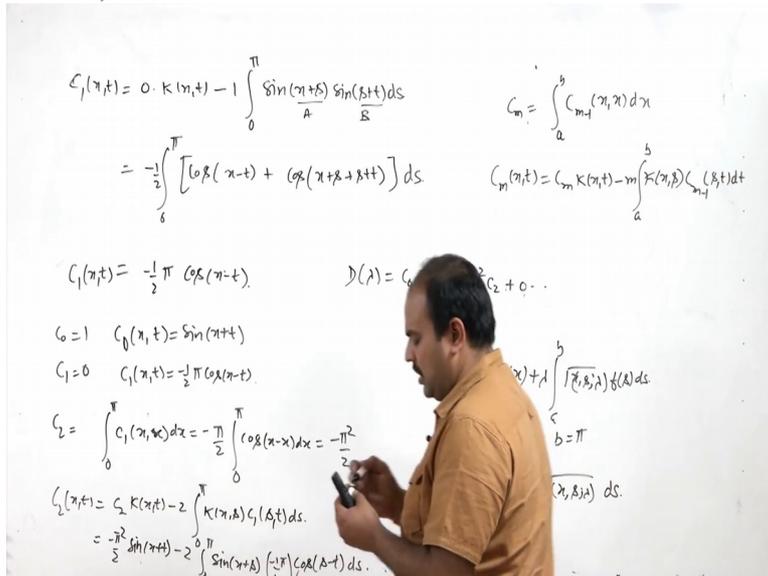
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c i and capital C i So c naught is already given as 1 and capital C naught x t is given in terms of kernel that is K of x t and then we can always use the relation These 2 relations are very very important that is small c m is equal to zero to a to b a to b c m minus 1 x comma x d of x This is very

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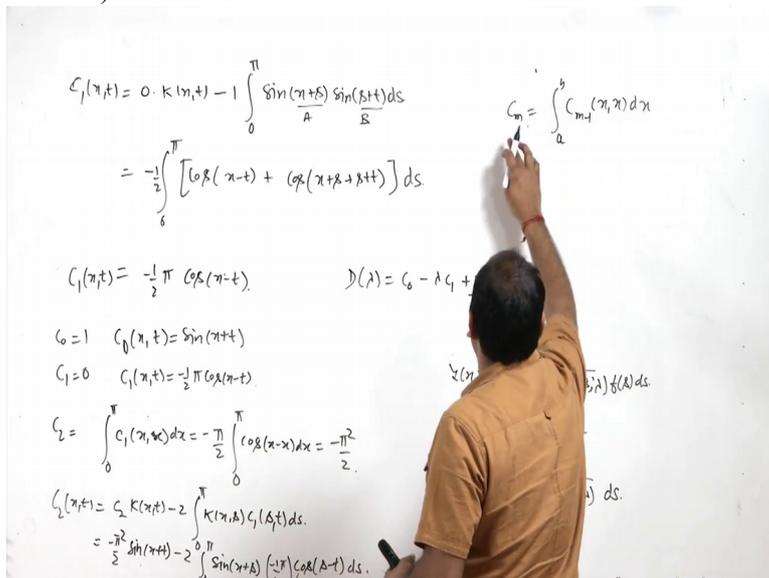
important And next one is c m x t is equal to small c m K of x t minus m times this a to b K of x of s and c m minus 1 s of t d t So if you know these two formula you can find out

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all the coefficients of D x t lambda and D lambda So here also I am not calculating but you can say that small c naught is given to you

