Course on Integral Equations, Calculus of Variations and their Applications
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Lecture 18
Fredholm alternative Theorem - 1

Hello friends, welcome to the lecture. In today's lecture we will discuss (())(0:24) Fredholm theory for solving Fredholm integral equation. So in the initial first thing is to discuss this kind of problem that if you remember we have solve this kind of problem this is a Fredholm integral equation of second kind where the kernel K x, t is given in terms of separable kernel.

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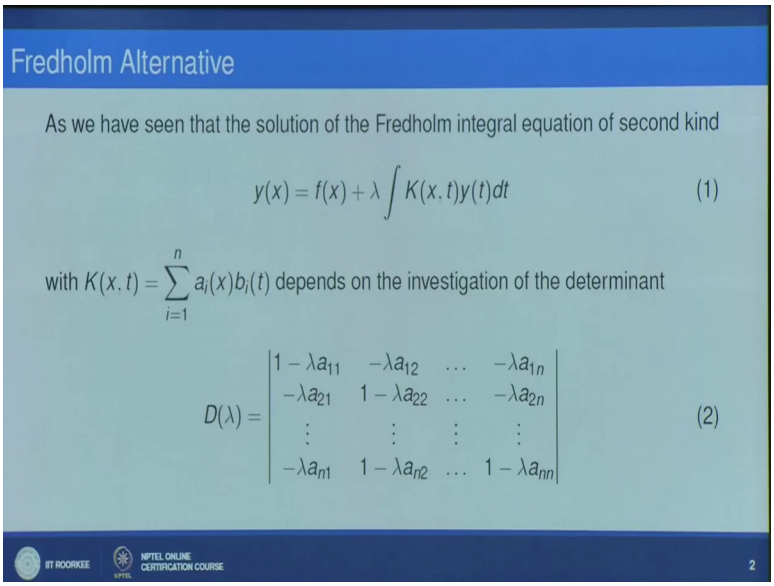

So K x, t kernel is a separable kernel and we have seen that the solution of equation number 1 is basically related to the solving this kind of determinant which we have obtained by putting the value K x, t here in the equation and then separating in terms of a i x and b i t and keeping solving this this we have already discussed so from there we can get that the solution of this equation depend on the determinant of this kind d lambda which is the determinant of the coefficient, okay?

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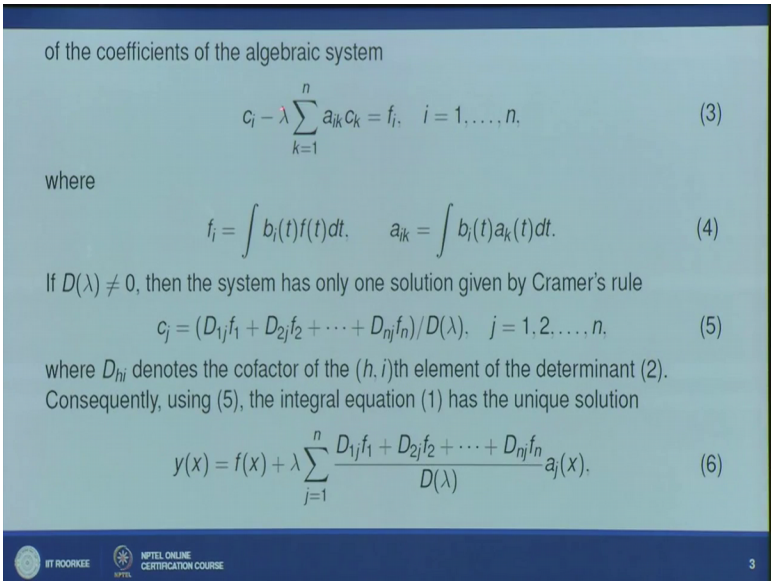

So here this is the system y of x by putting this and assuming your C i as a constant then we can have this kind of system of equation and when you solve this kind of equation where C i is your b i t yt dt. So C i t is given as integral b i t yt dt and a i k is defined as b i t a k t dt and f i is defined as b i t ft dt. So if you look at we need to find out only the coefficient C i constant C i if you can solve for then we can write down our solution yt in terms of this C i.

