Integral Equations, Calculus of Variations and their Applications
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Lecture 15
Construction of Green's Function-2

Hello friends here we will continue in this lecture we are going to continue the previous talk that is on Green function.

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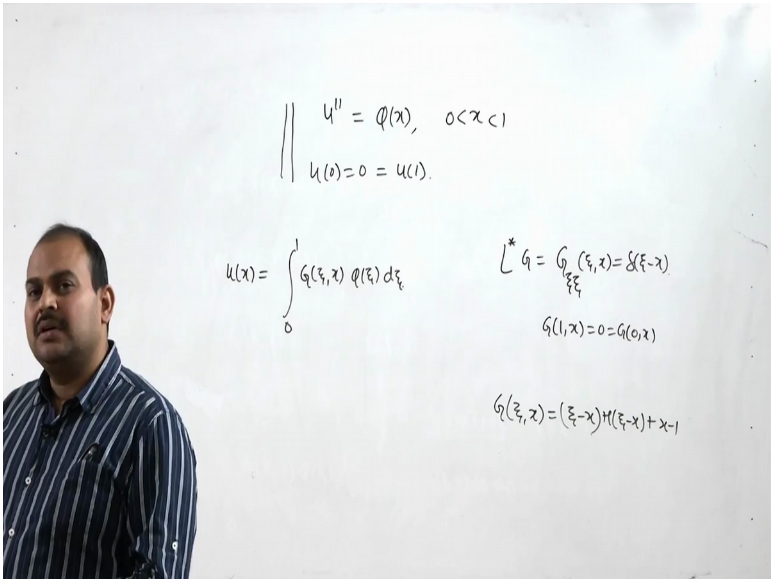

So in previous lecture we have discussed this particular problem u double dash equal to phi of x where x is between 0 to 1 and u of 0 equal to 0 and u 1 equal to 0. So we try to find out the solution u x in this integral form G xi, x phi xi d xi and where G xi, x is defined earlier. Now we try to look at the more complicated example here if you look at we simply solved our Green function by sayings that G will satisfy this equation that L star G which is nothing but G xi xi xi, x equal to delta xi minus x and G 1, x equal to 0 equal to G of 0, x.

So here we have find this G xi xi xi, x we simply integrated it and we have achieved G xi, x as xi minus x H of xi minus x plus x minus 1. So here this is quite easy because our operator is quite easy we simply integrated. Now let us discuss a little bit little bit more complicated problem where this integration is not so straight.

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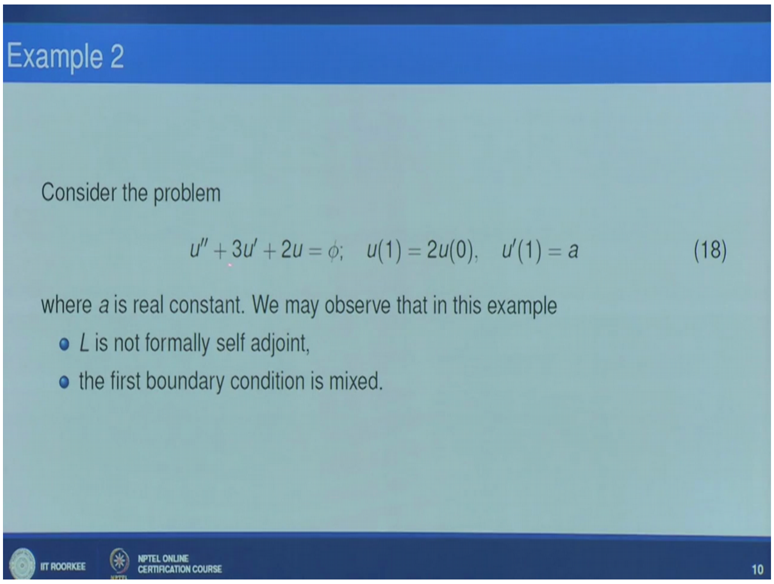

So let us consider the second problem here we have seen u double dash plus 3u dash plus 2 of u equal to phi, where u 1 is equal to 2 of u not and u dash 1 is equal to a. So in previous problem we have considered the boundary condition which is which are homogeneous boundary condition but here we have considered a non-homogeneous condition and the first boundary condition is a mix boundary condition that it involve both u 0 and u of m. And if you look at here this L is not given as formally a self adjoint, if you look at your a is simply 1 and B is 3 and B is not given as A dash. So it is not a self adjoint equation.

