Integral Equations, Calculus of Variations and their Applications
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Mod 06 Lecture Number 21
Classical Fredholm Theory Fredholm First Theorem-I

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Hello friends, welcome to today's lecture. So if you remember we solving this problem y of x equal to f of x plus lambda into b K of x of t y of t d t

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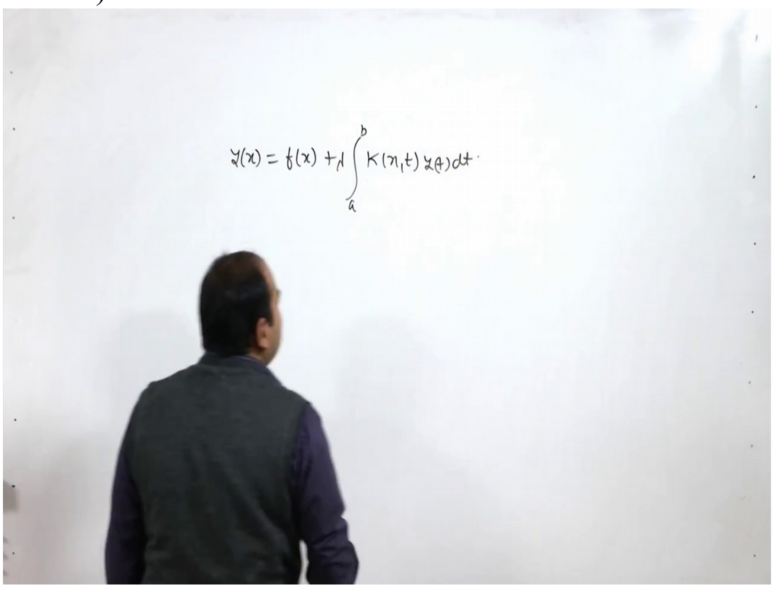

and here we are assuming, we have assumed that this K x t and f x are

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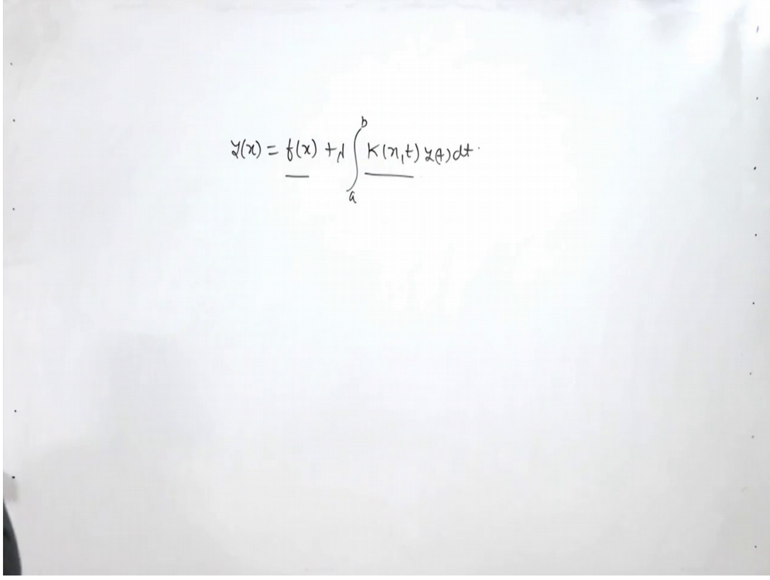

same l d o functions and we try to find the solution of this problem as y of x is equal to f of x plus lambda a to b gamma x of t lambda f of t d of t

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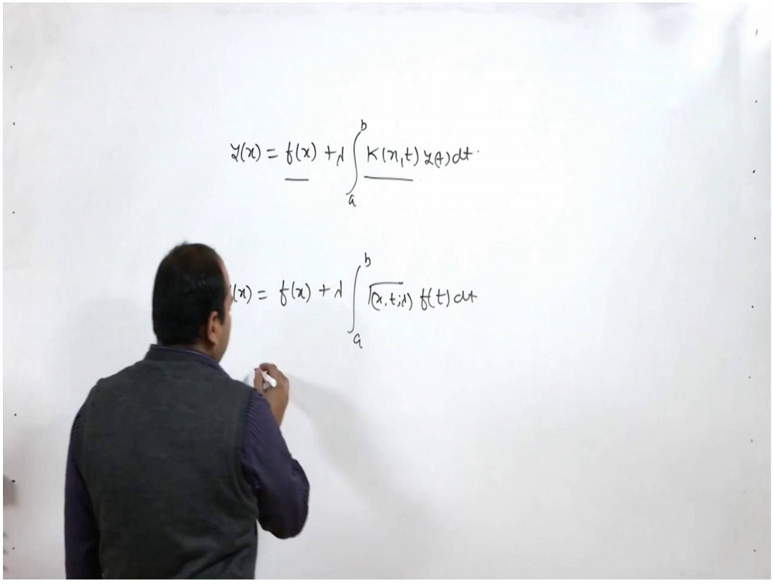

where gamma x of t lambda is given as D x t lambda divided by D lambda provided that D lambda, lambda is such that D lambda is not equal to zero

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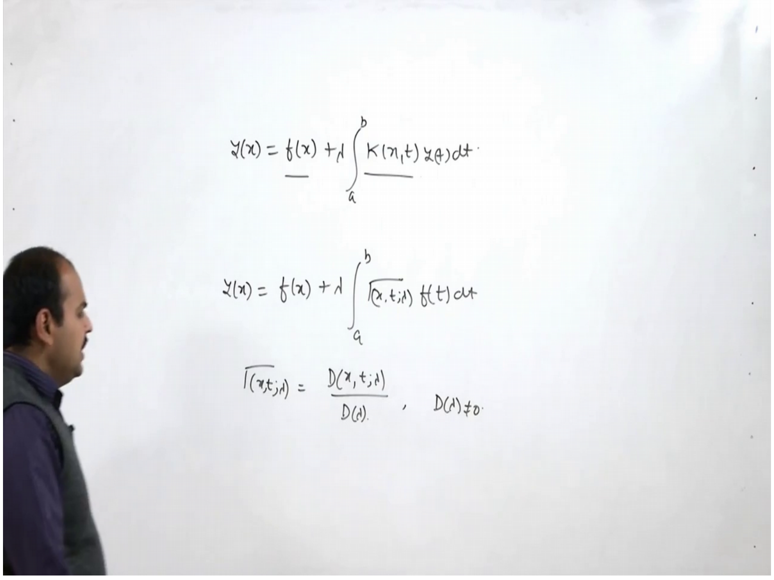

and we have seen in the previous lecture we have obtained the expression of D lambda and in this lecture we try to find out the expression for resolvent kernel given in this particular form.

So we try to find out this expression for D x t lambda such that we can have the solution here. So please recall that

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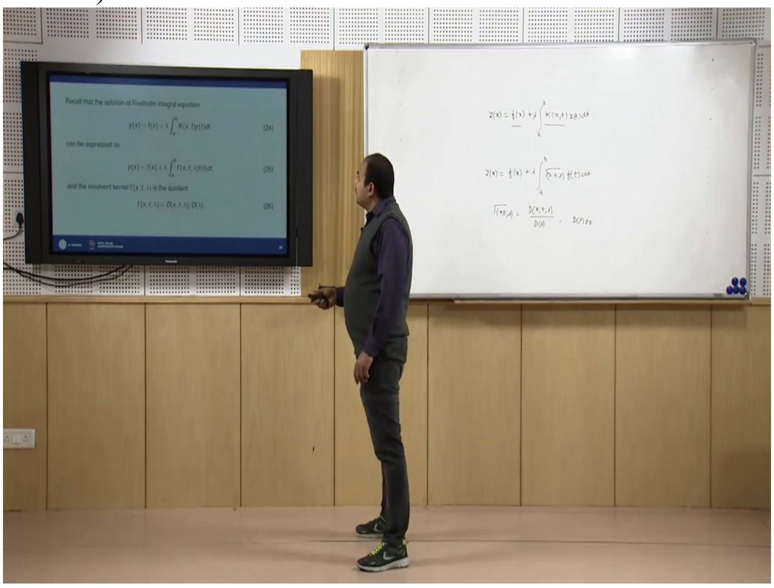

the solution of this Fredholm integral equation can be written like this. So this portion

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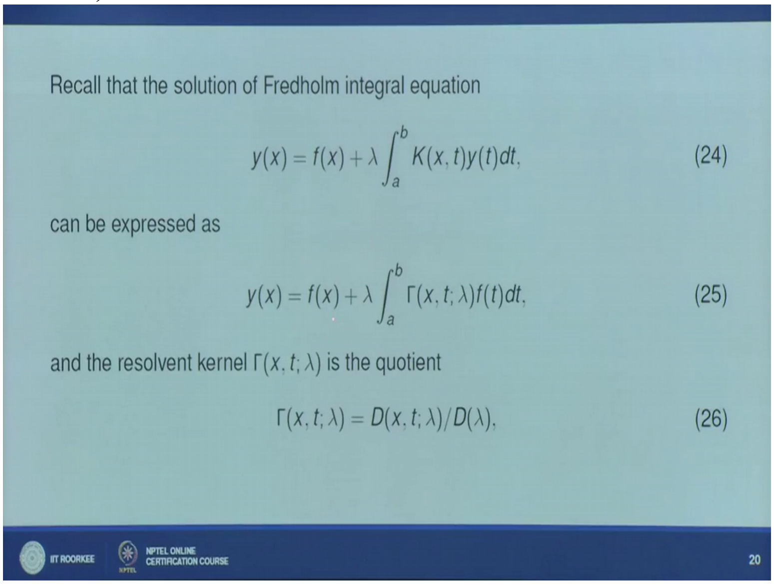

we have already discussed for the case when K x t is given in terms of separable kernel or we have already solved for, say method of approximation. So here we have seen here that the solution is given by this and gamma x t lambda is given by the quotient here. So we have seen the D lambda. The only thing we want to see that is there exist any resolvent kernel of this kind so that our solution can be written like this.

So here

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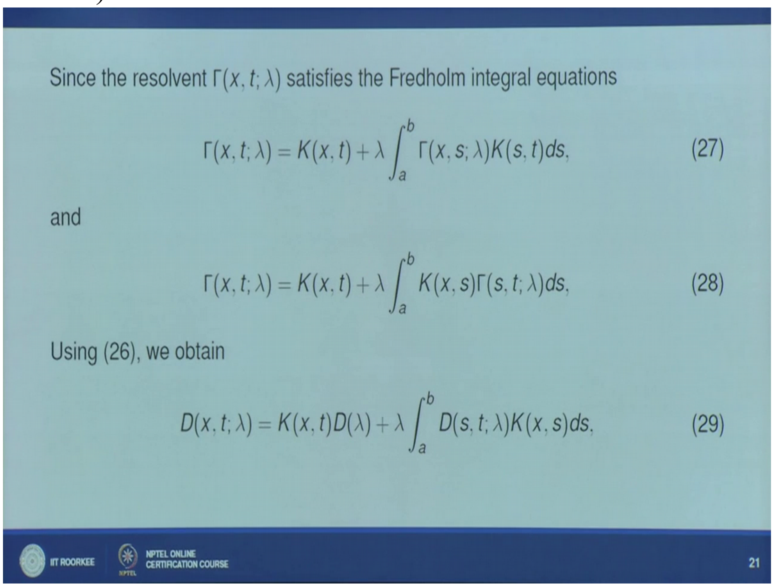

if we remember, we have seen that this resolvent kernel satisfies the Fredholm integral equation. So it means that

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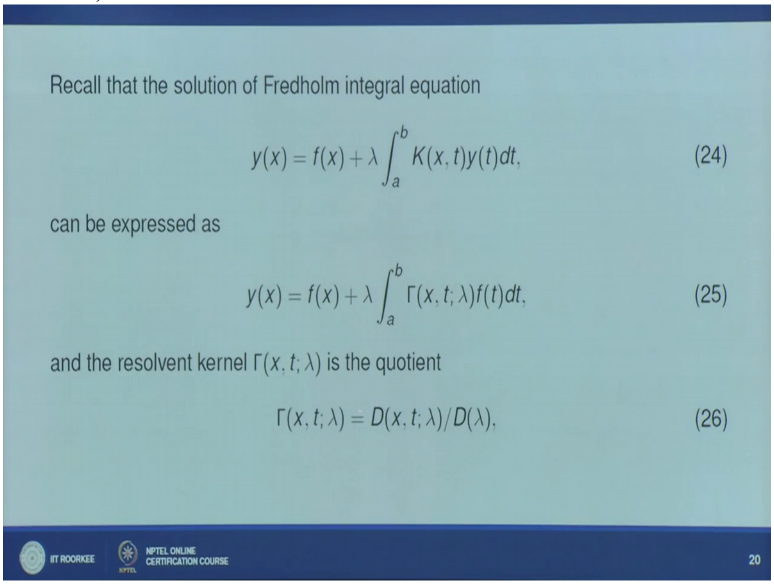

we want to find out a resolvent kernel gamma x t lambda such that it is given by this ratio of D x t lambda divided by D lambda and satisfy

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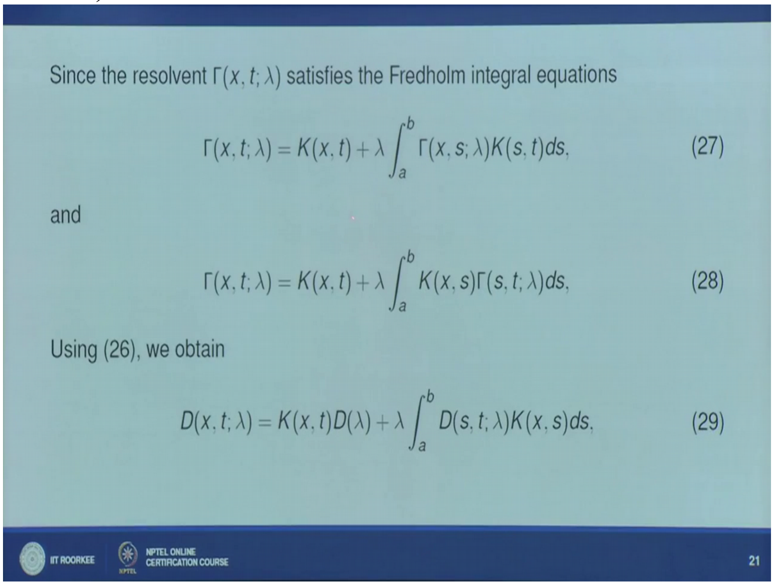

the following Fredholm integral equation. And if your remember this, we have already given in terms of, say in the lecture