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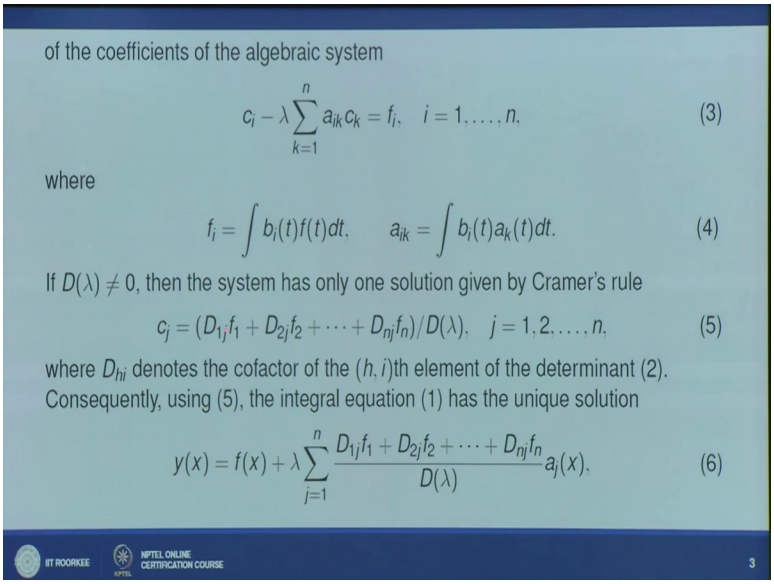

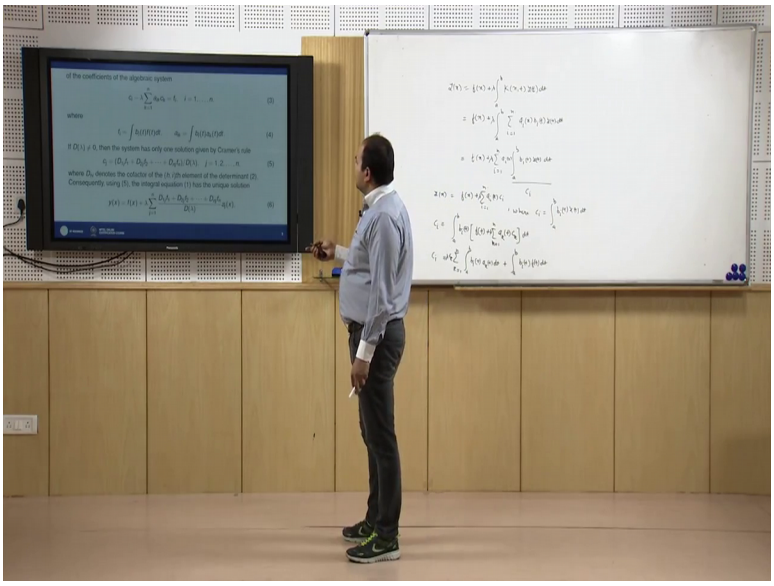

So here we know that the coefficient matrix is given in terms of d lambda and here f D lambda is not equal to 0 then we can get solution like this let me write this again in an explanation. So here we have this y of x is equal to f of x plus you can write it a to b K of x of t y of t dt, then what we try to do here we are just writing the expression for K x, t so it is summation i equal to 1 to n let me write it what we have assumed here, here i to n a i x b i t.

So here we have a i x and b i t y of t dt when you expand this and you can write this as f of x summation i equal to 1 to n and a to b then this is with respect to t so I can take out this a i x out so a i x then this is b i t y of t dt. So by assuming this as C i then I can write this as y of x is equal to f of x plus summation a i x C i, i is equal to 1 to n. So it means that if you can find out the C i where C i is where C i is nothing but a to b b i t yt dt.

So means that if you know the value C i here then you can write down the solution yx here. So to find out this C i we use the solution from and put it here and then we can have a relation like this C i as a to b b i t now in place of yt I am writing this as ft plus summation now here you take k equal to 1 to n a k t C k, right? dt. And if you expand this you will get C i equal to and this is what k equal to 1 to n I am writing this term out b i t, a k t so here we can say that it is C k integral of a to b b i t, a k t dt plus a to b b i t f of t dt, right?

So if you look at this if you look at this equation number 3 is basically the same thing. So here I am just writing here by the way here I forget to mention this lambda one lambda here, so if you put lambda here then we have a lambda here, lambda here, and if we write it here then this is a

lambda here and you can write it lambda here, okay. So if you write if you use the notation given here and as equation number 4 I am using f i as b i t ft dt so it means that if I use notation b i f i for this and for this we are using a i k then this is reduced to equation number 3.

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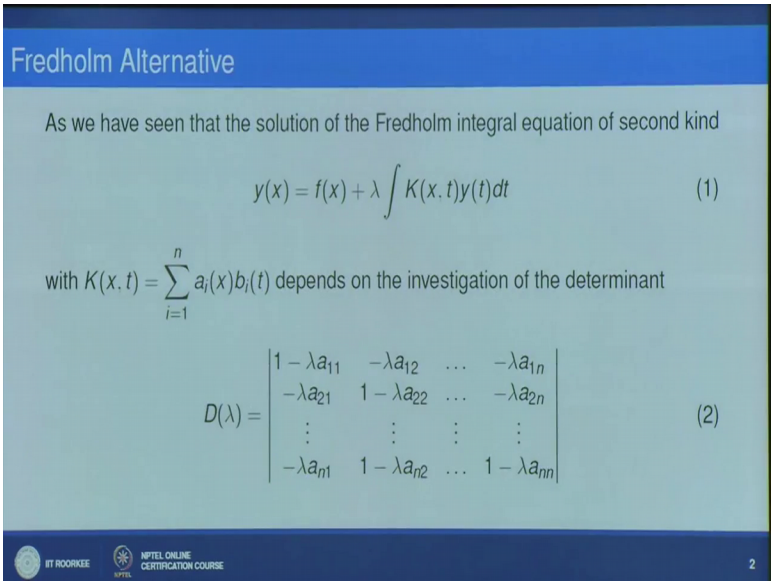