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Example 3

Solve the integral equation

$$y(x) = 1 + \int_0^1 (1 - 3xt)y(t)dt.$$

Here

$$c_0 = 1, \quad C_0(x,t) = K(x,t) = 1 - 3xt$$

$$c_1 = \int_0^1 C_0(x,x)dx = \int_0^1 (1 - 3x^2)dx = 0$$

$$C_1(x,t) = c_1K(x,t) - \int_0^1 K(x,s)C_0(s,t)ds$$

$$= 0 - \int_0^1 (1 - 3xs)(1 - 3st)ds = -3xt + 3/2(x + t) - 1.$$

c 1 is you can calculate from the help of capital C naught x t And it is coming out to be zero Similarly from this recurrence relation you can find out C 1 x t and you just plug in all the values here and you can see that this is coming out to be minus 3 x t plus 3 by 2 x plus t minus 1 So capital C 1 x t is known to us now Then we can calculate small c 2 And small c 2

$$c_2 = \int_0^1 C_1(x, x)dx = -\int_0^1 \frac{1}{2}(-3x^2 + 3x - 1)dx = -\frac{1}{2},$$

$$C_2(x, t) = c_2 K(x, t) - 2\int_0^1 K(x, s)C_1(s, t)ds$$

$$= -\frac{1}{2}(1 - 3xt) + 2\int_0^1 (1 - 3xs)(-3st + 3/2(s + t) - 1)dt = 0.$$

Since $C_2(x, t)$ vanishes, from (45) we find that $C_3 = C_4 = \ldots$ and $c_3 = c_4 = \ldots$ also vanish. Therefore

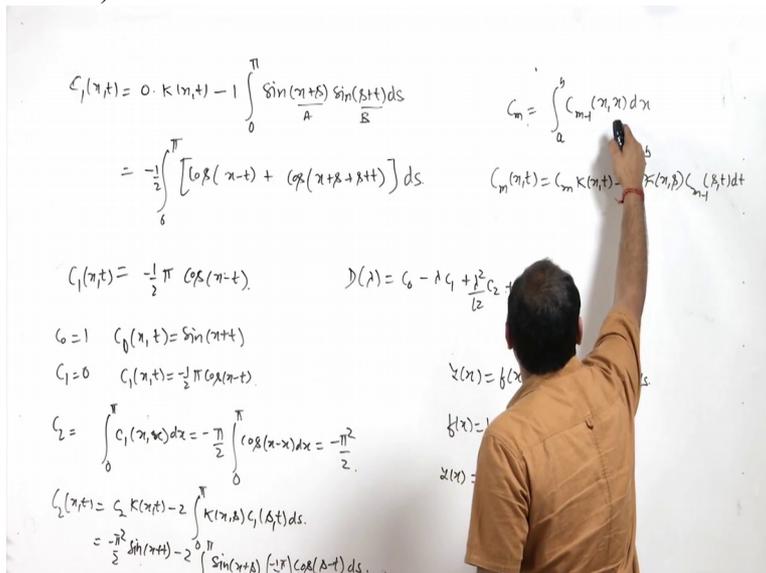$$\Gamma(x, t; \lambda) = \left[\sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} C_m(x, t)\right] \bigg/ \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} c_m$$

$$= \frac{2}{3}[4 - 3(x + t)].$$

it is equal to zero to 1 c 1 x comma x d of x and just plugging the value of c 1 x x and you calculate and it is coming out to be minus 1 by 2 And once we have small c 2 capital C 1 x t then use the recurrence relation to find out C 2 x t and which is given by small c 2 K x t minus 2 times zero to 1 K of x s c 1 s t d s Just plug in all the values and it is coming out to be zero And as soon as we are getting one value of say capital C i x comma t as zero then you can see that from the two recurrence relations this and this you can see that once your c m minus 1 x comma

c m minus 1 x comma t is zero then small c m is zero and from this relation you can see that C m is also zero So it means that as soon as your c m minus 1 x comma t is equal to zero then