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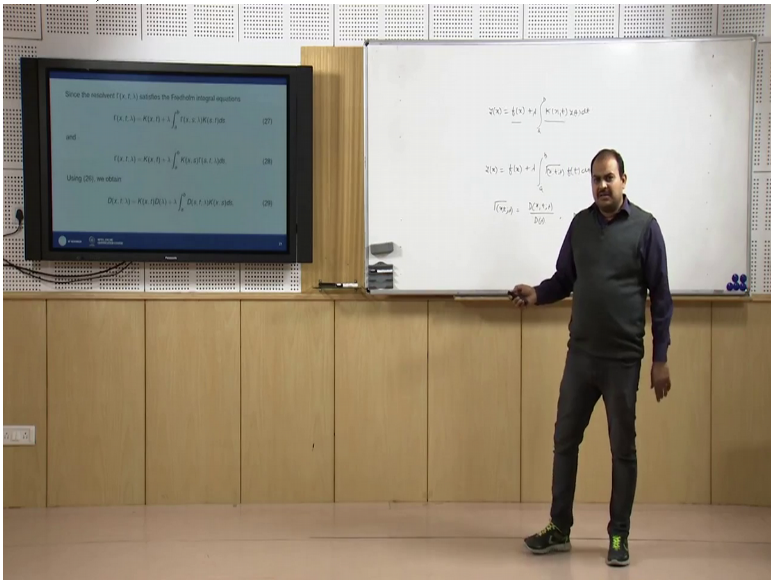

when we discussed the method of successive approximation, there we also proved these two relations 27 and 28, not proved it is given in terms of this. So here this gamma x t lambda satisfies the Fredholm integral identities given as

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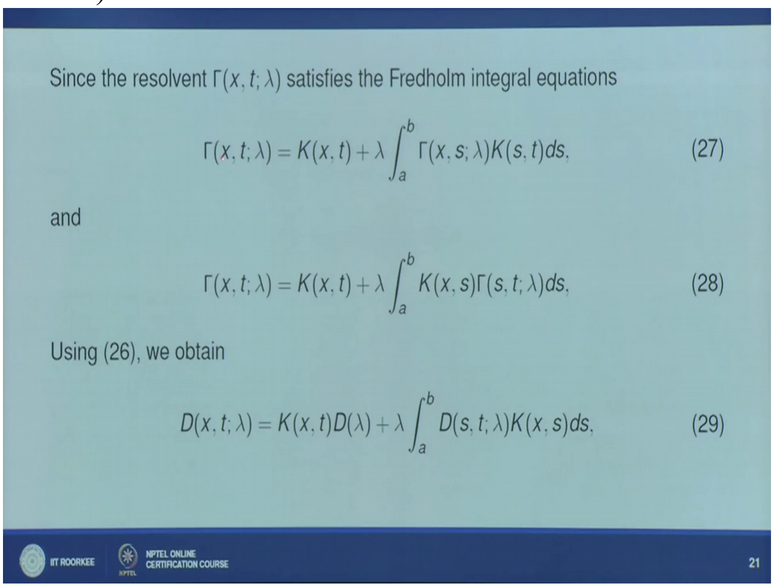

these; gamma x t lambda is equal to K of x t plus lambda a to b gamma x s lambda K s t d s and the other identity is given as gamma x t lambda as K of x t plus lambda a to b K x s plus gamma x t lambda d of s. We are going to use this identity here.

So here if I take this

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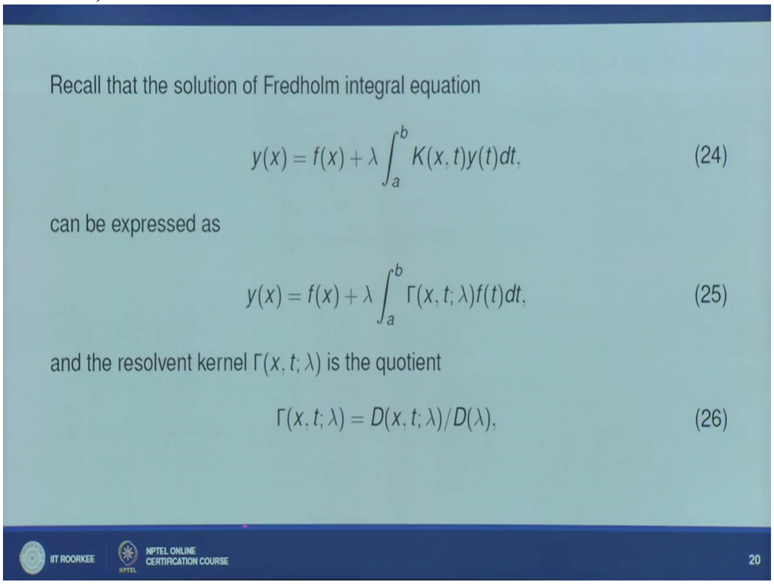

notation that gamma x t lambda is given by ratio of these two quantities, then you can

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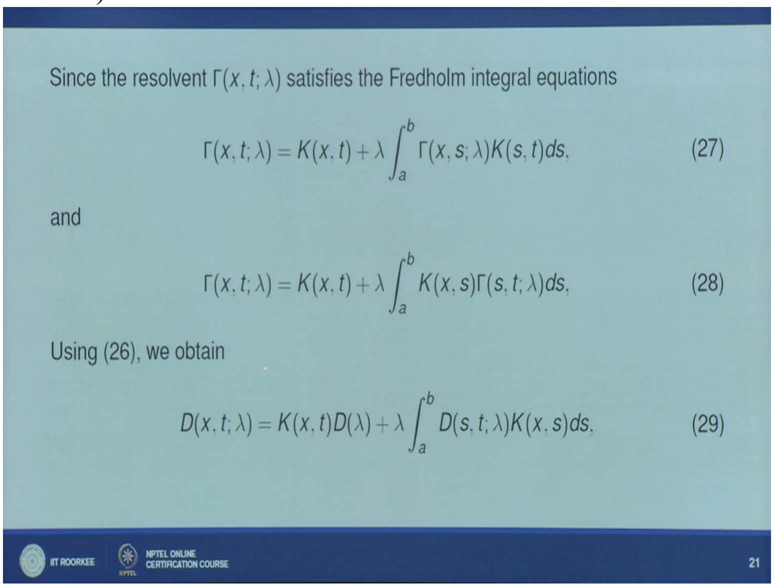

write gamma x t lambda as D x t lambda divided by D lambda. So you can take D lambda the other side and you can write this as

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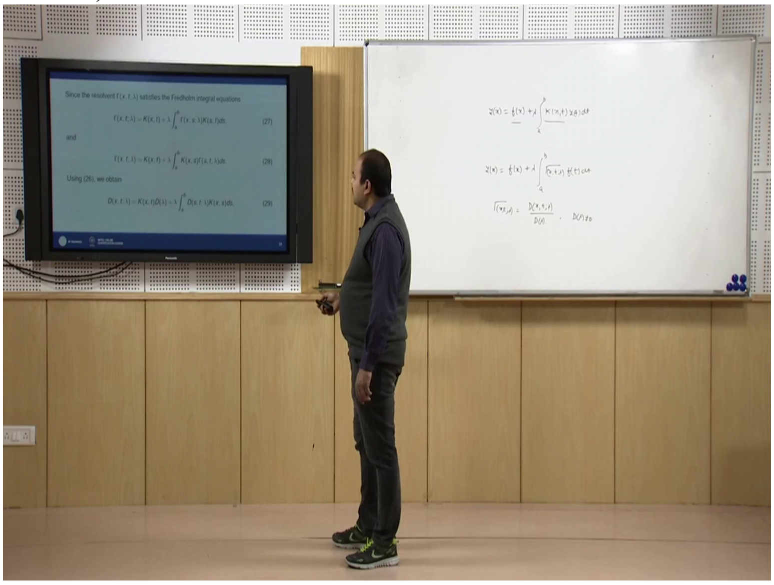

D x t lambda as K x t D lambda plus lambda. Here also we have used the notation that gamma x t lambda as D x t lambda divided by

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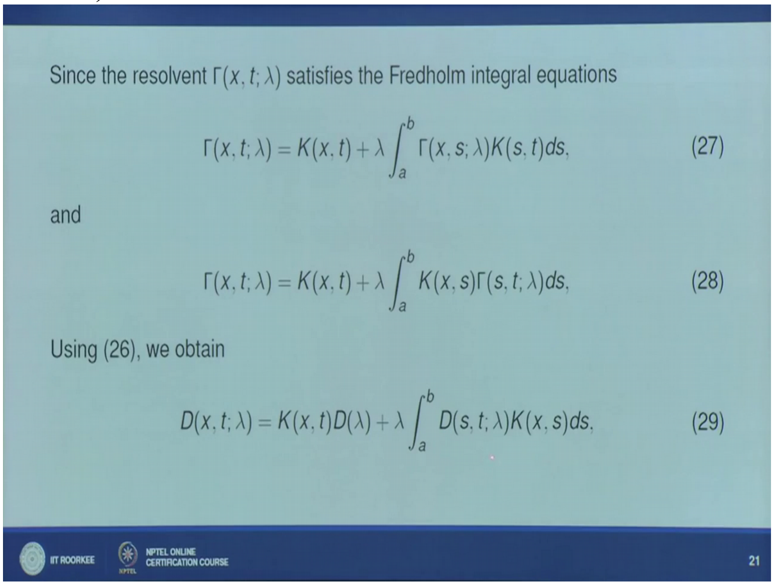

D lambda. So when you simplify, so this equation number 28, you can write it like this. So here when you write it this, then I can write here that your equation that gamma x t lambda satisfy is that gamma x t lambda is equal to K of x t plus lambda times a to b K of x s s t gamma x s lambda D of s. Now here I am writing this as this. So here I can write

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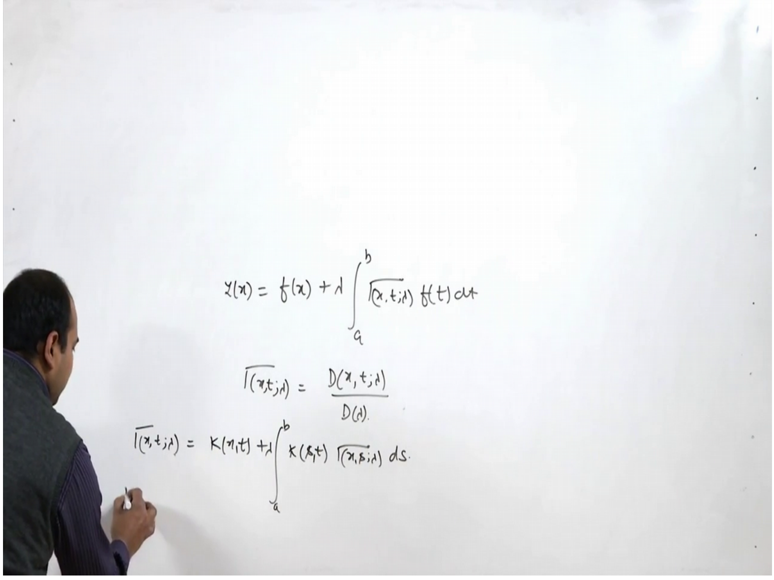

D of x t lambda divided by D lambda equal to K of x t plus lambda times a to b. Here I am writing D x s lambda x s lambda divided by D lambda and K of s t D of s.

So D lambda you can multiply

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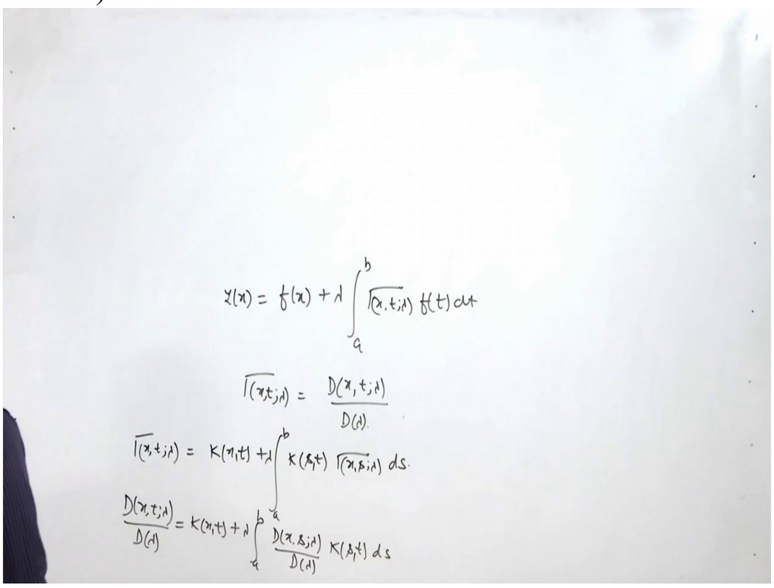

to all and you can get that equation number 29 is valid.

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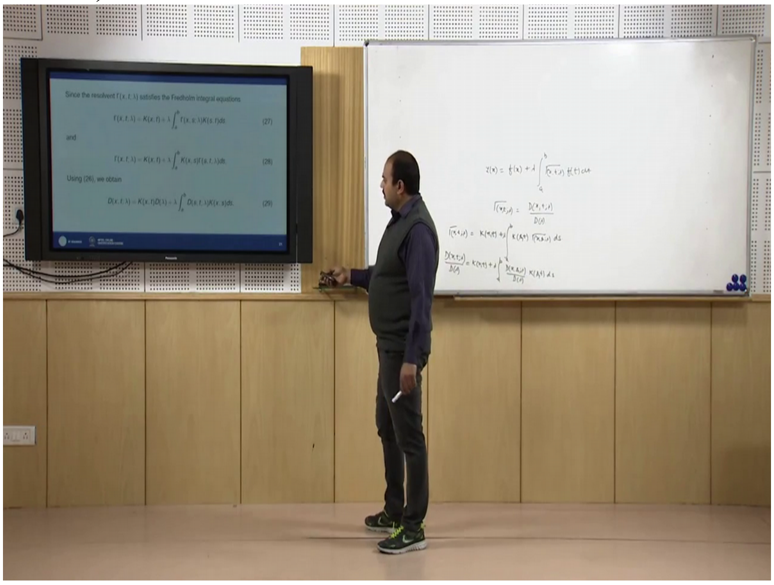

So here D x t lambda is written

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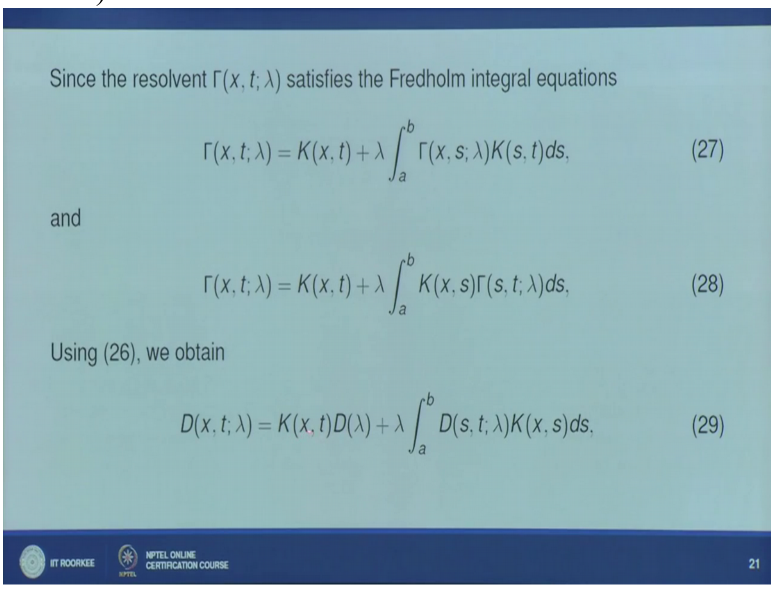

as K x t D lambda plus lambda times a to b D s t lambda K x s d of s. So it means that