So this we can solve by system of algebraic equation who's coefficient matrix is given as d lambda this we have already discussed in case of in the lecture of separable kernel. So here the main thing is that how to find out these C i.

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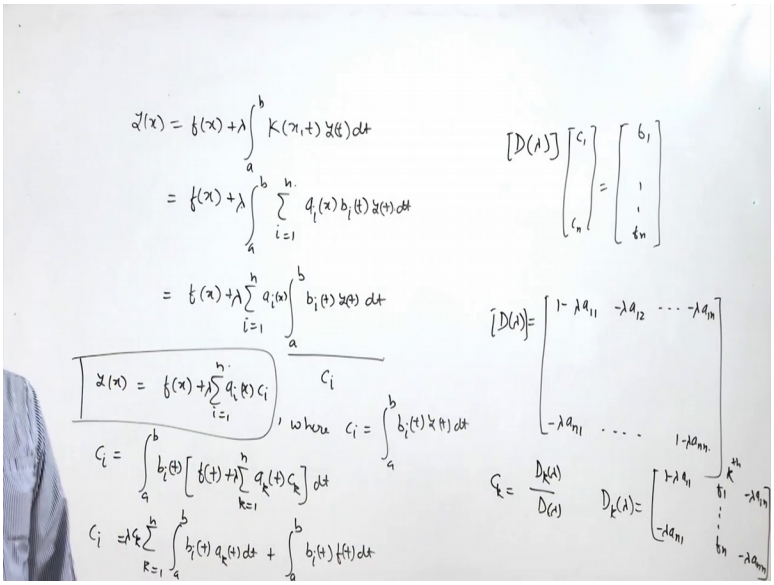

So here if you look at the d lambda so d lambda is basically what so we can write this as d lambda matrix of d lambda into C 1, 2 say C n equal to say f 1 to say f n, right? And then you can get your C i as what? So how to get this C i? So here your d lambda let me write it what is d lambda here here d lambda is given as it is determinant basically so matrix will be given as this 1 minus lambda a 11 minus lambda a 12 and so on minus lambda a 1 n and here we have minus lambda a n1 and so on 1 minus lambda a nn.

So this is your matrix of d lambda so I am just denoting this symbol as matrix of corresponding to d lambda, right? So here I can find out your C k as what, so C k I can get from D k upon d d lambda. So D k lambda divided by d lambda by Crammer's rule. So with this D k lambda so D k lambda is nothing but you simply replace the kth column by your f 1 to f n so it means that D k lambda I am writing here as say 1 minus lambda a 11 and at kth position kth position I am writing f 1 to say f n and this is minus lambda a n1 and here we have minus lambda a and 1n and here we have minus lambda a nn.

So to find out this D k lambda we can get our C k once we have C k we can put it in our equation this and we can get our solution. Now I am saying that this process we have already discussed but we are discussing it again in a different manner so that it will generalized to find out a general theory for a general kernel.

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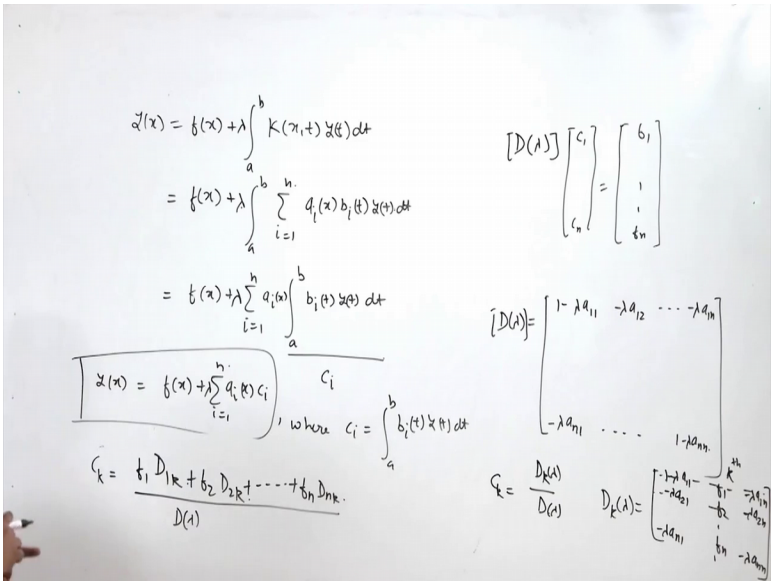

So here to find out this C k let me write it how to expand this, so here we say that your C k can be written as, so here we can write it C k as what? So C k I am writing we are expanding this determinant with respect to kth column so when you expand it, it will be what? You simple write it f 1 and then it is D k1 which I am writing it here this we want to get that C j I am writing in terms of f 1 to f n with the help of cofactors here.