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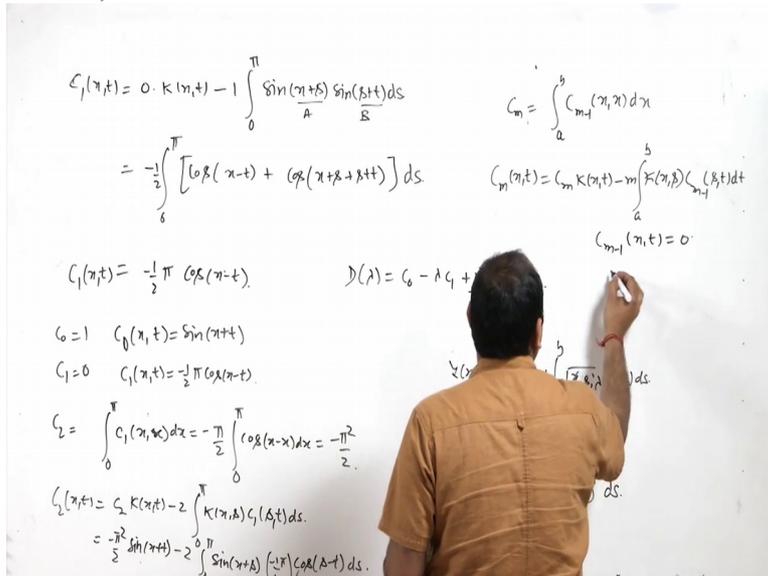
this implies that c m is equal to zero and this relation shows that c m x t is equal to zero and since because of these 2 relations you can see that all others are also zero small c m plus 1 and so on all are simply vanishing So it means that in this particular case when c m minus 1 x t is zero then all other higher coefficients are

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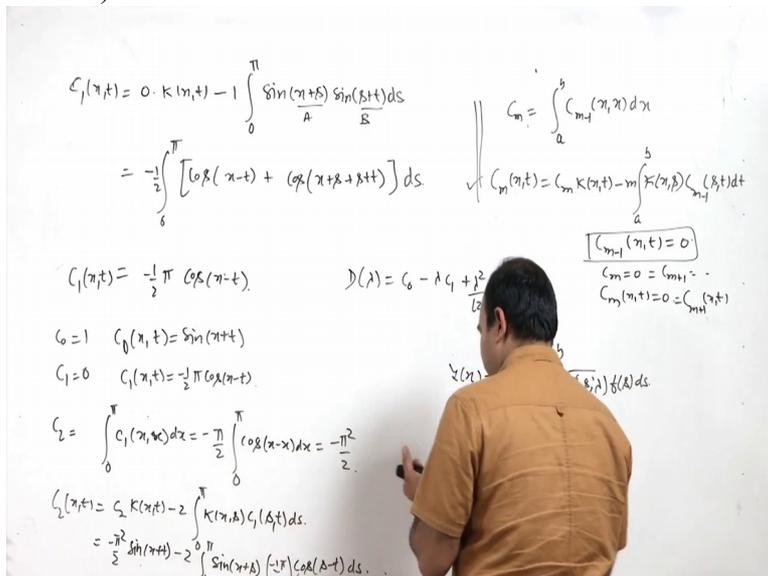
simply vanishing And your infinite series is truncated to finite series So here in this case also

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$$c_2 = \int_0^1 C_1(x,x)dx = -\int_0^1 \frac{1}{2}(-3x^2 + 3x - 1)dx = -\frac{1}{2}.$$

$$C_2(x,t) = c_2 K(x,t) - 2\int_0^1 K(x,s)C_1(s,t)ds$$

$$= -\frac{1}{2}(1 - 3xt) + 2\int_0^1 (1 - 3xs)(-3st + 3/2(s+t) - 1)dt = 0.$$

Since $C_2(x,t)$ vanishes, from (45) we find that $C_3 = C_4 = \ldots$ and $c_3 = c_4 = \ldots$ also vanish. Therefore

$$\Gamma(x,t;\lambda) = \left[\sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} C_m(x,t)\right] / \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} c_m$$

$$= \frac{2}{3}[4 - 3(x+t)].$$

since c 2 x t is coming out to be zero so all other higher order coefficients are simply vanishing So gamma x t lambda is given by ratio of two finite series and in this case it is coming out to be 2 by 3 4 minus 3 x plus t Now with the help of gamma x t lambda you can find out