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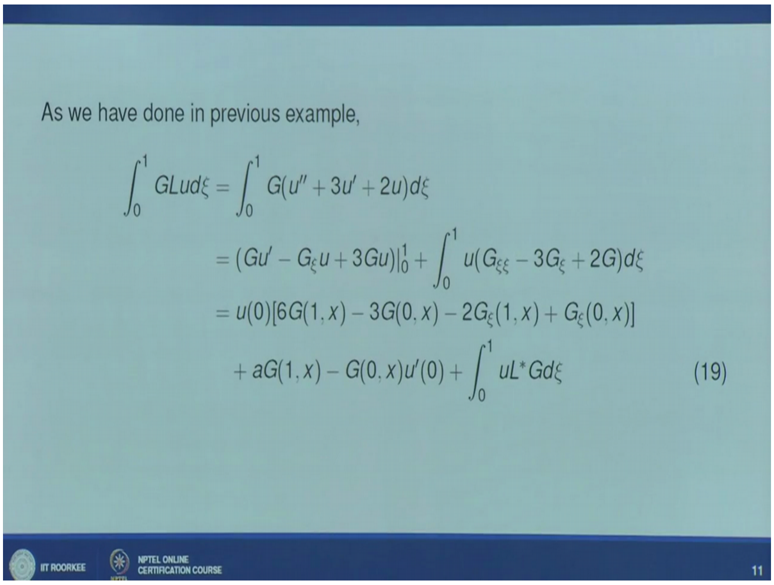

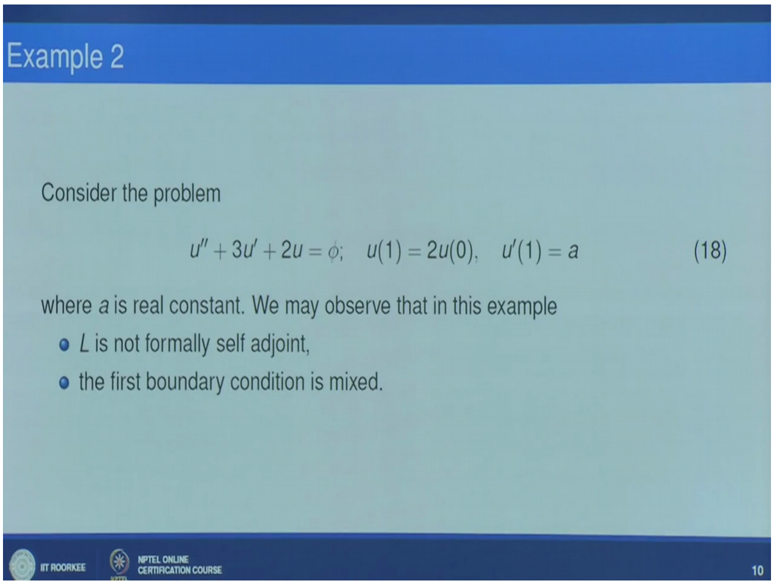

So to solve this we adopt the same method and we what we try to do is we define L as L as u double dash plus 3u dash plus 2 of u. So we multiply by Green function G and integrate between 0 to 1. So if you do the same thing we have 0 to 1 G u double dash plus 3u dash plus 2 of u d xi and as we did in previous lecture we transfer the derivative of u double dash and u dash on this G and this is the process integrated here and at the end we are going to get this.

Here I have utilized this condition that u of 1 is equal to 2 of u not, if you utilize the condition it is simplified as u of 0 within bracket 6 G 1, x minus 3 0, x minus 2G xi 1, x plus G xi 0, x plus aG 1, x minus G 0, x u dash 0 plus 0 to 1 uL star Gd xi. Now if you follow the if you want to get our solution then here we simply say that L star G is going to be delta xi minus x and we define G 0, x as 0.