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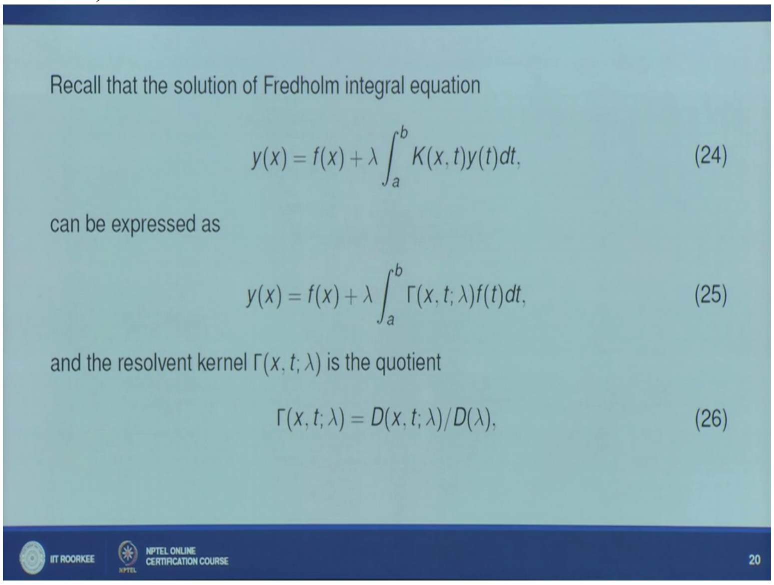

if our solution which we want in this particular form, y x equal to f x plus lambda a to b gamma x t lambda f t D t where we want that this resolvent kernel can be written in this particular form. Why we are choosing this, because we have already seen that in case of separable kernel your, your solution can be written like this. So here we want that in case

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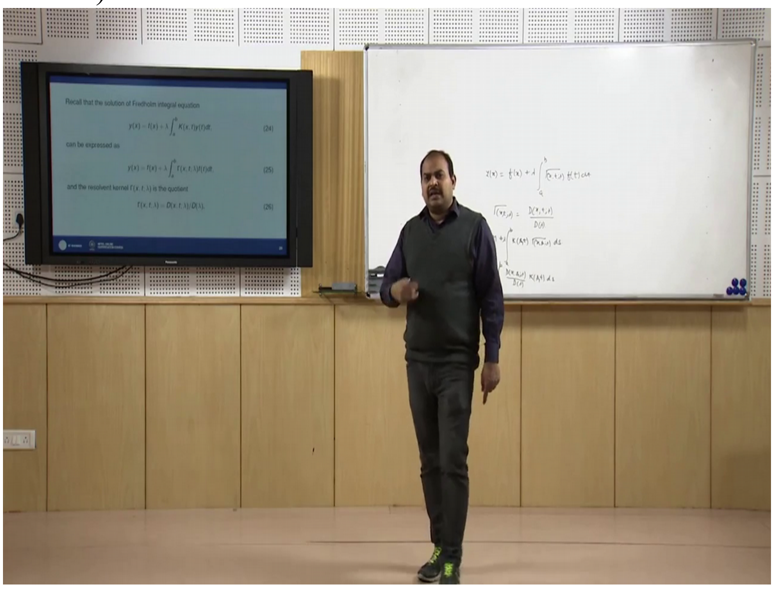

when we have a general kernel, can we have the similar kind of solution so that when we take x t as the kernel which is separable, the same results are identical. The result obtained in separable case and result obtained here are identical. Or we can say that we are generalizing the result which we obtain for separable kernel.

So it means that to find out this

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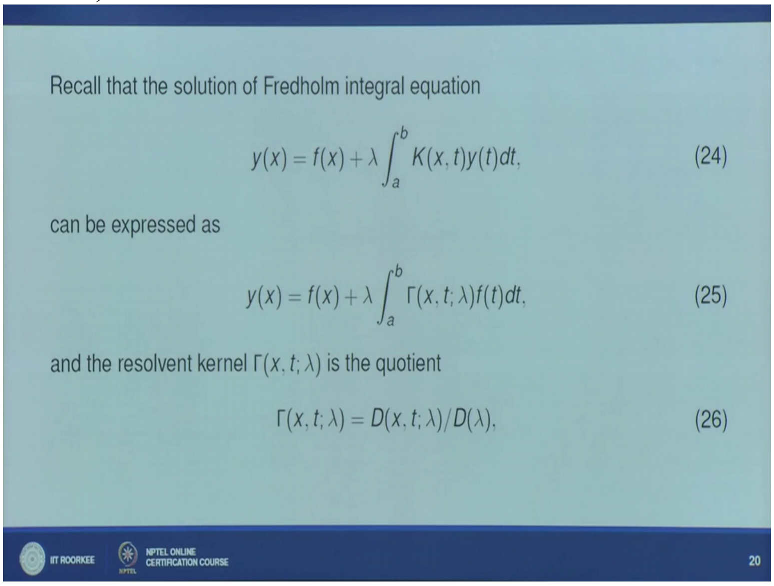

we try to find out D x t lambda, because D lambda which we have already obtained. So then we try to find out the identifying the expression satisfied by D x t lambda and we have seen that

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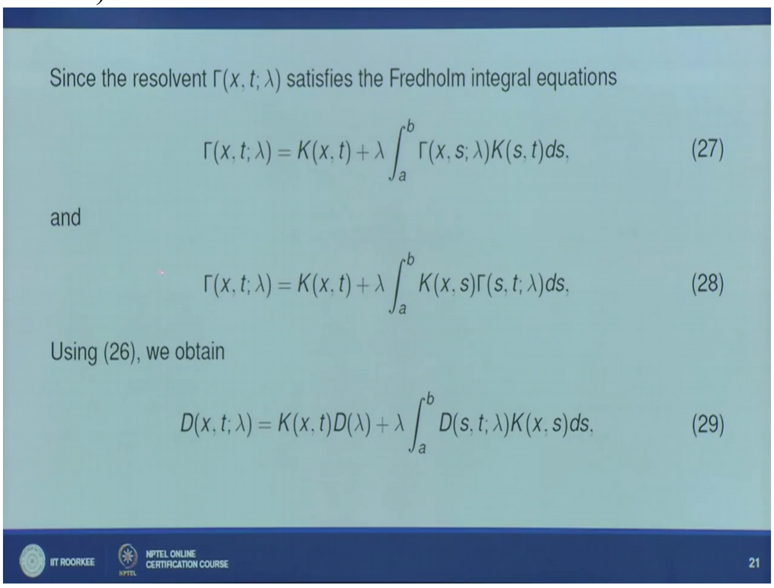

if gamma x t lambda which satisfy these Fredholm identities then you can say that D x t lambda is something which satisfy this equation 29. So this is an integral equation satisfied by D x t lambda and with the help of this equation number 29, we we try to find out D x t lambda. So here

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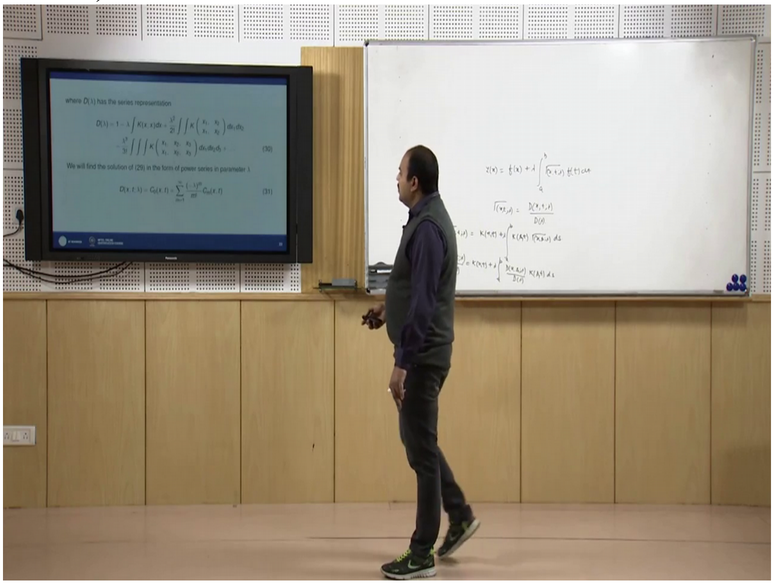

we can say that here, if D lambda which is, which we have obtained like this, then you can say this is kind of power series in terms of lambda. So this gives that if D lambda is a power series in terms of lambda then D x t lambda will satisfy this equation number 29 will also be written in terms of power series in terms of lambda and in a similar way which we have already mentioned that the convergence of D lambda can be discussed in a similar way we can also discuss the convergence of D x t lambda.

So let us assume first of all that the solution of equation number 29 can be written in terms of power series in the parameter lambda. So looking at the expression given in terms of D lambda as

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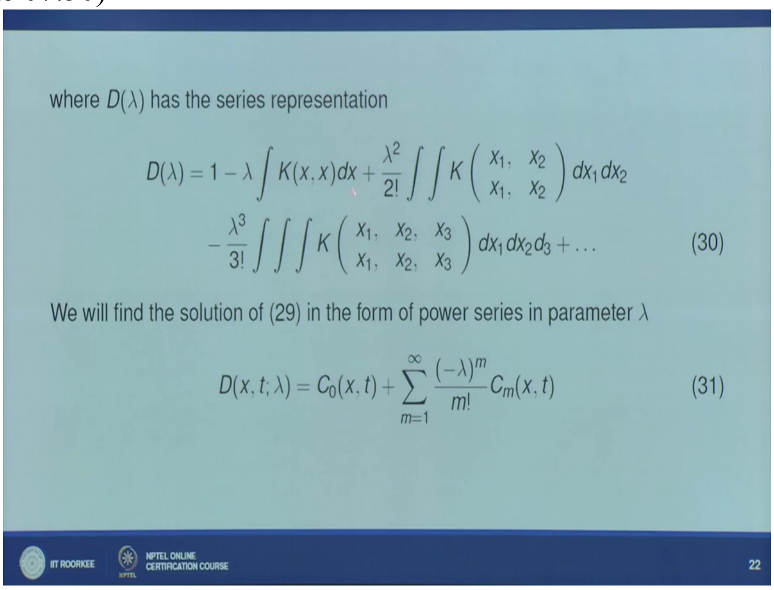

a power series in terms of lambda we assume that D x t lambda is also written in terms of power series in terms of lambda. So here we take the coefficient like this. C naught x t to C m x t m equal to 1 to infinity, we want to find out this C m x t coefficient C m x t such that D x t lambda satisfy the equation number 29.

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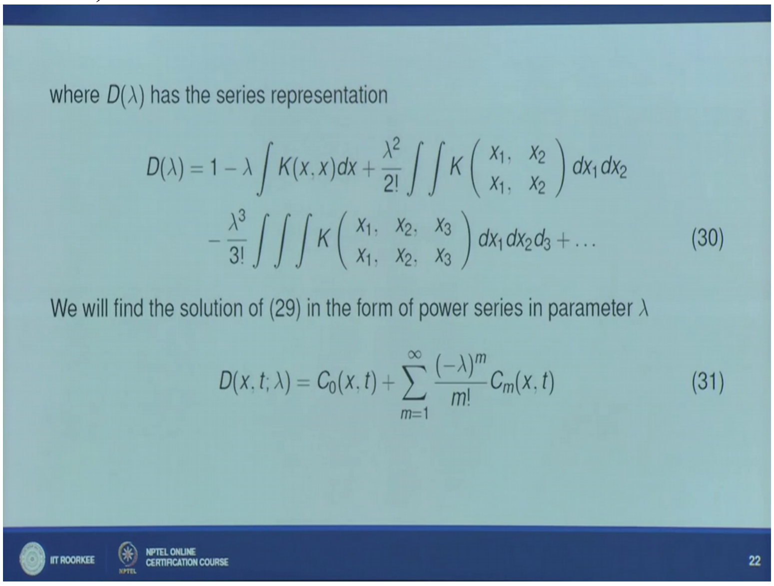

So here let us further simplify this notation. Let us write this

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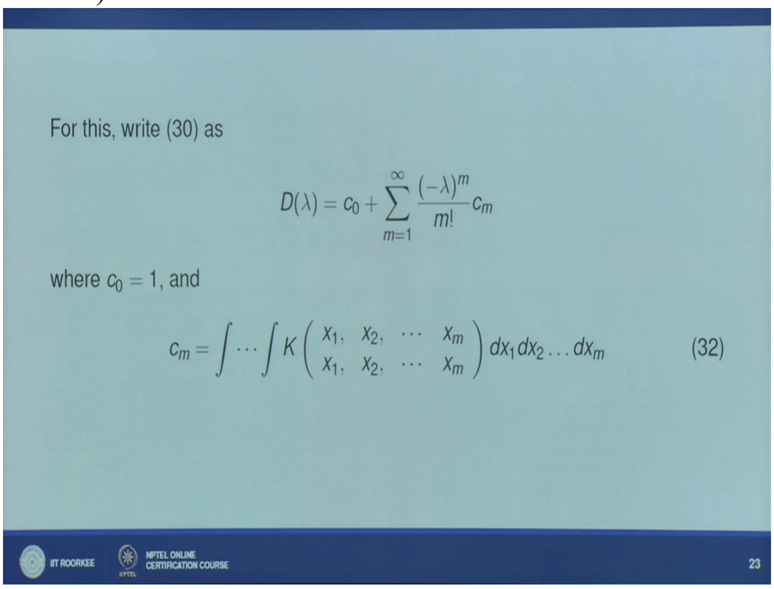

D lambda as C naught plus m equal to 1 to infinity minus lambda to power m factorial m C m where C m, if you look at what we have denoted here, c m is the coefficient of minus lambda to power m divided by factorial m and it is given by this, this m-fold integral of this quantity K x 1 to x m and

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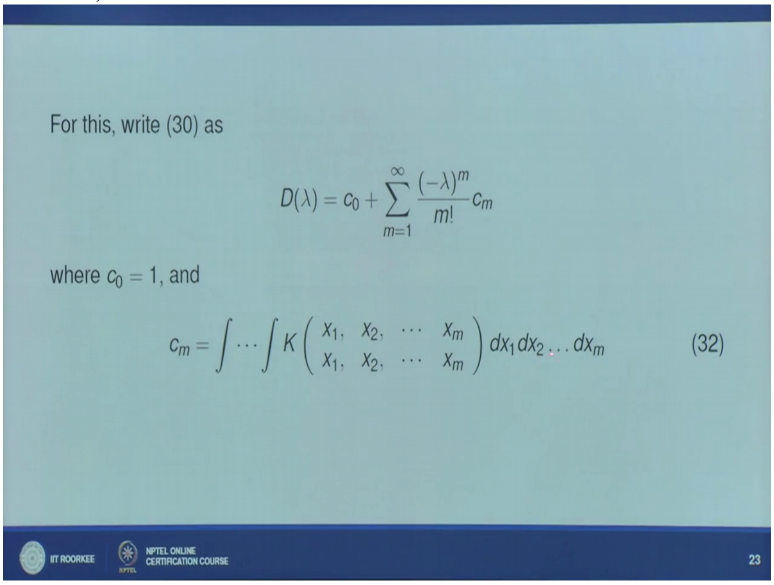

d x 1 d x 2 d x m. So here c naught is given by 1 which is obvious from this. So here we just writing D lambda in terms of this power series.