So we want to write it C k so C k will be what? You start with this f 1 and then this and this kth column is gone then what will happen here f 1 and we write it here minus lambda a to 1 minus lambda a to n, so what you will get? This is deleted and this is deleted, so what we have is this minus lambda a to 1 so kth column is simply gone minus lambda a to n and so on. So this is a determinant of this so this is represented as the cofactor corresponding to position a k 1 k.

So this I can write it here as d of 1 k, so here if you take this row out and this column out whatever left we say that it is cofactor corresponding to the position 1 k. So I am writing this as D 1 k so I can say that C k is written as f 1 D 1 k plus f 2 D 2 k, right? So it means that when you consider f 2 and then you remove this and this then whatever left is the cofactor of corresponding to position 2 k. So I am writing f 2 D 2 k and so on, so this I can write it f n D of nk, right?

And so C k is divided C k is given as this divided by d lambda. So what we are doing here we are just writing the expression of D k lambda in terms of cofactors corresponding to 1 k, 2 k and so on. So we can write C k as this so once we have C k we can obtain this C k we can put it here

and we can get our solution. So what we are trying to write here so this is the expression C j given in terms of 5 so that in this way we can find out all the constant C 1 to C n in this manner.

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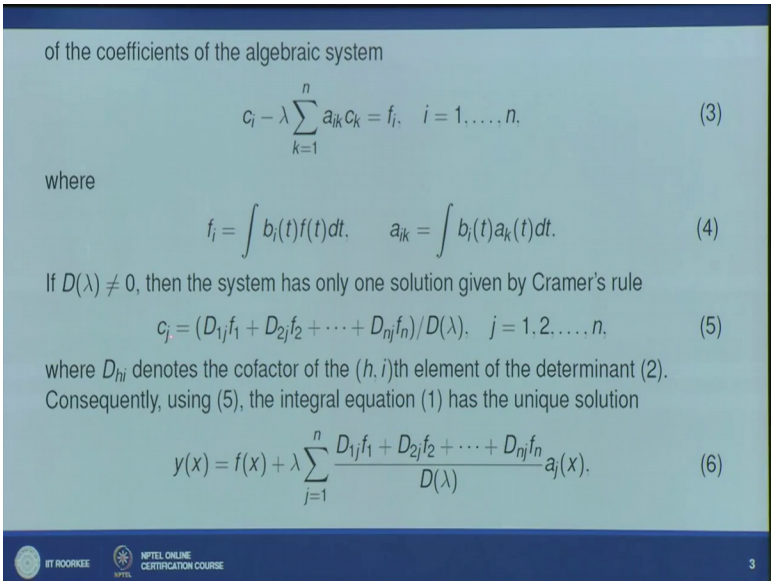

So if you use these values of constants you can put it back into solution and we can write down the solution like this, so let me write it our solution here.

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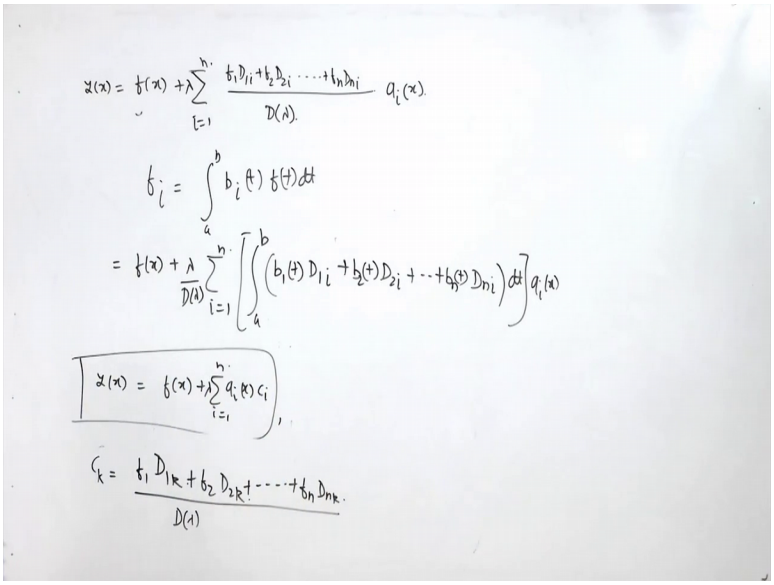

So here we are writing this solution as y of x is equal to f of x as it is plus lambda times summation i equal to 1 to n and a i x I am taking out C i I am writing here so this is your a i x

here and C i I am writing here f 1 D 1 i 1 i f 2 D 2 i and so on f n d of n i and divided by d of lambda here.