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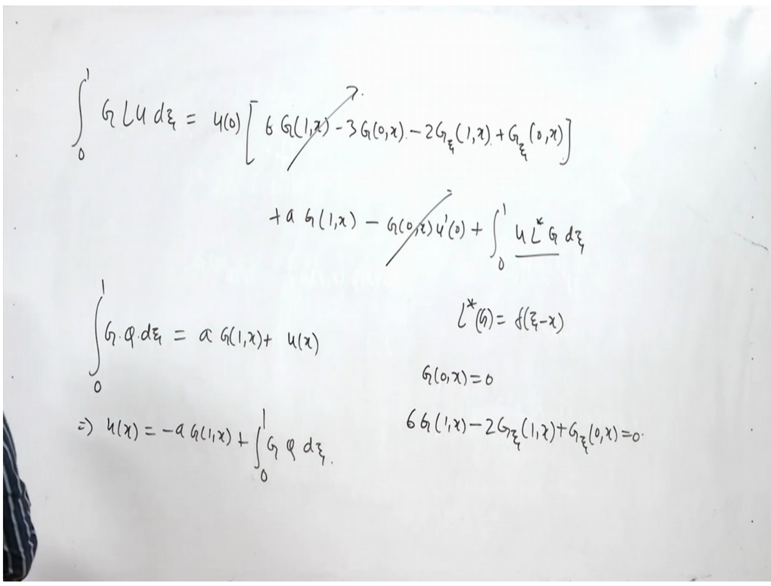

So here we have write this 0 to 1 G Lu d xi is equal to u of 0 within bracket we have 6 G 1, x minus 3G 0, x minus 2G xi 1, x plus G xi 0, x plus a G 1, x minus G 0, x u dash 0 plus 0 to 1 uL star G d xi. So here to get our solution we assume that (u) this L star G is going to be delta xi minus x and the condition on this G boundary condition is G of 0, x is equal to 0 so that this term is simply vanished and we assume that this condition since u 0 is not given so we assume that 6 G 1, comma x minus 3 so this term is already vanished. So here we can write it minus 2G xi 1, comma x plus G xi 0, x is equal to 0.

So we try to find out our Green function which satisfy this condition that L star G is equal to delta xi minus x and G 0, x is equal to 0 and this is equal to 0. So if we assume this this term is gone, this term is gone. And you can write it here 0 to 1 G Lu is given as phi d xi is equal to a G 1, x plus 0 to 1 uL star G. But if you take L star G as delta G minus x then this is going to be u of x, so your u x is given as minus a G 1, x plus 0 to 1 G phi d xi. So we need to calculate this G.

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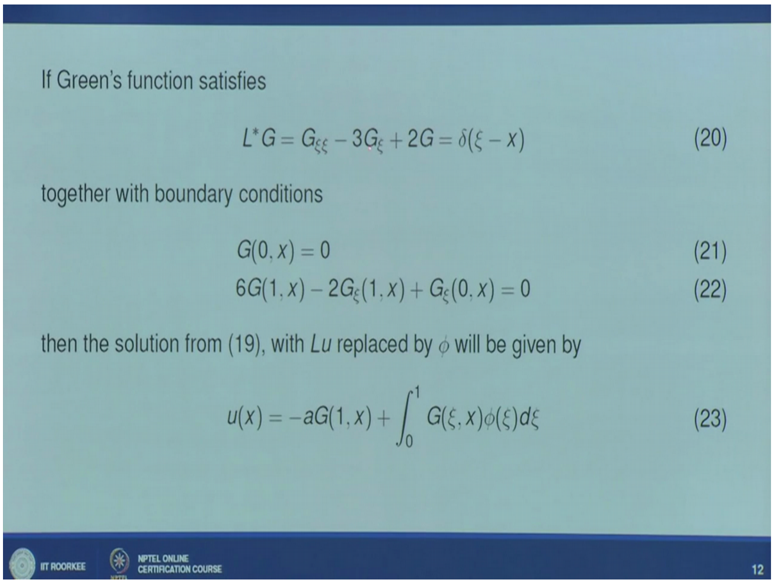