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The solution is given by

$$y(x) = f(x) + \lambda \int_0^1 \Gamma(x,t;\lambda)f(t)dt$$

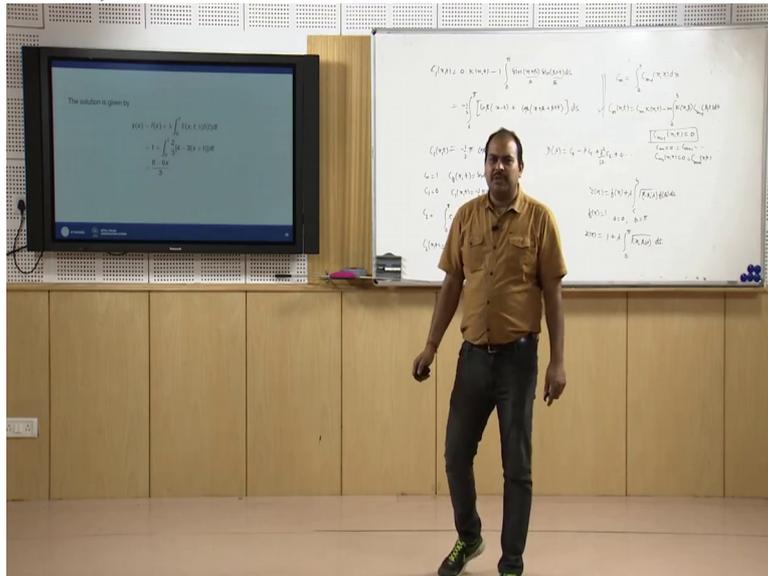$$= 1 + \int_0^1 \frac{2}{3}[4 - 3(x+t)]dt$$

$$= \frac{8 - 6x}{3}.$$

the solution here y of x equal to f x plus lambda zero to 1 gamma x t lambda d t here Now f x is simply 1 lambda is 1 here and you can find out gamma x t lambda f t d t like this So when you plug in all the values you can get y of x equal to 8 minus 6 x by 3