Now with the value given as D lambda

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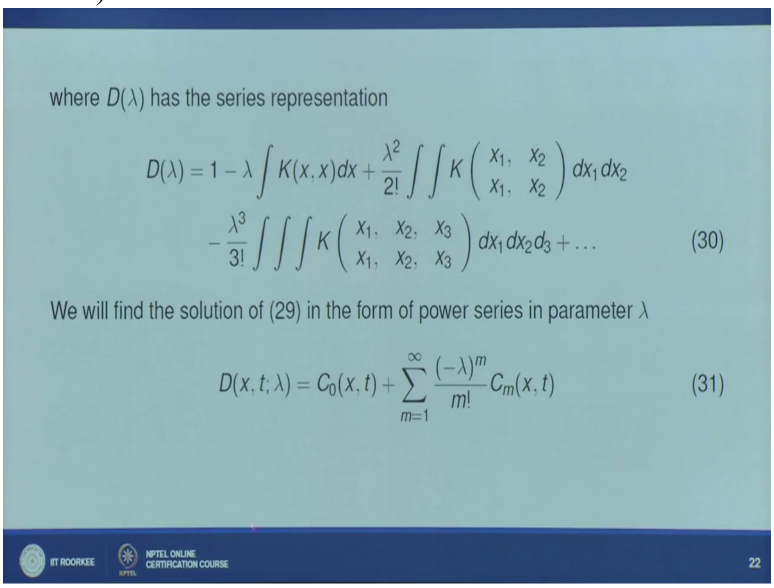

and the assumption that D x t lambda will be of this kind, we try to find out the relation which this C m x t satisfied and hence we tried to find out the expression the C m x t satisfy.

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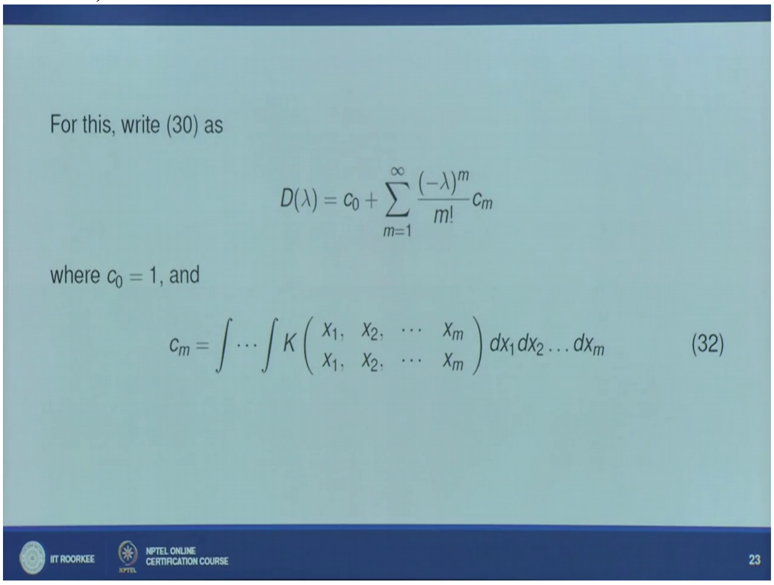

So here to do this, what we try

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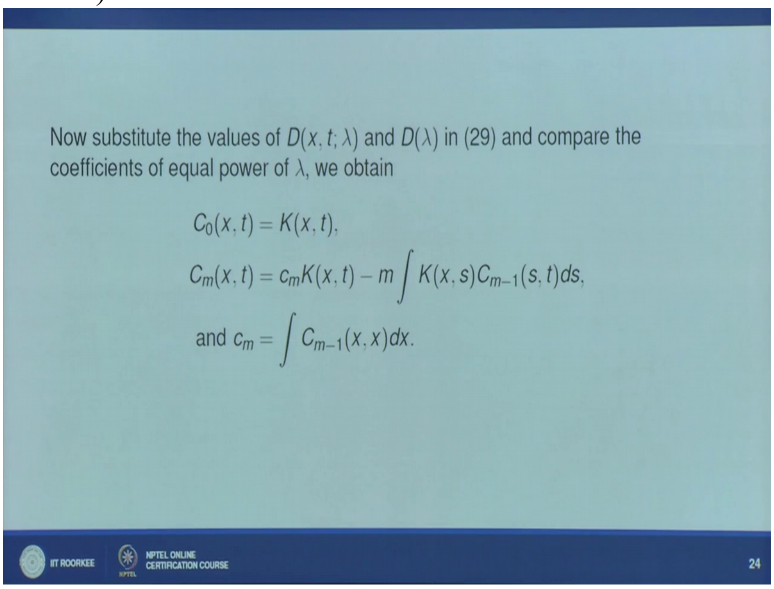

to do here we write down

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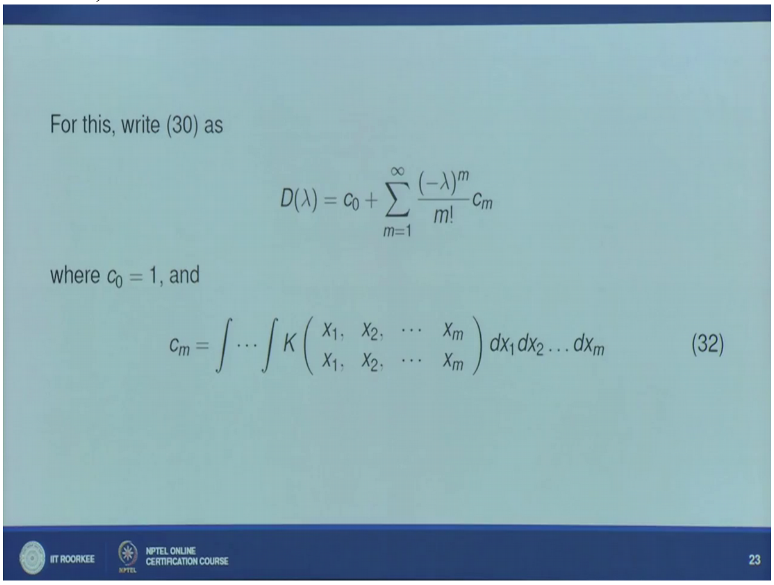

the expression D lambda and D x t lambda in this equation number 29.

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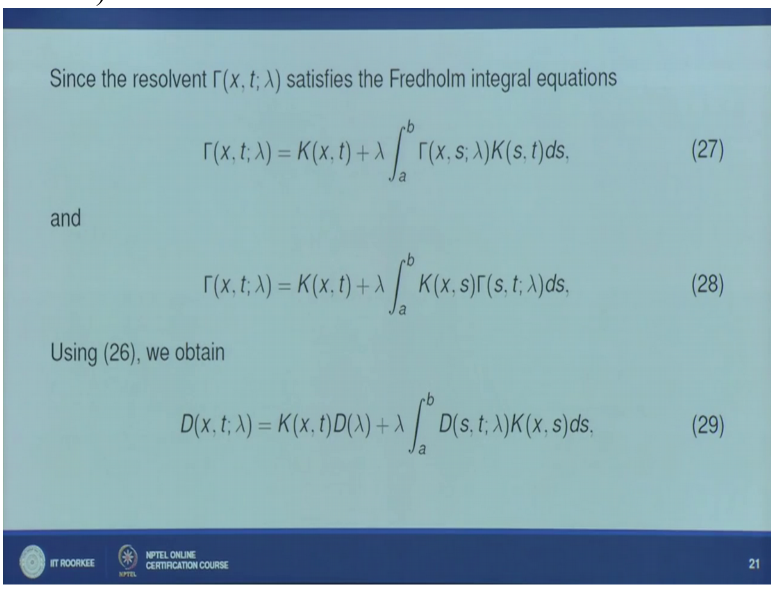

So if you look at what we try to do, let me write it here. Just I am giving you hint, you can do it on your own. So here I am writing D lambda as c naught c naught plus summation minus lambda to m factorial m and c m, small c m.

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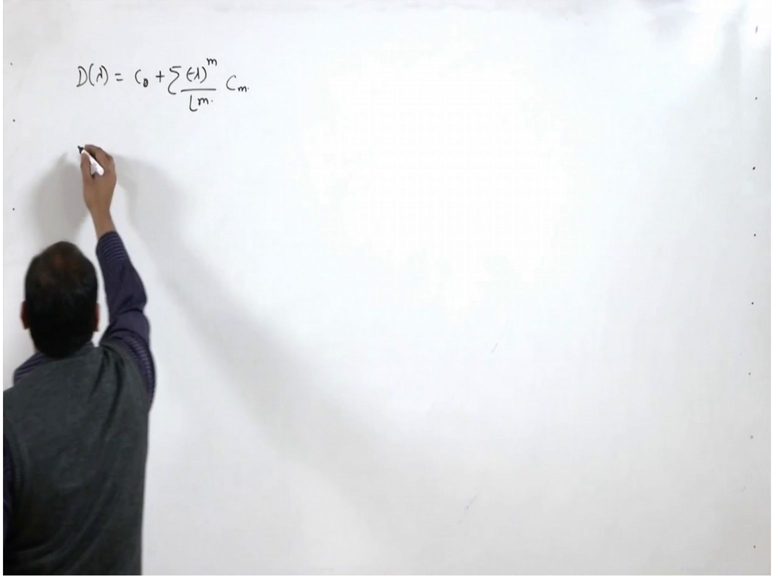

And D x t lambda it is C naught x of t plus summation minus lambda to power m factorial m, m is from 1 to infinity. Here also m is equal to 1 to infinity and this is capital C m x of t. And we want to find our C m x t because c m, c naught, c naught is 1 here and c m is already obtained as this, so here your C m is

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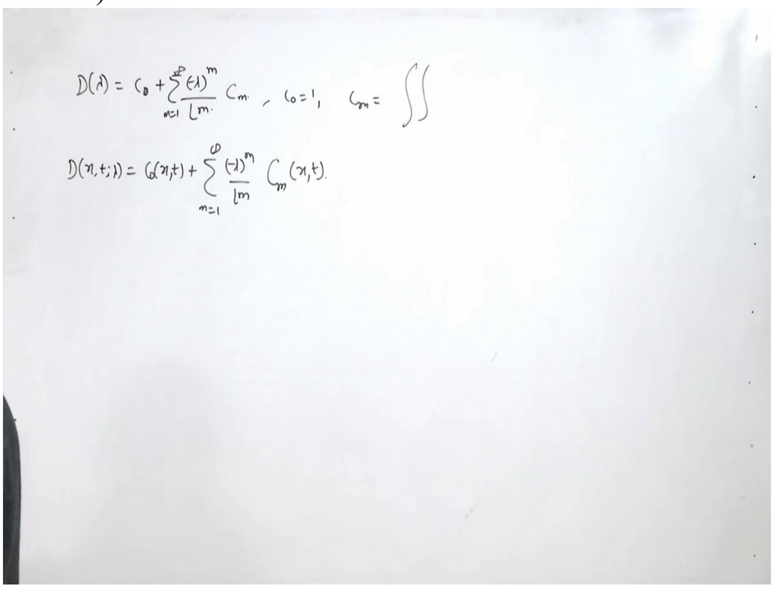

given by this m-fold integral.

So here this is m-fold integral, m integral and K of x 1 to say x m and x 1 to x m, notation we have already discussed. So

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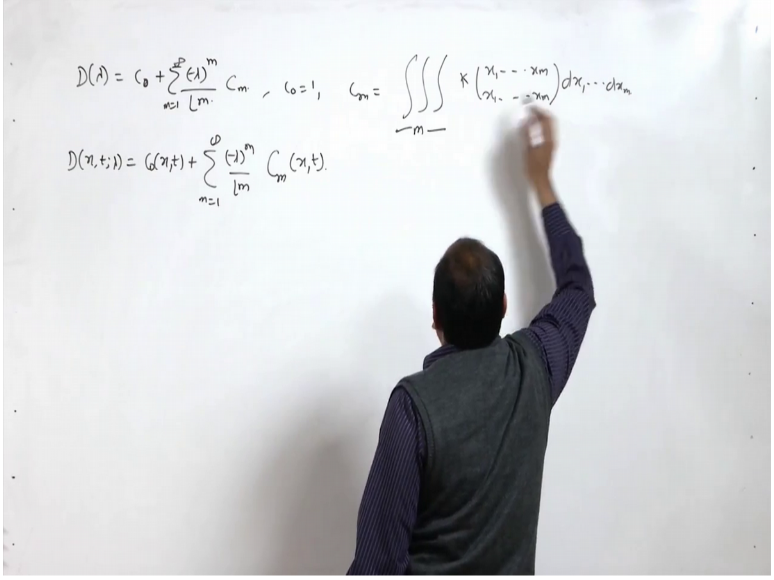

with the value c naught, C m known to us, we try to find out D x t lambda, it means we need to find out C naught x t. Now we know that

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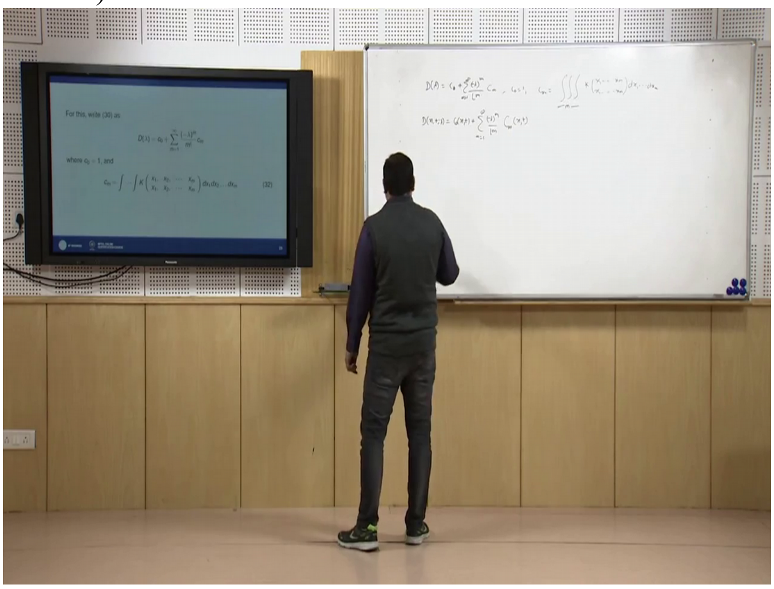

this D x t lambda satisfy this relation, this thing. So here D x t lambda is satisfying this relation. D x t lambda is equal to K x t D lambda plus lambda times a to b t of s t lambda and K of x s D of s. Or