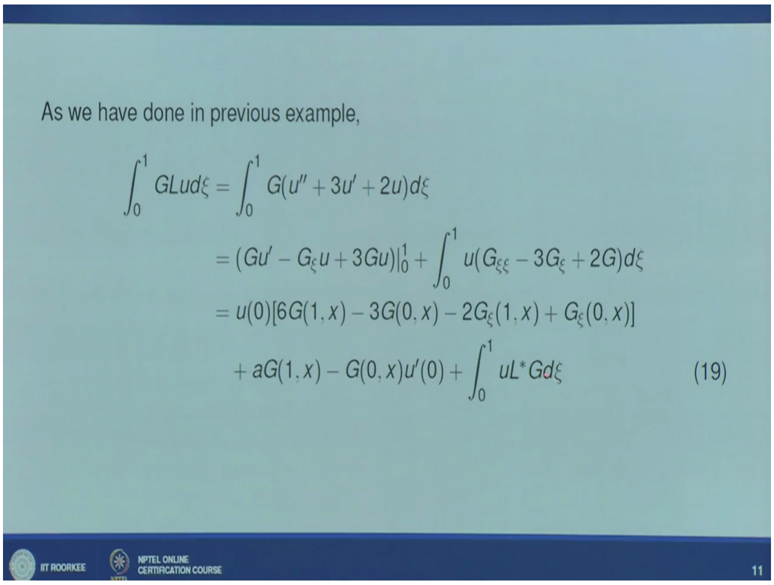

So we have seen that if our Green function satisfy this property that L star G which is defined as G xi xi minus 3G xi plus 2B G equal to delta xi minus x along with the condition G 0, x equal to 0 and 6G 1, x minus 2G xi 1, x plus G xi 0, x equal to 0 then the previous this is going to be u of x and it is equal to minus of aG 1, x plus 0 to 1 G L u d xi which is given as u of x equal to minus aG 1, x plus 0 to 1 G xi, x phi xi d xi.

But if you look at here this G which is solution of this differential equation is not easy to solve because it is not done by direct integration.

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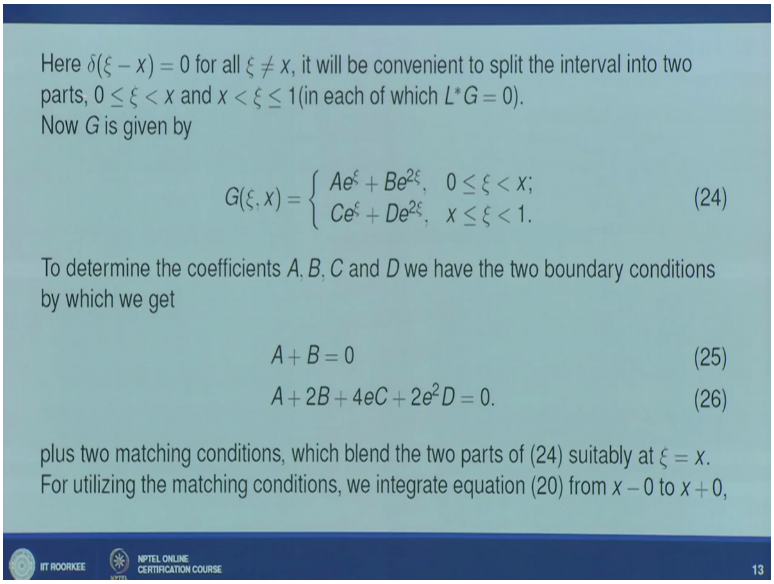