So with the help of discussed theory we are able to find out the solution of Fredholm integral equation of second kind like this So in next lecture we will see the case when D lambda will be equal to zero Because when if you look at here we have defined gamma x t lambda as the ratio of 2 infinite series that is D x t lambda divided by lambda Let me write it here Here your solution is what we have just seen that if we have equation like this y x equal to f of x plus lambda a to b K of x t y t d t then
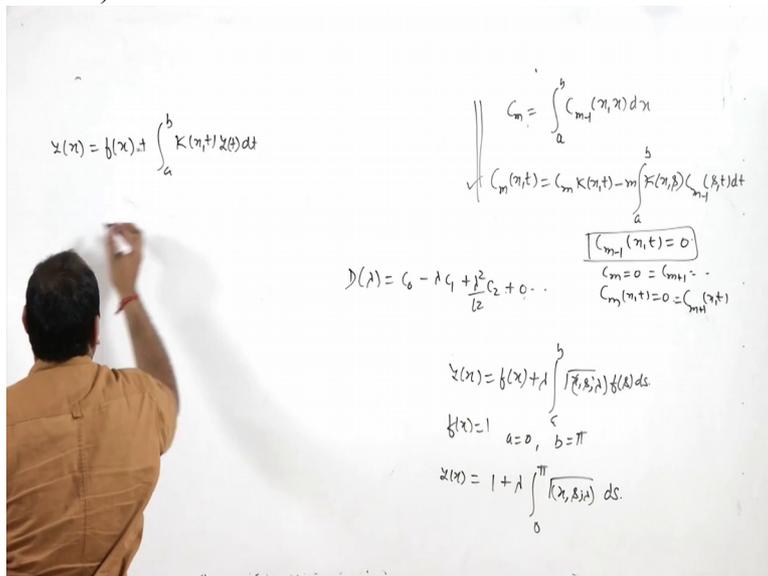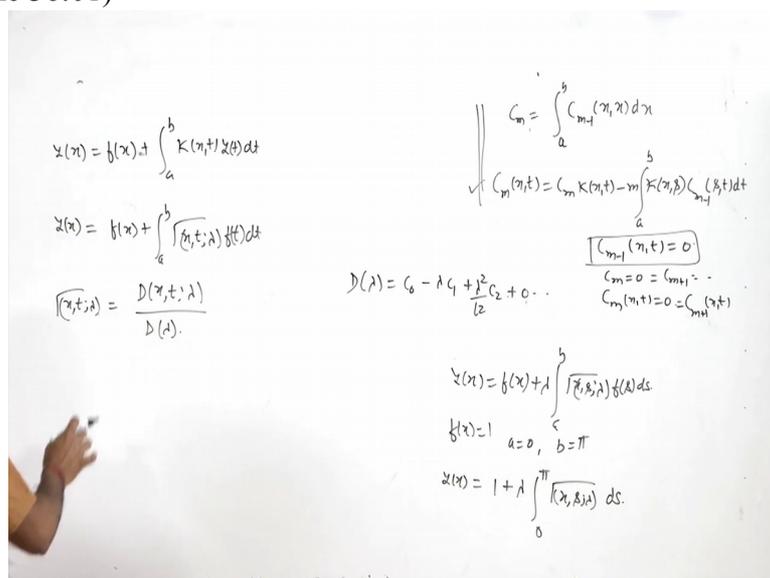
solution in this lecture we have seen that the solution is coming out to be f of x plus a to b gamma x t lambda f of t d t where
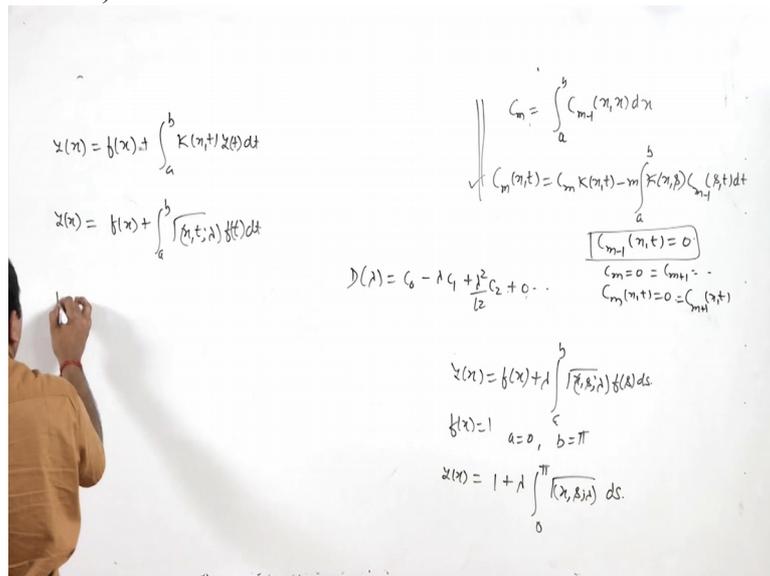
gamma x t lambda is defined as the ratio of two infinite series given by D x t lambda divided by D lambda So it means that

this solution is valid only when this D lambda is non zero It means that the lambda which is present here this lambda is

not a root of this Fredholm determinant D lambda So it means that if lambda is such that D lambda is not equal to zero then this solution will give you this expression will give you a unique solution of this Fredholm integral equation But what happens when lambda is such that D lambda is equal to zero That case we are going to discuss in next lecture So thank you for listening us We will meet in next lecture thank you