So here so solution can be written in terms of 6 now if you further use the value of f i, what is f i here? fi is given as a to b b i t f of t dt. So if you use this notation this we have already assumed that f i is this then you can further simplify this and you can write this as f of x plus lambda times summation i equal to 1 to n and I am taking d lambda out so d lambda here and this I can write it as ft you can take it out so it is a to b b i I am writing f 1 so b i t and this is D 1 i plus and so on. So f 2 is what? So here it is b 1 so f 1 is b 1 t D 1 i t, sorry D 1 i plus D 2 t D 2 i and so on D n t d ni and this is with respect to dt and whole it is a i x which we are representing it in next equation that is here.

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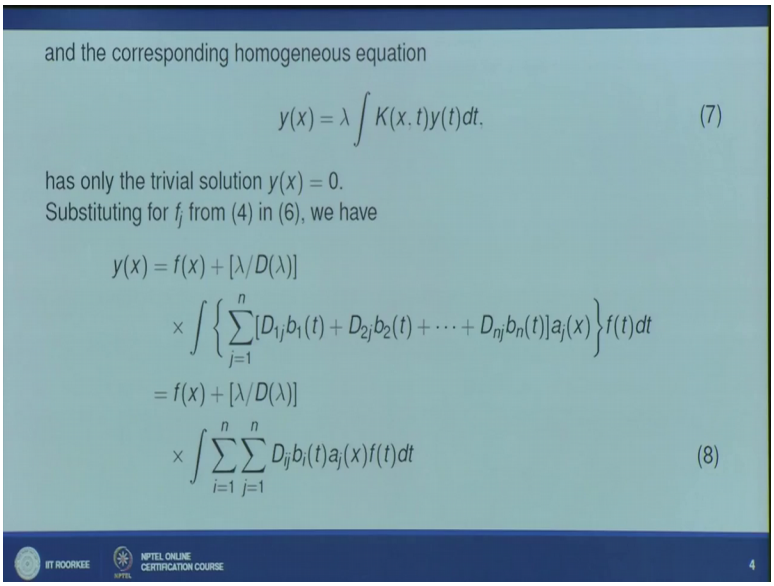

So when we put the value of this f i t it can be written in this form that summation j equal to 1 to n D 1 j D 1 t plus D 2 j D 2 t and so on and if you write down in a simpler form then this is nothing but summation i equal to 1 to n j equal to 1 to n D i j D i t a j x ft dt, okay.

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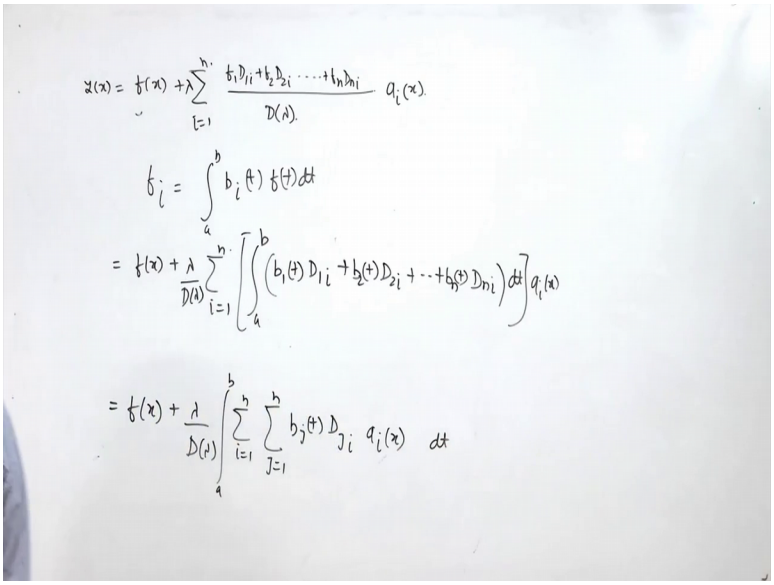

So here I can write this as in a summation form I can write simplify this as so here this I can write f of x plus lambda upon d lambda and here summation you can take inside no problem a to b i equal to 1 to n now this summation I can write as summation say j equal to 1 to n and this I can write it b j t D j i and this is so this is written like this and you have a i x and this is our dt.