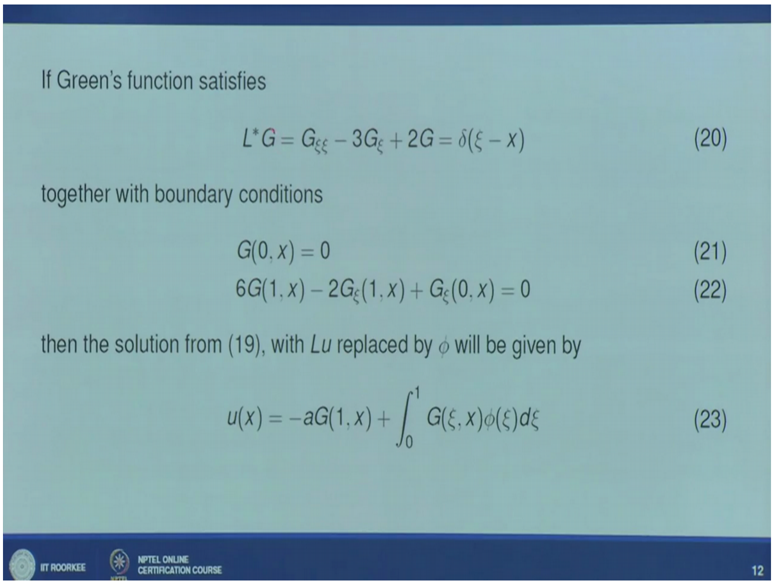

So here we use the property of Dirac delta function that delta xi minus x is 0 for all xi not equal to x. So what we try to do is we split our interval into two part, one is between 0 and x and between x to 1. So in each interval your delta xi minus x is equal to 0, so we can say that G is going to be a solution of this homogeneous equation L star G equal to 0.

So it means that we can get the solution we can get the expression of G by solving L star G equal to 0 and if you look at your G star is a simple order linear order differential equation with constant coefficient. So we can easily find out the solution and it can be written as that the general solution of this problem is given as constant times of e to power xi plus constant times of e to power 2 xi.

So we can write G as G xi, x as Ae to power xi plus Be to power 2 xi between 0 to x and between x to 1 it is another constant Ce to power xi plus De to power 2 xi. Now we can find out these constant A, B, C and D with the help of boundary condition we have assumed. So boundary condition is what boundary condition we have assumed that G of 0, x equal to 0 and this is the boundary condition.

So we if we apply these two boundary condition we can have this relation that A plus B equal to 0 and A plus 2B plus (4e to power) 4eC plus 2e square D equal to 0. Now along with this we have two matching condition also which is which we can obtain by the problem itself that L star G equal to delta xi minus x. So to get that matching condition what we try to do we simply integrate the L star G equal to delta xi minus x from x minus 0 to x plus 0.

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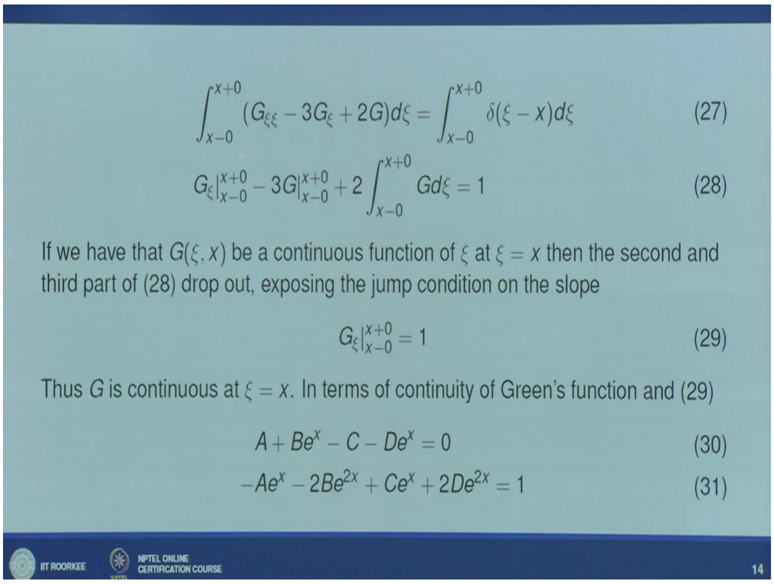