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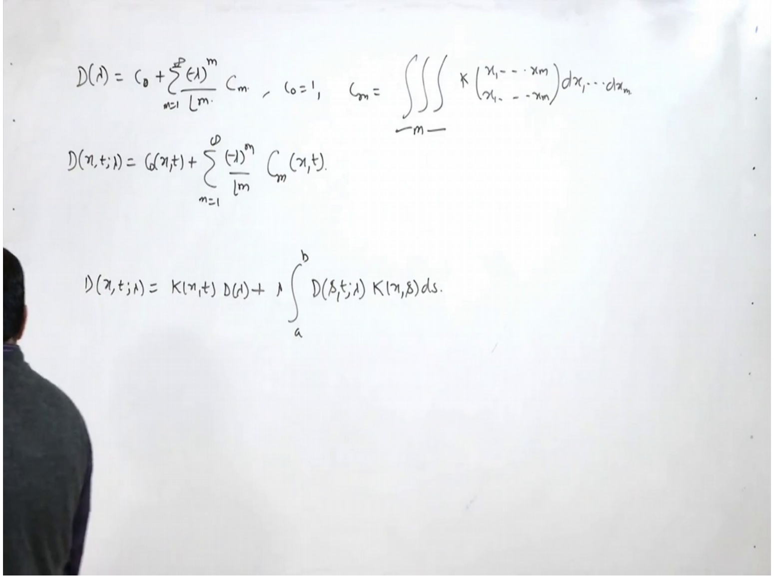

you can write it K of x D s t lambda. So here let me write it here, C naught x of t plus summation m equal to 1 to infinity minus lambda to power m, let me, yes, minus lambda to power m upon factorial m C m x of t minus, I am just taking this K x t here and you can write it here no problem; this is what this I can write C naught plus summation m is from 1 to infinity minus lambda to power m, this is minus lambda, factorial m C m plus lambda times a to b and here I am writing D x t lambda. So you can write it K x s here, no problem and then D s t lambda in the same way, so it is C naught, I am writing s t plus summation m is equal to 1 to infinity minus lambda to power m factorial m and this is what C m s of t and this is integration with respect to s here.

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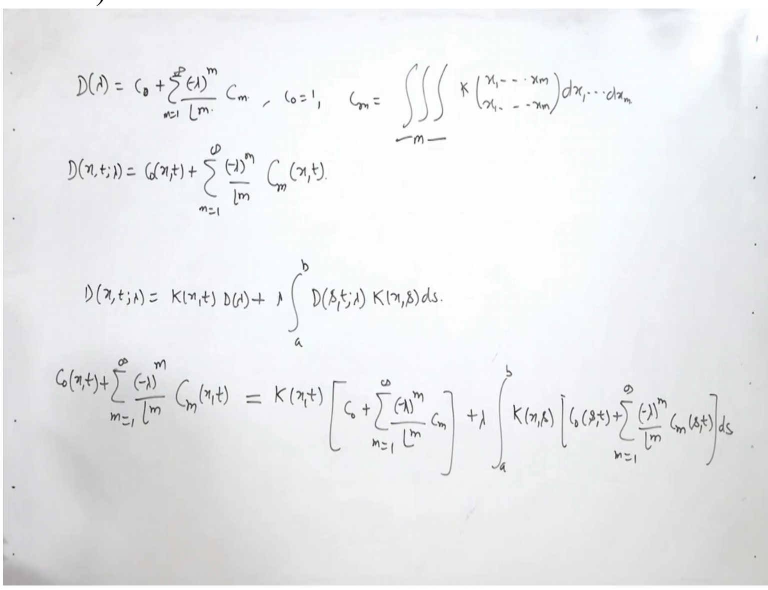

So if we look at the constant terms, C naught is simply 1, I can write it 1 here and

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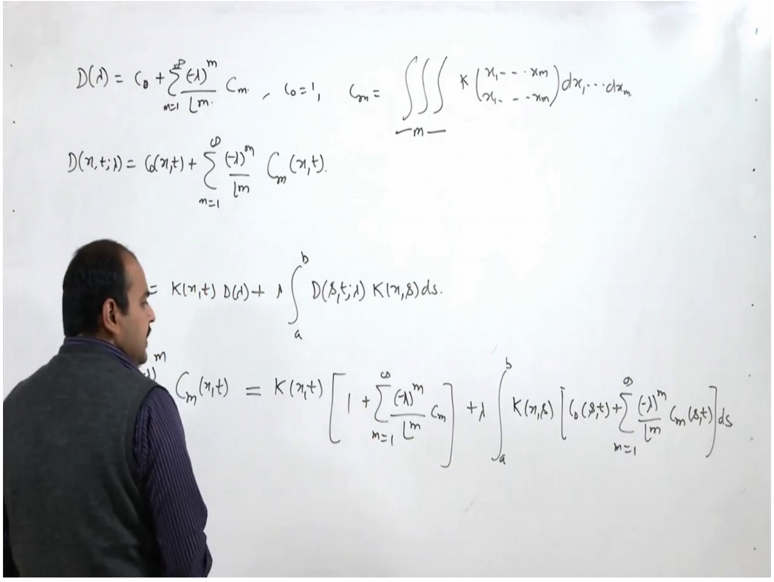

yeah, if you look at, the constant term is what? So constant term on both the side is, here it is C naught x of t and here, rest all containing lambda here and here also, we have constant term given as K of x t. So K of x t as C naught x t so if we compare the constant term on both the side, you can get

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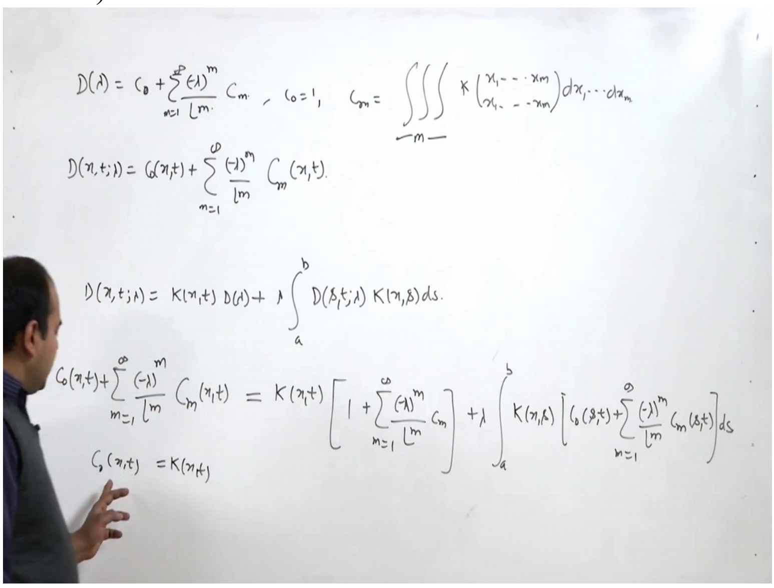

C naught as K of x t. Now to find out C m x t, so what is C m x t? C m x t, I can consider as the coefficient of minus lambda to power m upon factorial m. So we need to find out the coefficient of minus lambda to power upon factorial m. So here or you can say, let's find out

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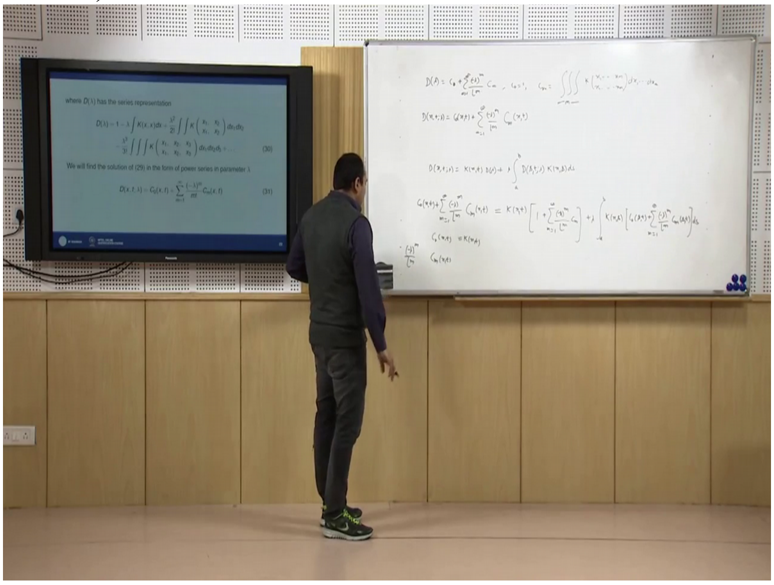

the coefficient of minus lambda to power m. So it is what, factorial m, so this C m x t into factorial m. So here I am just finding the coefficient of minus lambda to power m. So if you look at, here we have factorial m C m x t now equal to K x t we have, and here what we have is the coefficient of minus lambda to power m, that is C m, small c m divided by factorial m. I think something wrong here, this is divided, this is not in multiplication, it is divided factorial m. Ok, plus since we have lambda here. So to find out coefficient of minus lambda to power m, I need to find out only the coefficient of minus lambda to power m minus 1 times minus, 1 minus.

So here I can write this as minus, and what you will get, integral of a to b, you have K of x s, and the coefficient of minus lambda to power m minus 1. So here that will be what? That will be your C m minus 1 s of t d of s, is that Ok? So,

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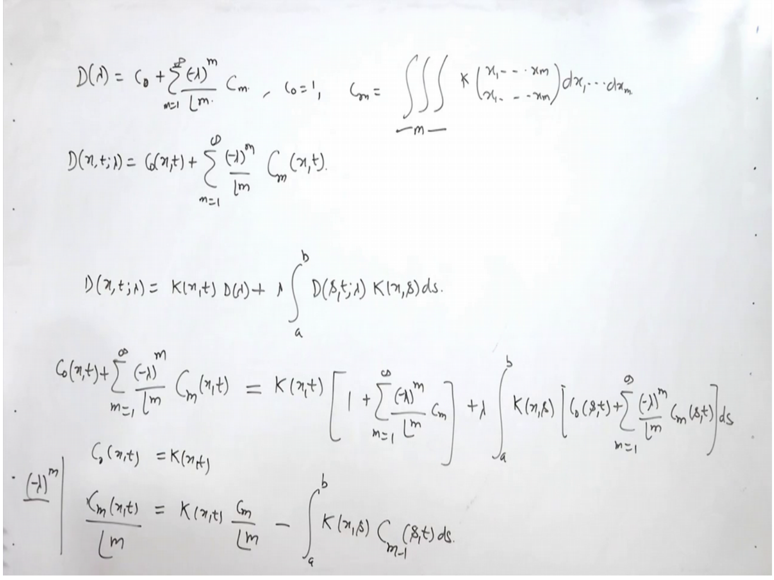

and yeah that is the thing. This is the coefficient, C m minus 1 s t is the coefficient of minus lambda to power m minus 1, is that Ok? So yeah, then upon factorial m minus 1. There is a factorial m minus 1 here, so now

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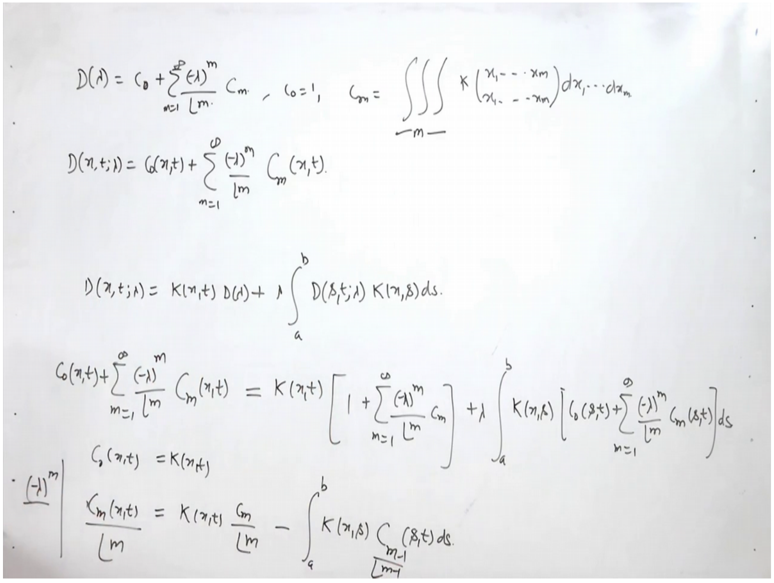

multiplied by factorial m here, on throughout. So you can say that, C m x t, I am just multiplying by, so this is factorial m, you cancel out. Here because of divided by factorial minus 1 we have this. So it means that your C naught x t, x t is K of x t and C m x t is satisfying this integral equation given as this. So here we have

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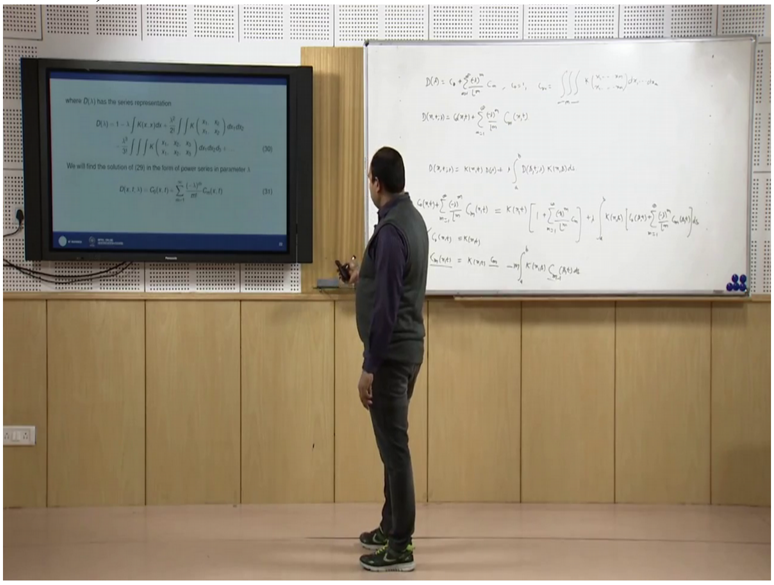

this coefficient written as like this. So C naught x t is equal to K of x t and C m x t is given as C m K x t minus

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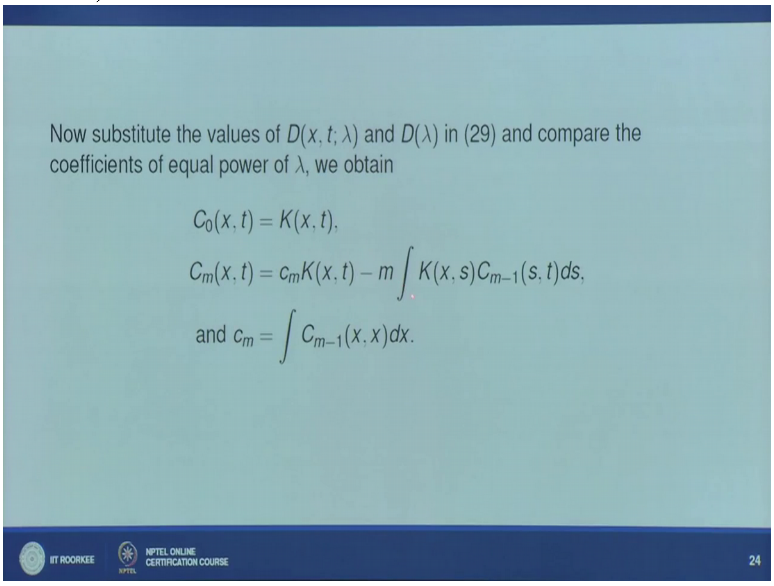

m K x s C m minus 1 s t d of s.