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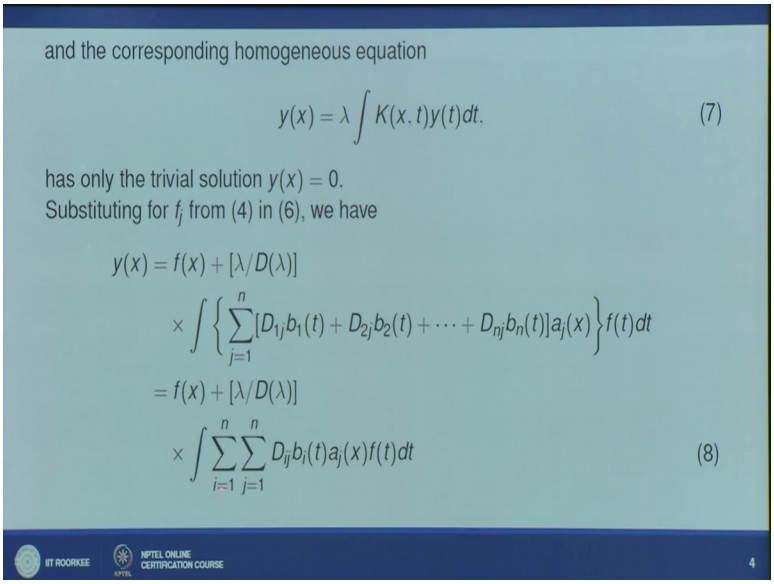

So that is what is written as equation number 8 so I am writing equation number 8 as y of x equal to f of x plus lambda upon d lambda into this which is integration a to b i equal to 1 to n j equal to 1 to n D i j b i t a j x ft dt which I have written here I think I have forgot this ft here

somewhere ft so what is ft here I putting this thing so there should be ft here so here we have ft, so we have ft here, is it okay?

And what is important here important here is that we can denote this whole thing as another variable which depend not only on x so it will depend on x and t so we are writing this as this thing we can write this as equation number 8 and then we further want to simplify this. So to simplify this equation number 8 we consider another determinant of n plus 1th order which is given in this particular format which we denote as D x t lambda so it is in first we have a i x it is at 1, 1 position it is 0 and the rest is a 1 to a n and in the first column first position is 0 and we have b 12 b nt.

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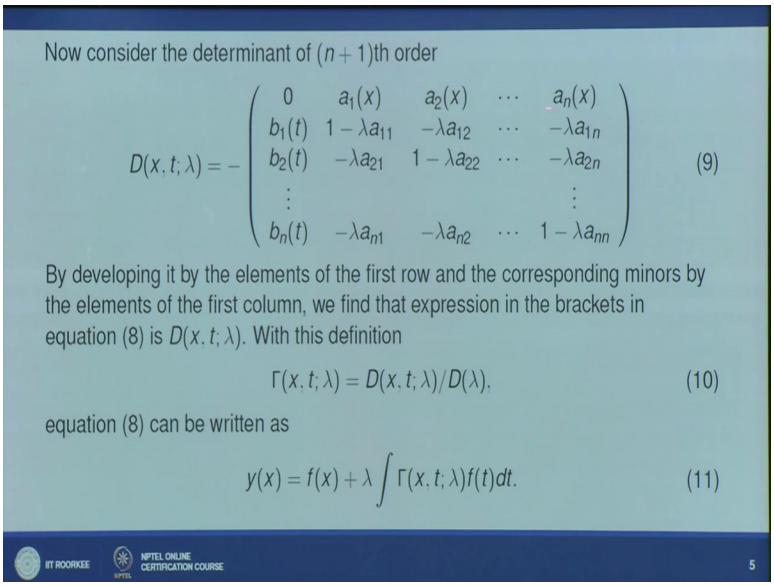

Now by using this if you expand this D x t lambda along your first row then you will see that the term written in this summation that summation i equal to 1 to n summation j equal to 1 to n D i j D i t a j x this can be written as the single expression D x t lambda. So how we can look at if you look at this a i x then it is what let me if you expand this if you find out this value D x t lambda then first thing is along 0 we are expanding along the first row. So first thing is 0 into this thing which is coming out to be 0 then we have a 1 x and then this first this is this column is gone this row is gone then what is left is your b 1 t b 2 to b n t and this thing minus lambda a 12 1 minus lambda a 22 and so on and if you look at it is nothing but this factor this is nothing but you can

write it a 1 and this factor, right? So and we want to show that whatever the expression given here is nothing but D x t lambda.