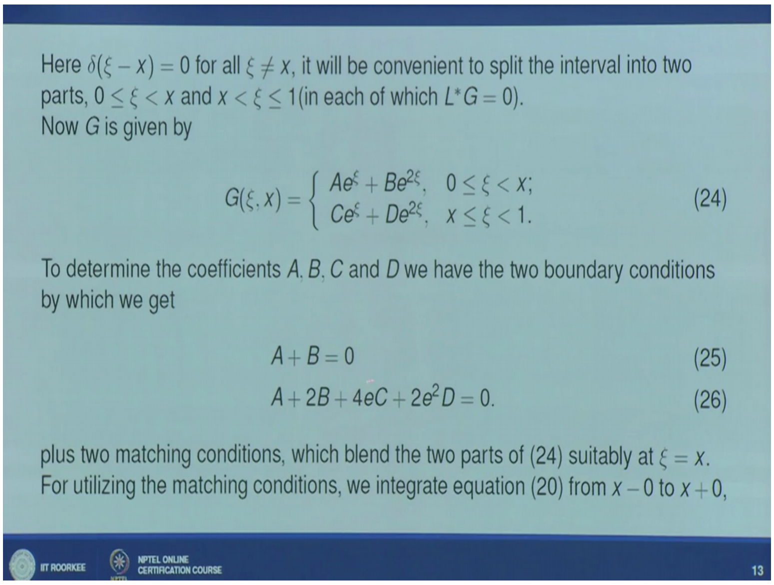

So which is nothing but (integration of x minus) integration of this term from x minus 0 to x plus 0. So here using the filtering property we have 1 here and this we can integrate. So I can have G xi evaluating at minus x minus 2 x plus and so on. Now here so far our G is completely arbitrary but to simplify our calculation what we can assume if we assume that G xi, x is a continuous function of xi at xi equal to x then this second term and the third term is going to vanish and we have one term left that is G xi from x minus to x plus is equal to 1. So here we simply say that this is a jump condition on the slope.

So here with this these assumptions we can say that our G is continuous at xi equal to x, so we have this condition that A plus Be to power x minus C minus De to power x equal to 0, this is done by using the continuity of G at xi equal to x and the jump condition at of G xi from x minus to x plus we have this last condition minus Ae to power x minus 2Be to power 2x plus Ce to power x plus 2De to power 2x equal to 1x.

So now we have four conditions this 25 and 26 that A plus B equal to 0 and A plus 2B plus 4eC plus 2e square D equal to 0 and two condition which we have obtained by assuming that G is continuous and G has jump discontinuity at the first order derivative.

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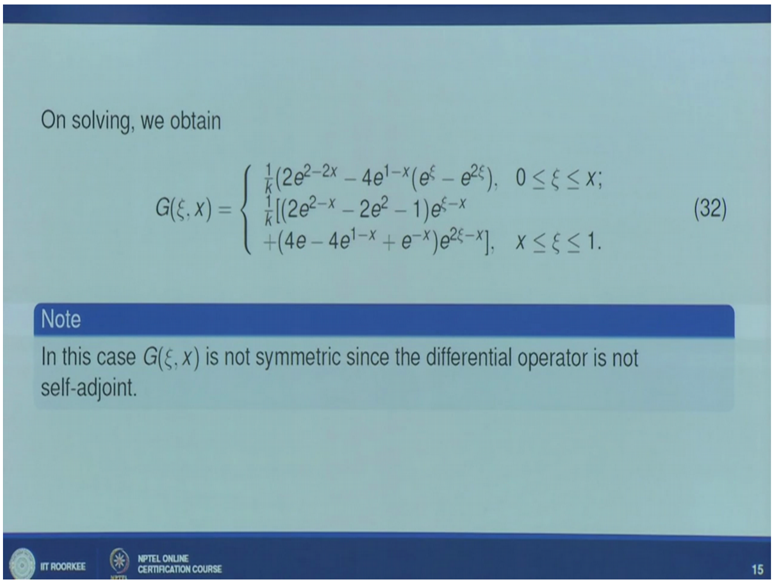

So if you use all these condition we can find out A, B, C and D and we can write down our solution as G xi, x equal to this expression. Now here we may observe that this G xi, x is not symmetric. But this is quite clear because the initial operator L is not a self adjoint operator. So this symmetricity will come when we have L as self adjoint operator. So in case of self adjoint operator our Green function is coming out to be asymmetric Green function, okay.