Now there is one more thing which we wanted to discuss as the relation between small c m and capital C m minus 1. So here if you look at, we try to find out with the help of C m x t the last expression that is small c m as given as integral of C m minus 1 x of s d of s. So here

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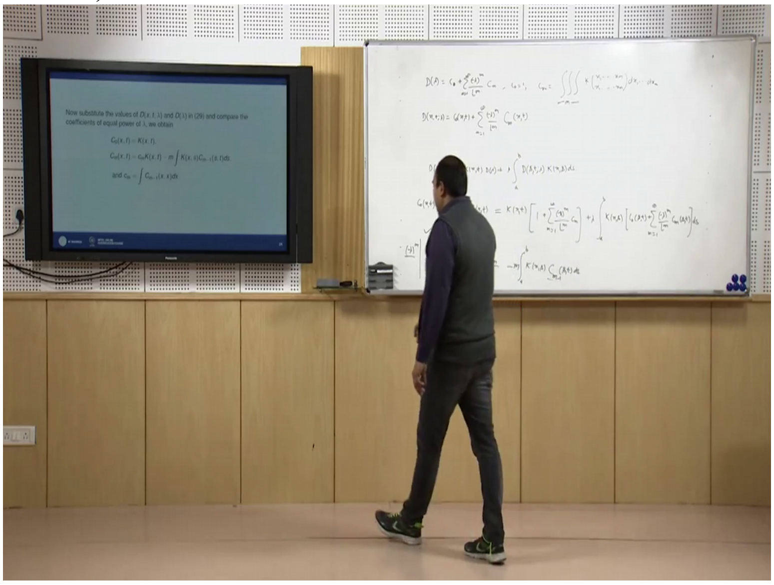

if you look at the expression here and try to find out c 1, c 2 and so on and see what it will be? Right? Now claim is that this, claim is that this C m x of t is given as, you can say that it is K of, K of x t x 1 to say x m and here we have x 1 to x m. And this is D x 1 to D x m. And here we have m-fold integral. I hope,

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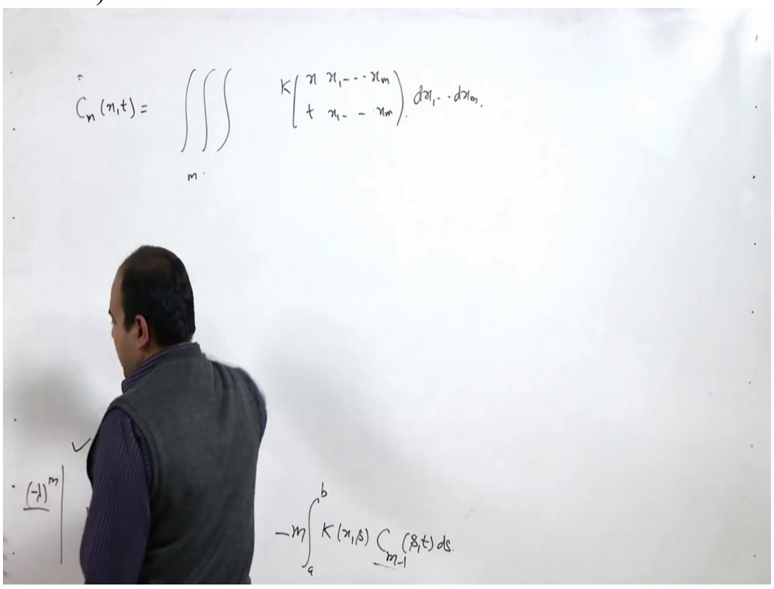

because this is given here in terms of this thing, this is the D x t lambda, d x 1 to d x m.

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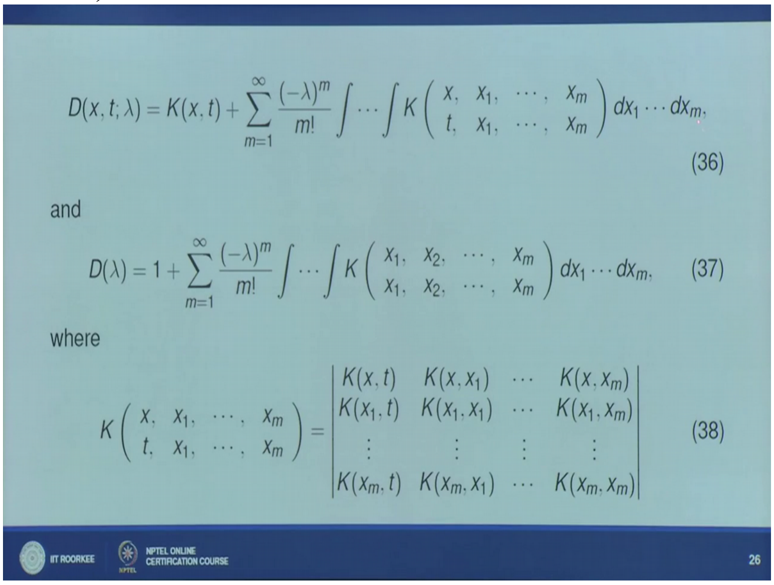

So here we claim that your C m x t

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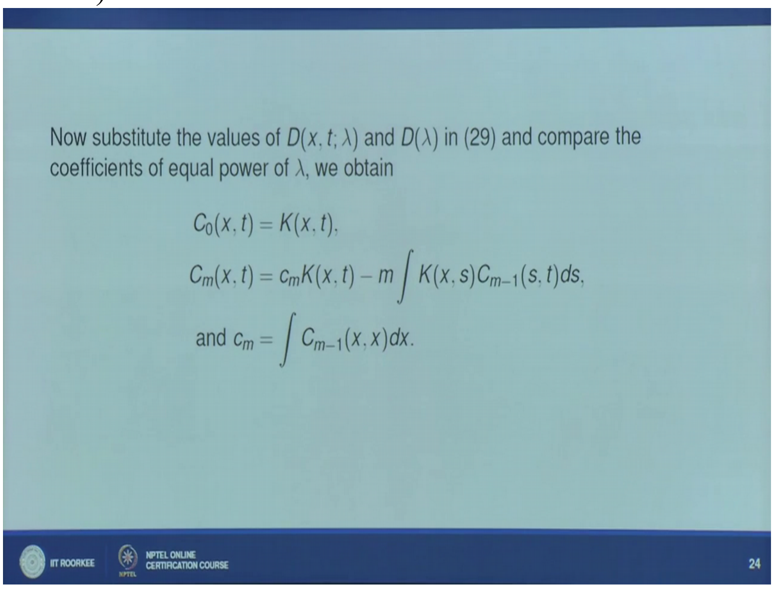

is given by this. So how we can show here. That if you look at, for m equal to 1.

Let us try to find out m equal to 1. So to find out c 1 x of t is equal to K of x t and C m, C m means c 1. Now what is the value of c 1 here? So if you look at the value of c 1 here? c 1 is we can obtain by c 1 is nothing

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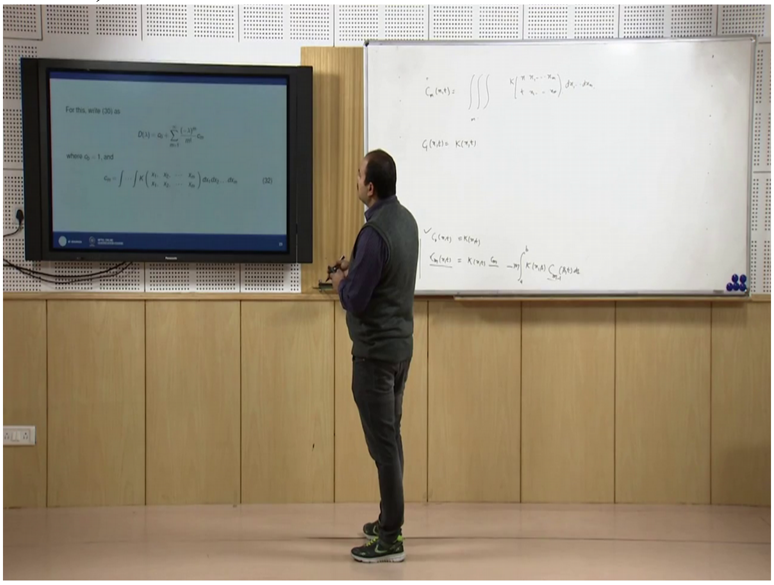

but the coefficient of minus lambda, minus lambda, so you can say that c 1 is nothing but integral of K x x. So c 1 is integral of K x x D of x. Is that Ok? minus m a to b and this is K of x of s. Now when m is equal to 1, then it is nothing but C naught s of t, and C naught s of t is nothing but K of s of t d of s. So for n equal to 1, you can say that c 1 x t

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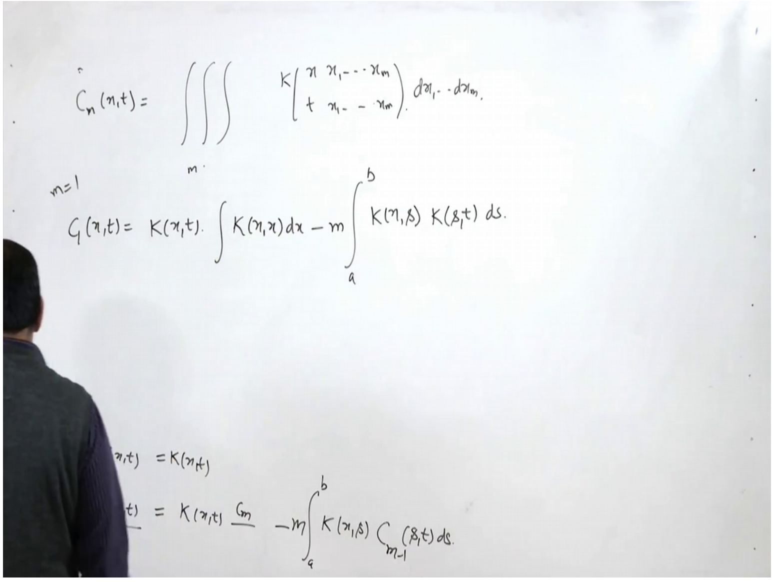

is given by this. And if you further simplify you can further simplify as this. Since this is integration with respect to d x here I can write this in terms of d s, so let us change the dummy variable here. Let us write in terms of s. So I can write this as integral K x of t. I can write this as K of s s d of s and m is 1 here basically, so m is simply 1, so I can remove this minus, I am taking this sign out, so I can write it here, yeah so here I can write it a to b and this is d s I am taking out, here we have minus K of x s, K of s of t, this is whole as d of s, right.

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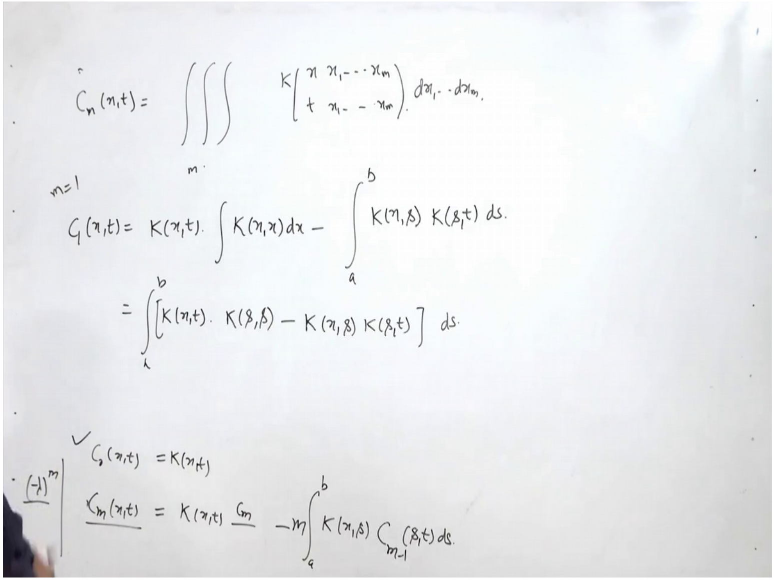

Now can we write it in a different version? So can I write it a to b and here I am writing this as K of x, Ok what we want is this, this notation. So this we want to claim. So we want to check whether it is this or not. So if you look at this expression is what? This expression a to b and this is K of x of t and K of x of s and determinant of this and here we have K of s t and K of s s. So we want to check whether this is written or this not. So if you look at, this is what, K x t, K s s. So K x t K s s minus K of x s into K s of t. So I hope that, that this I can write it a to b K of x s t s d of s, right? So this is for m equal to 1,

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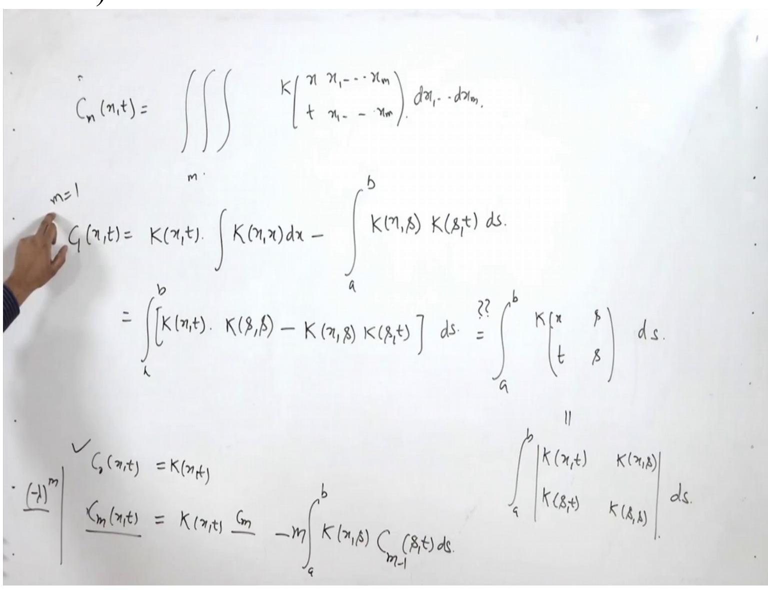

and we want to show a similar expression is valid for a general m, right?