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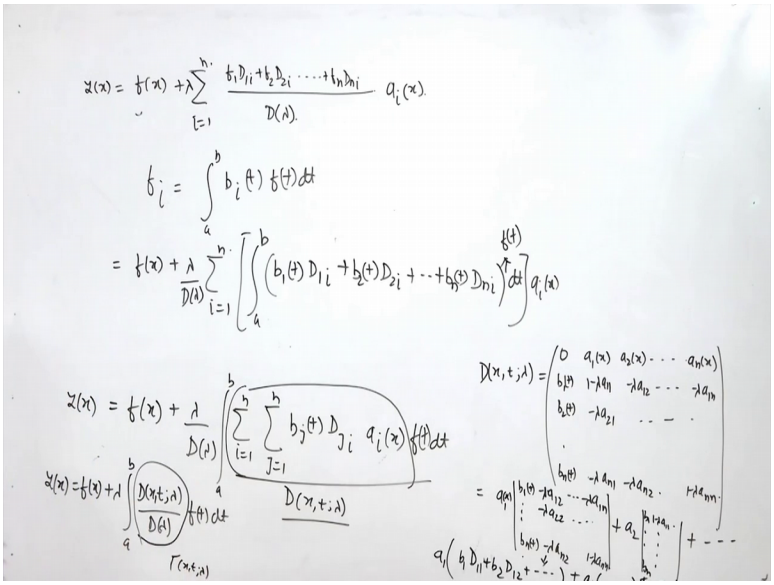

So for that let us write down D x t lambda so I am just writing D x t lambda and try to show that whatever expression is written this is this is nothing but D x, t, lambda, so this we want to show that this is your D x t lambda so let us see how it is. So we are defining D x, t, lambda is 0 a 1 x a 2 x and up to a n x and here we have b 1 t b 2 t and so on here it is b n t and then we have expression for 1 minus lambda a 11 minus lambda a 12 and so on minus lambda a 1 n and here we have minus lambda a 21 and so on.

So last one is minus lambda a n 1 and here we have 1 minus lambda a nn, right? Now we try to expand this D x, t, lambda and try to show that we are getting this. So we are expanding this D x, t, lambda along the row, so if you expand this along the row this will not contribute anything and then we look at this a 1 so here when you write this as then we have a 1 and then what is left here then here we have this row out and this row out what we, sorry this is gone and this is gone.

So we have determinant of this b 1 t and up to b n t and here we have 1 say minus lambda a 12 minus lambda a 1n and here we have what is this this is gone, so here we have minus lambda a n2 so here we have minus lambda a n2 and 1 minus lambda a nn. Now plus minus a 2 so here we have a 2 and so on, right? Now we try to identify what it is written here, now if you look at if you want to find out say determinant of this then it will be what? It will be now this time when

we expand this determinant we expand along the row along the column first column that is b 1 t to b n t here we do it along the first column, okay.

Let us see what is a 2 here, so here when you take this a 2 then we have b 1 to say b n here and then we have what we have this 1 minus lambda a 11 and so on we will write it. Now if you look at this what is this so here expanding this we will have what? We will have I am just expanding this so here we have b 1 and then whatever is left is nothing but your what it is nothing but D 11, right? So that is this expression and similarly you can write so here we have so here we can get the coefficient of a 1 is what a 1 is D 1 D 11 and when you and then you have b 2 and then this row and this row is gone this row and this column is gone and then we write it then it is D 12 and so on, right?

And similarly when you do it for a 2 then you will get the similar expression. So this is little bit calculative so you try to expand it. So what we are trying to do it we are trying to expand this along the first row and then we have this and then we find out so determinant of this along the first columns and when you do this you will get that a 1 and whatever we have is b 1, D 11, b 2, D 12 and so on. Similarly you can find out the coefficient of a 2 and a 3 and so on.

And when you simplify this it is nothing but this summation i equal to 1 to n j equal to 1 to n b j t d j i a i x this you will get. So it means that this expression this you are getting as the determinant of d x, t, lambda. So when we do this then this expression y of x can be written as f of x plus lambda a to b we are writing this as D x, t, lambda divided by d lambda f of t dt, right? So this we are summarizing in this matrix so here this D x, t, lambda is define in equation number 9 and we are solving this by developing it by the elements of the first row and the corresponding minus by the elements of the first column, we find that expression in this bracket is nothing but D x, t, lambda.

So it means that our solution can be written as this yx equal to fx plus lambda a to b D x, t, lambda upon d lambda ft dt. Now here if we denote this as say gamma xt lambda then we can write our solution in terms of equation number 11. So here I am assuming here we are just denoting D x, t, lambda divided by d lambda as gamma xt lambda known as resolvent kernel and we can write our solution yx as f of x plus lambda a to b gamma xt lambda ft dt.

So here if you remember we are dividing this is ())(24:04) D x, t, lambda by d lambda. So here I am assuming that lambda a such that we do not have say 0 d lambda a equal to 0. So here I am just assuming that lambda is such that d lambda is not equal to 0 in that case our solution is given in terms of y of x equal to f of x plus lambda a to b gamma xt lambda ft dt, where gamma xt lambda is defined like this.