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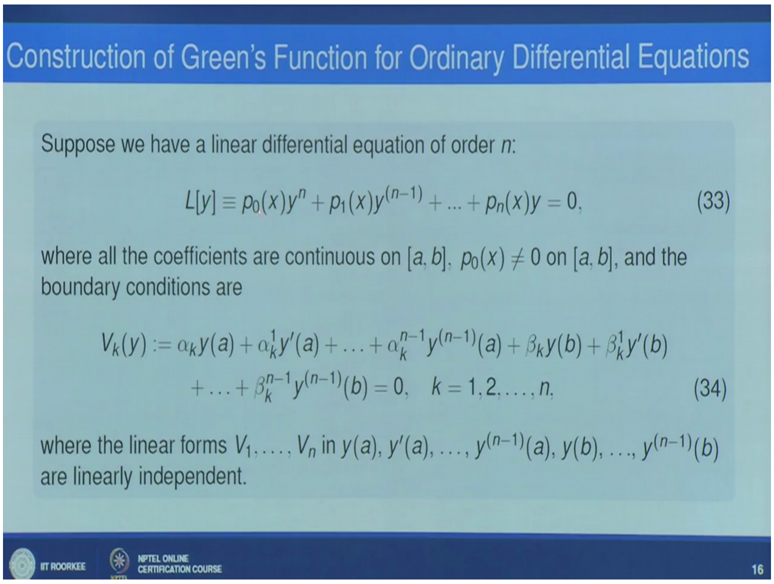

So now let us generalize this concept of constructing the Green function to a general nth order linear differential equation. So for this let us consider nth order linear differential equation defined as L y operating L y defined as this p not x y n plus p 1 x y n minus 1 and so on. Here we have assumed that all the coefficients p not x, p 1 x, p n x all are continuous on this close interval a, b and additionally we assume that p not x is nonzero in this interval a to b and boundary conditions are here we have assumed that these are mixed boundary condition this is general most boundary condition we have assumed.

So V k y is the boundary condition this is n boundary condition given k from 1 to n. So here this alpha k y a plus alpha 1 k y dash a and so on these are the boundary condition defined at the starting point that is a and beta k y b plus beta 1 k y dash b and beta n minus 1 k y n minus b these are the boundary condition given at the point p and this is given as 0 and here also we assumed that all these boundary conditions are linearly independent.

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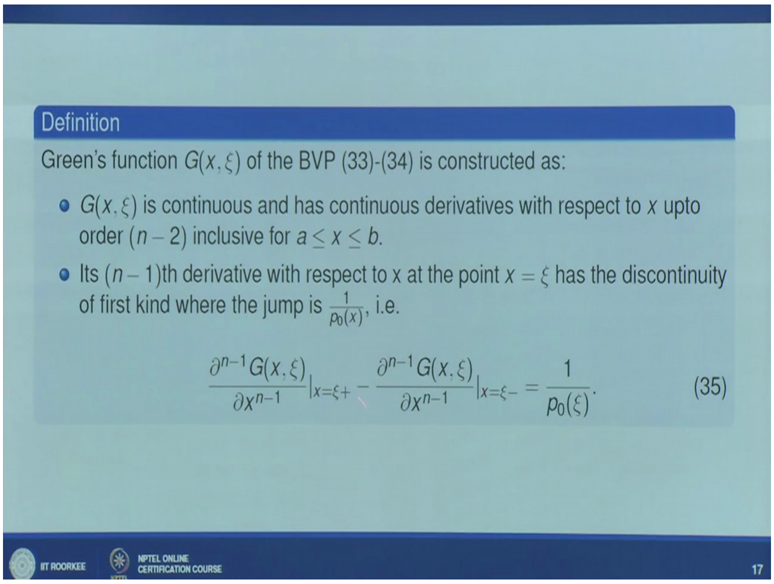

So for this we construct our Green function satisfying these following condition, so first condition is that G x, xi is continuous and its n minus 2th derivative is also continuous in this interval a to b, right. And its n minus 1th derivative has a jump discontinuity of order 1 by p not xi.