So for that what we try to do here, we try to verify that if C m x t is given by this, then it will satisfy this equation, recurrence relation given by this. So for that this is a quite detail one, so I am not doing it. But I can say that what you try to do here, if you write down this is what, this is m-integral and this I can write this as determinant of m here, so you can write it K of x t K of x, x 1 and so on. K of x, x m and here we have K of x 1 t and so on. So if you express, if you expand this along the first column, you d x 1 to d x m. And if you expand this with respect first column and simplify, you can see

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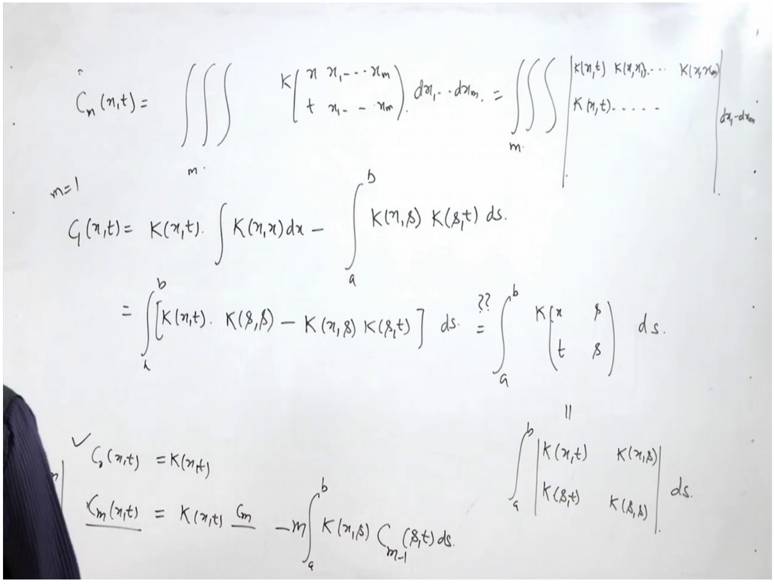

that this is given by this. Again I am not discussing the proof but if you want to have the proof it is given by W. V. Lovitt. So there you can find the proof of this, W. V. Lovitt and the title of the book is Linear Integral Equation.

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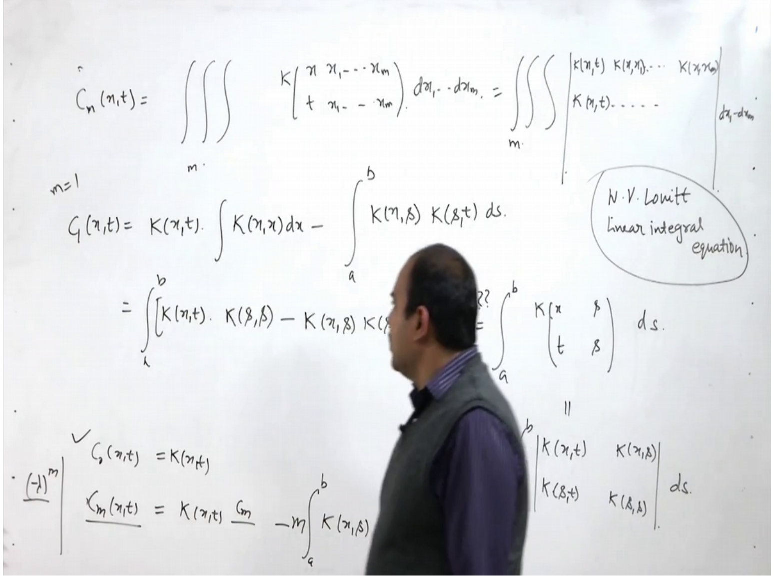

So, so we can simply say that for m equal to 1, this c 1 x t is satisfying this relation and for general m you can show that the expression given here is satisfying this. And for detailed proof of this, you can refer this book, W V Lovitt; title of the book is linear Integral Equation. So in this book, not only you can find out the expression for C m x t, but also you can say that the convergence of D x t lambda is also discussed. As we have seen that the convergence of D lambda is discussed. So right now

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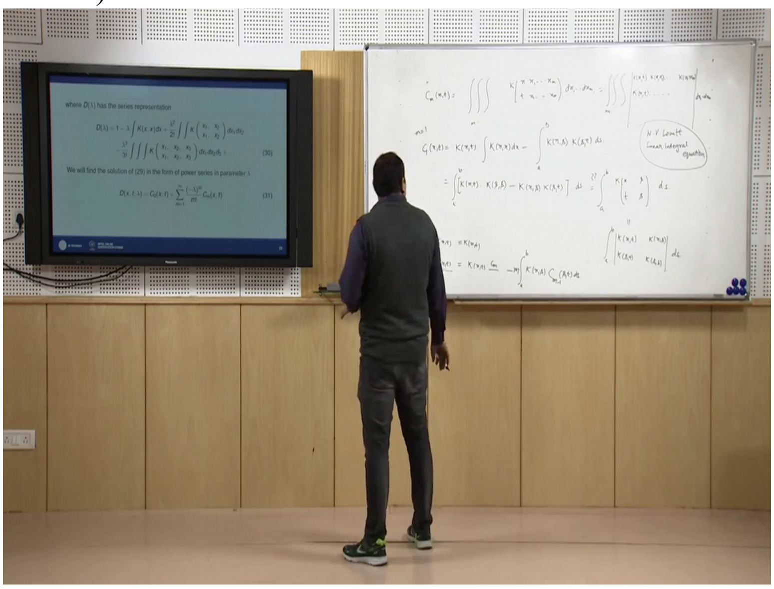

since it is involving I am not writing the proof but I am assuming that your C m x t is given by this and C naught x t is given by this and C m x t can e obtained by this. In fact we really don't require the expression for computational part. What we need from this is the expression for C m.

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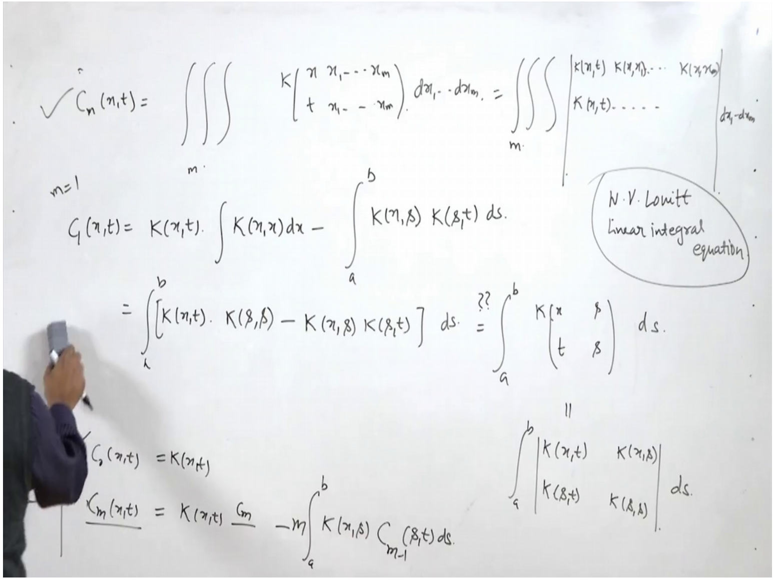

So if you look at here, here your C m x t is given by this. Now I can say that small c m plus 1 is given by what? Small c m plus 1 is the coefficient of minus lambda to power m plus 1 divided by factorial m. So this is the coefficient of minus lambda to power m plus 1 divided by factorial m plus 1, m so this is coefficient. So coefficient of this in D lambda and my claim is that this is nothing but your C m x, C m x x d of x.

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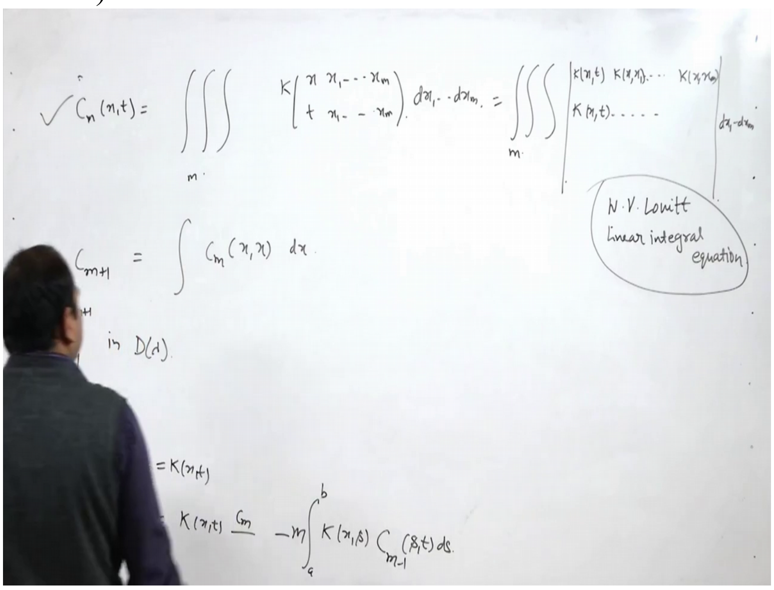

So for that, it is quite obvious.

So C m x t you already know. So you simply write it C m x x, so here t is given by x here. And then if you write it here this is nothing but your C m plus 1. So I am just writing it here. So this is what, this I can write it here. I can say that one this is, one integral is given by this and we have m plus 1 integral. So m plus 1 integral and this is what? K of x x, so x, x 1 to x m here and x, x 1 to x m here. We already have d x 1 to d x m and there is one more, that is d of x. Now what we can do, here we simply

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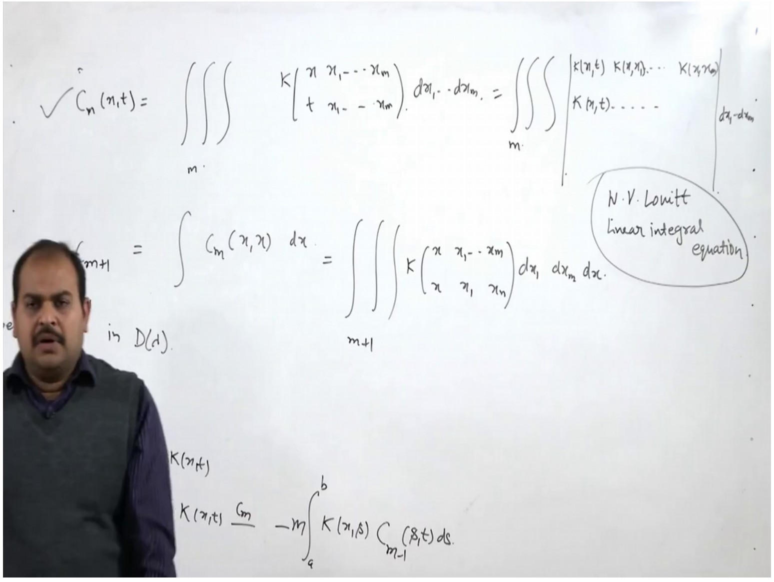

replace this x as m plus 1. So here if you change the dummy variable x by x m plus 1 then I can write this as D x m plus 1 and this is x m plus 1 and you can shift this first column to last column and this minus 1 to m plus 1 will come out and you can say that this is nothing but your C m plus 1, Ok. So you can say that, Ok, so here you can say that this is nothing but the expression C m plus 1. So this we require.

So we can say that what we have obtained here with this small calculation, you can say that

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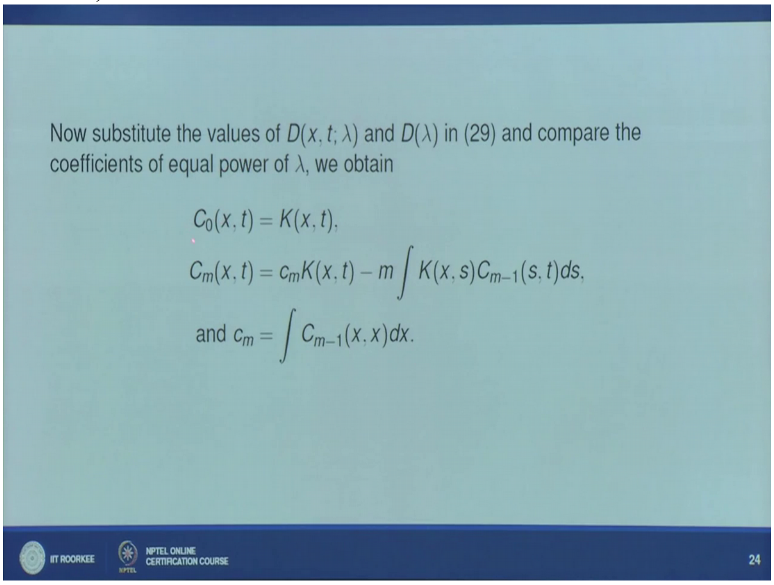

especially for D x t lambda and D lambda and we can say that you can obtain C naught x t as K of x t and that capital C m x t satisfy this recurrence relation given as C m x t is equal to small c m K x t minus m K x s C m minus 1 s t D s and you can also prove that small c m can be written as integral a to b c m minus 1 x s d of s as we have pointed out.