So if you know d lambda if you know D x, t, lambda which is defined like this then you can find out our solution yx in this particular format, right? Now with the help of this we try to find out say solution and here we are saying that if d lambda is non-zero then we have a unique solution here but if d lambda is 0 means we have a lambda which is the 0 of d lambda or in some sense we can say that lambda is the ())(25:02) value of the coefficient matrix d lambda then we have in that particular case we may have solution or we may not have any solution.

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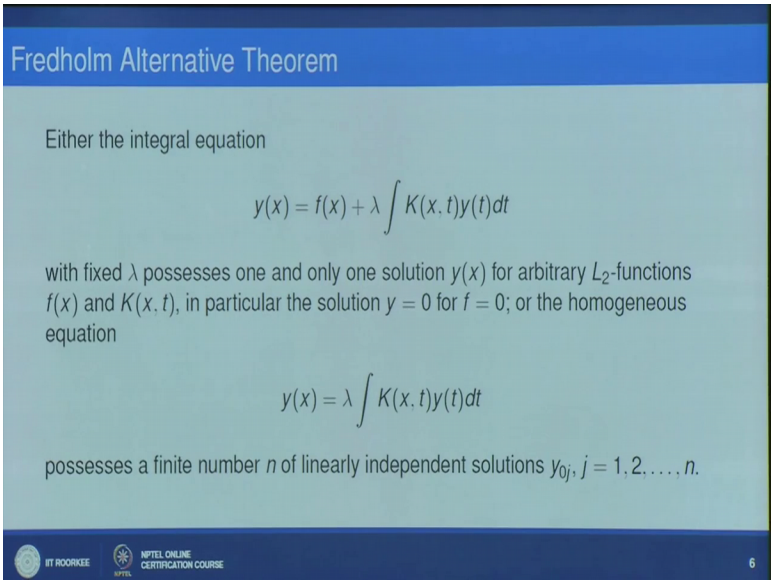

So this we have already discussed but again I am repeating it in this case we may have 0 solution or we have ())(25:23) solution. So this we can write it here like this so either this integral equation yx equal to fx plus lambda a to b k xt yt dt with fixed lambda poses one and only one solution that is corresponding to d lambda not equal to 0 and that is similar to say that when f is 0 then y is 0. So it means that when d lambda is non-zero then you can simply say that f is 0 it means that your C i's are all 0 and you can write down your solution as y equal to 0.

So here I can say that if d lambda is non-zero and you do not have ft here then all this part is 0 and we can have y as 0 solution. So we can say that either this will have a unique solution one and only one solution or the homogeneous part will have trivial solution but if d lambda is equal to 0 then we have to simplify and we can say that we have to do further analysis and we can say that we have homogeneous equation this may poses a finite number n of linearly independent solution.

What is this y not j from j from 1 to n this is the nothing but the eigen function corresponding to this homogeneous part so this we can always discuss.

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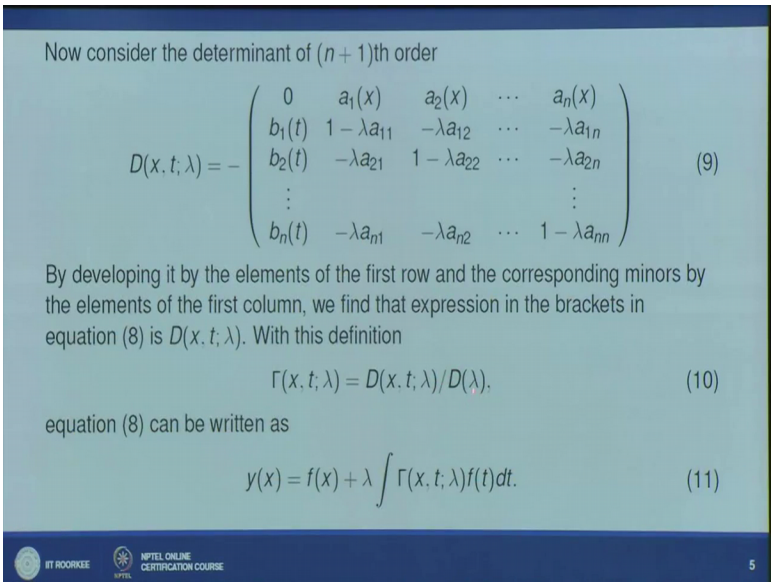

Now please remember here here I am dividing by d of lambda so it means that I am assuming that d lambda is never 0 so it means that this expression 11 is true for all those lambda for which d lambda is non-zero. Now what happen when d lambda is 0? So it means that it may happen that lambda is the root of d lambda or you can say that lambda is an eigen value corresponding to d lambda.

In that case what should be the condition on your f or what should be the condition, so that we will discuss in next lecture so here we will conclude that if we will conclude that if d lambda is non-zero the solution is given in this and this is kind of a unique solution and we try to find out when d lambda is 0 then what will happen, okay so thanks for listening us we will meet in next lecture, thank you.