Now we want to summarize before using

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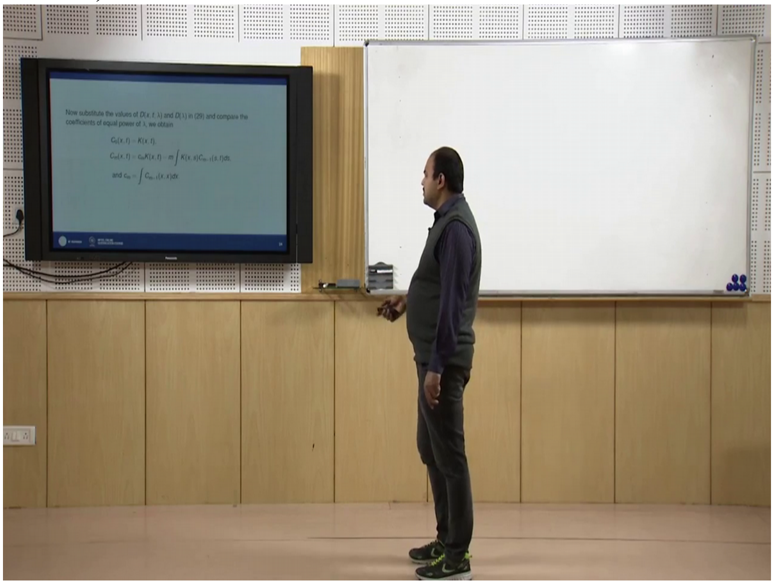

this result to find out your solution y of x, that is summarize the discussion which we have discussed in the following theorem which is known as Fredholm first theorem which says that this

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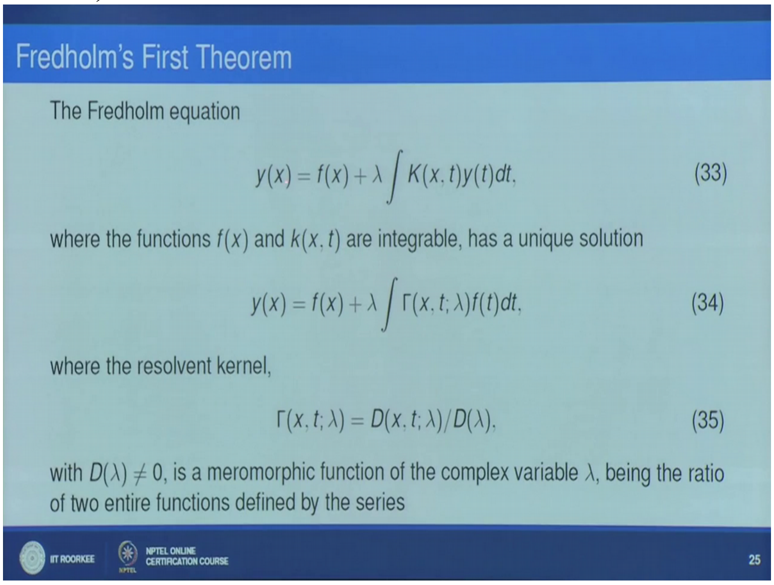

Fredholm equation y x equal to f of x plus lambda a to b K x t y t d t where the function f of x and this kernel K x t are integrable, then this has a unique solution y of x equal to f of x plus lambda gamma x t lambda f t d t and where this gamma x t lambda is denoted as resolvent kernel and it is given as ratio of 2, entire function D x t lambda and D lambda and here we have assumed that D of lambda is non-zero so it means we are assuming that the constant lambda is such that D lambda is non-zero is a meromorphic function of the complex variable lambda here. It means that this gamma x t lambda is a identical function and it may have only

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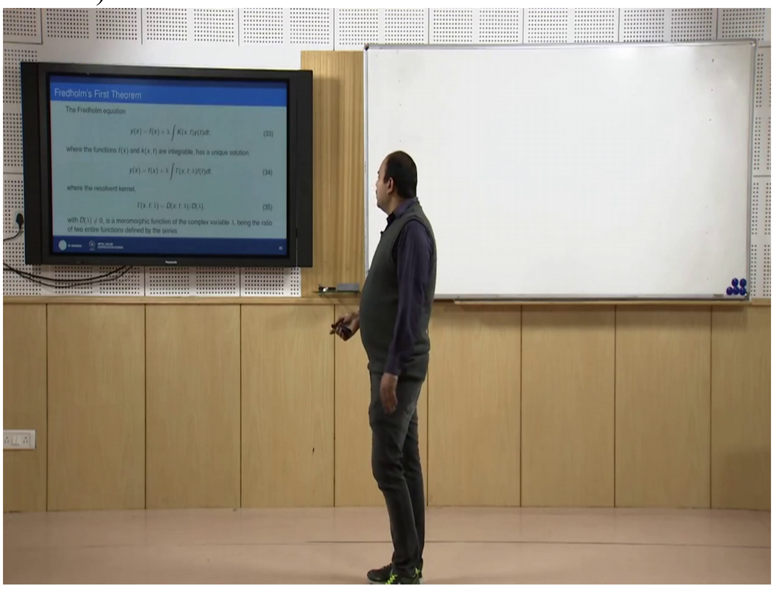

similarity where this D lambda is equal to zero and this similarity is nothing but pole, pole kind of similarity, Ok and here we have just seen that

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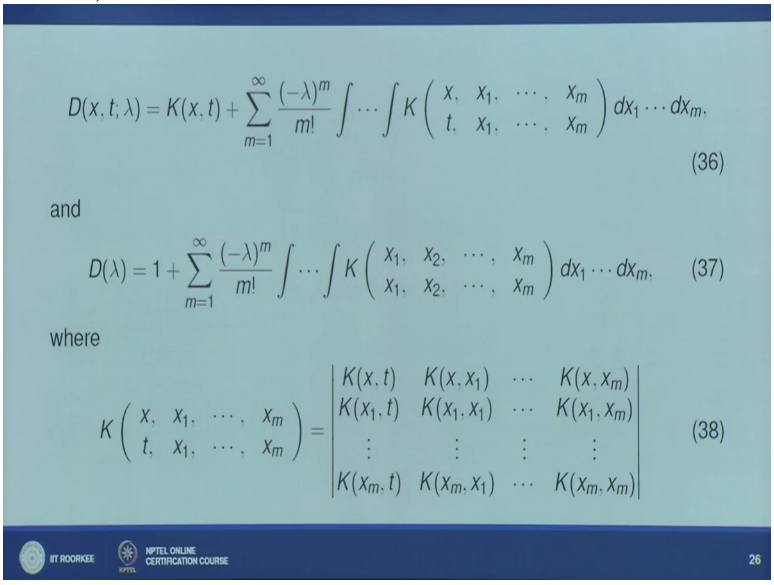

D x t lambda is given as K of x t where K of x t

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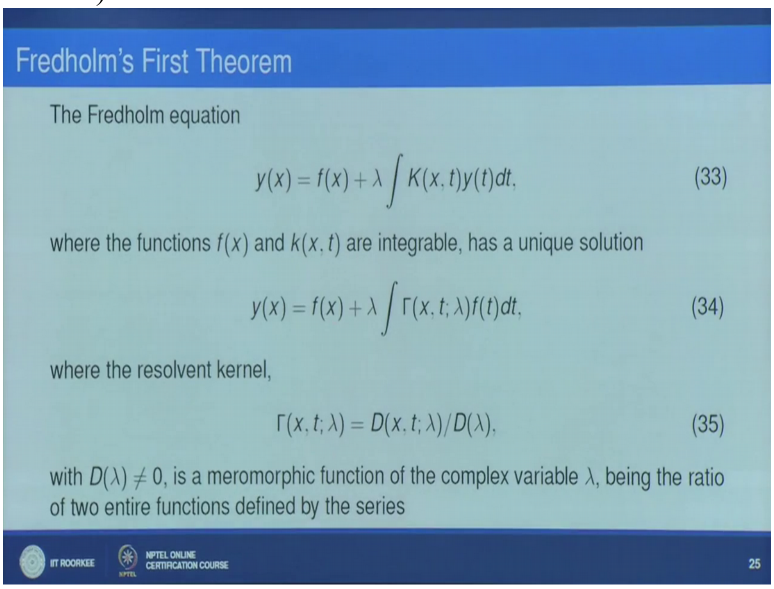

is nothing but

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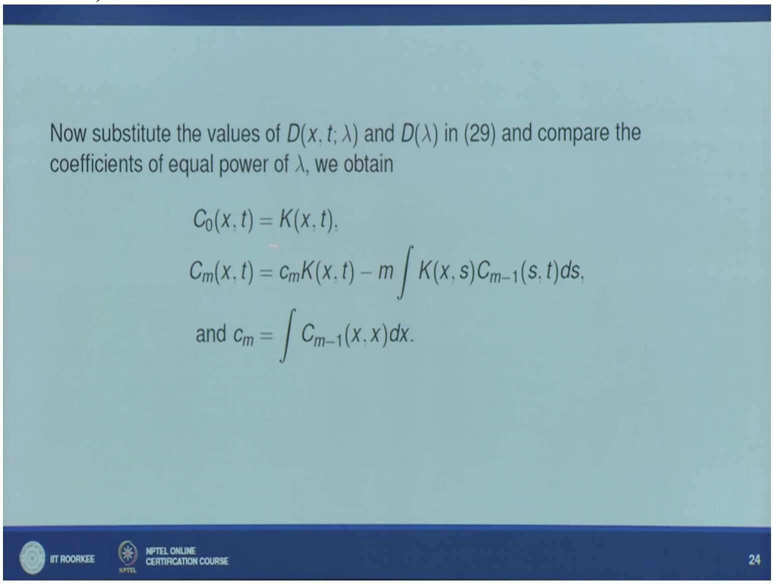

C naught x t right and you can verify that your C m x t can be given

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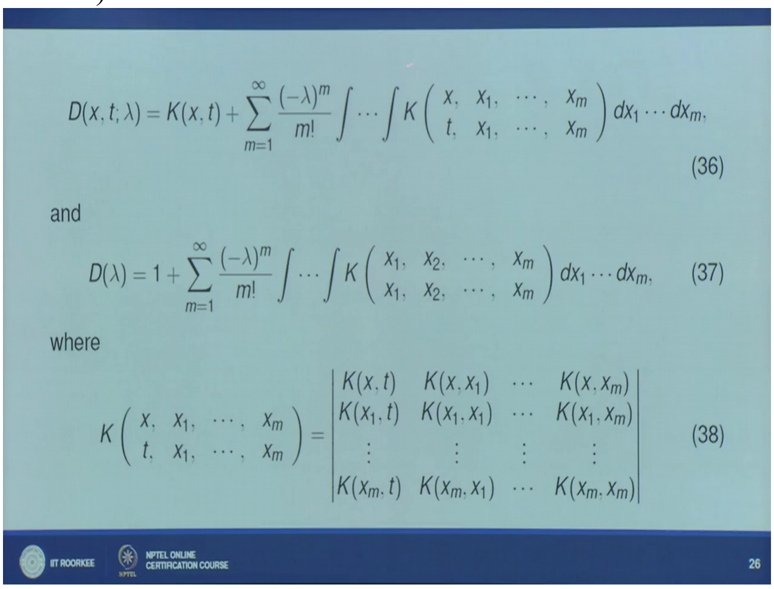

by this m-fold integral which is given as m-fold integral K x so first, if we look at the first variable x and t, rest is same as your D lambda, right? So here it is K x t here, x 1 to x m, x 1 to x m, D x 1 to D x m. So integral m-fold integral is Ok. So only difference between this small c m x t and capital C m, small c m and capital C m x t is that in capital C m x t this is one additional column is corresponding to x and t, right and here we don't have that. So small c m and capital C m x t is defined as this. D x t lambda is given by this and small d lambda is given by this where this symbol K x t, this symbol is denoted by this determinant and we call this as Fredholm determinant and this equation number 36 is known as Fredholm, this D lambda is known is known as Fredholm first series and this is known as Fredholm second series. So it means that once we are able to find out this D lambda and D x t lambda, your solution

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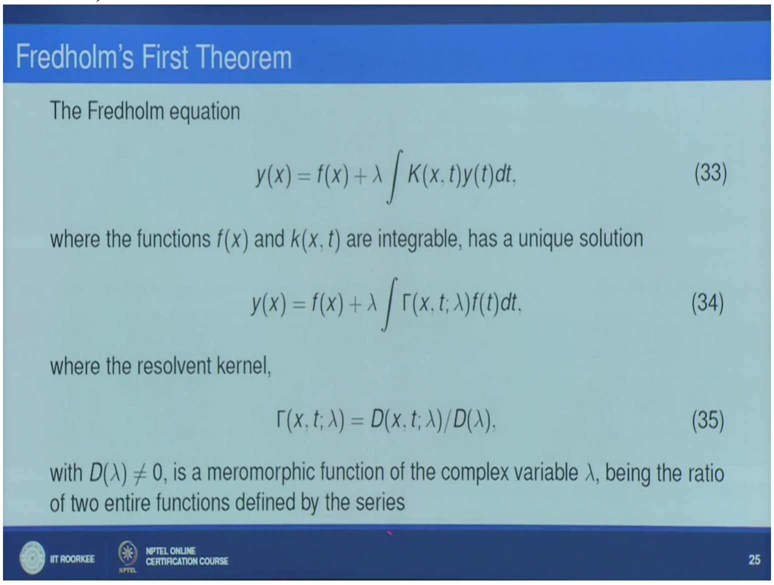

can be given by this.

So in next class, in next lecture we

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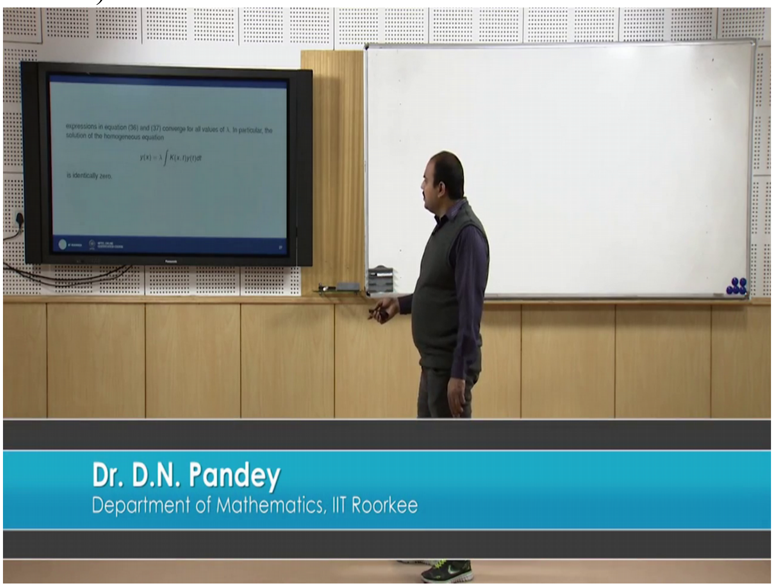

try to find out, say solution of this and if you look at, here we have assumed that D lambda is non-zero, so we say that the

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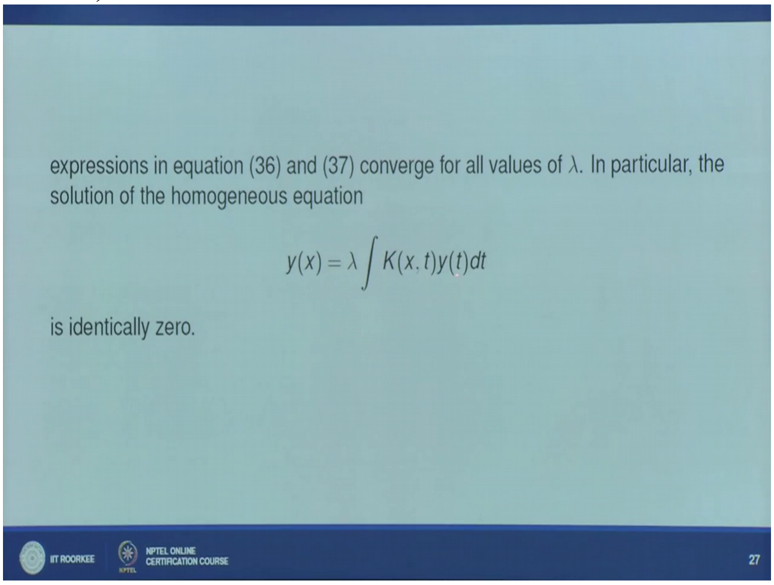

homogenous Fredholmic equation will have only solution which is a trivial solution. So in next lecture we try to find the example where we can apply our theory and also we try to prove the uniqueness part. So in next lecture, we try to apply our theory to find out the solution and solution which is a unique solution. So we will discuss in next lecture. Thank you very much for listening us, thank you.