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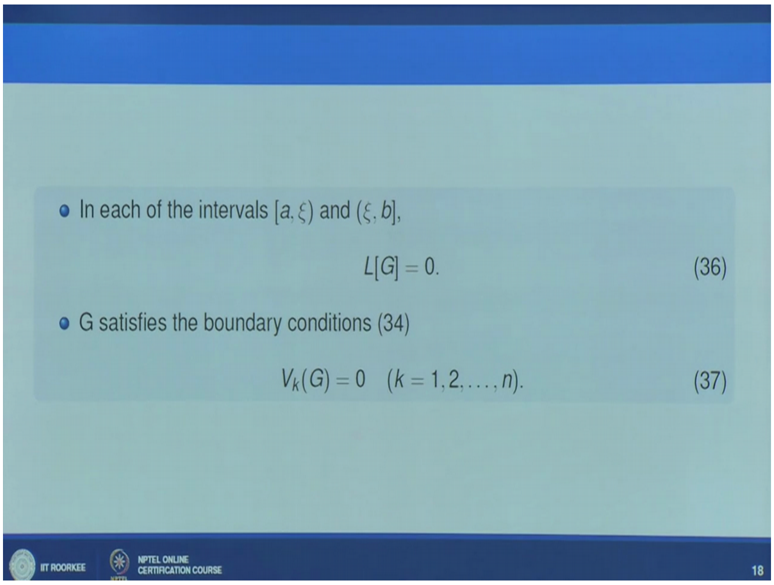

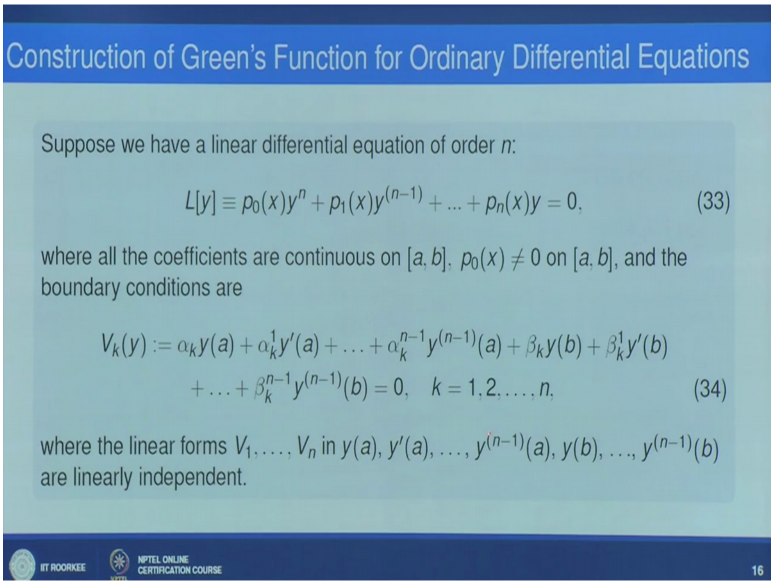

And third condition is that this Green function satisfy the linear differential operator that is L G. So it means that G is a solution of linear differential operator L G equal to 0 when xi is not equal to x. So it means that in each interval a to xi and xi to b L G is the solution of this operator L, so L G is equal to 0 when interval is a to xi and xi to b. And last condition is that G satisfy the boundary condition given as this given by this V k y equal to this.

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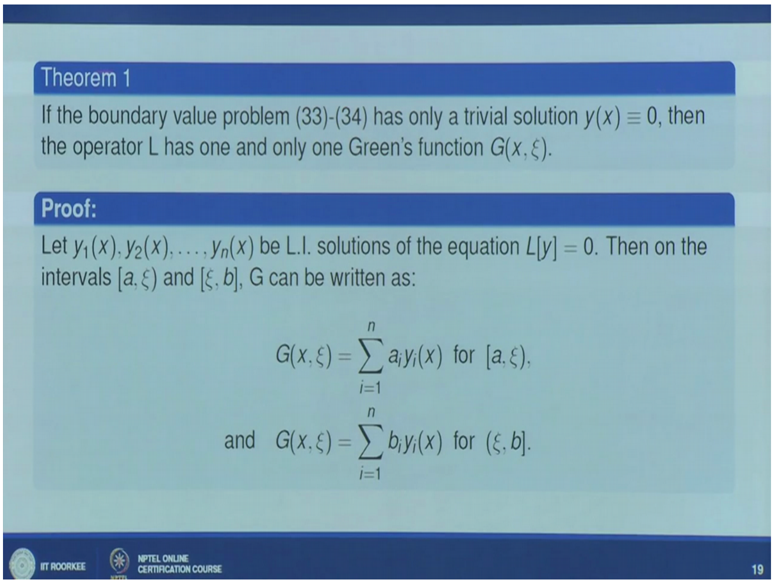

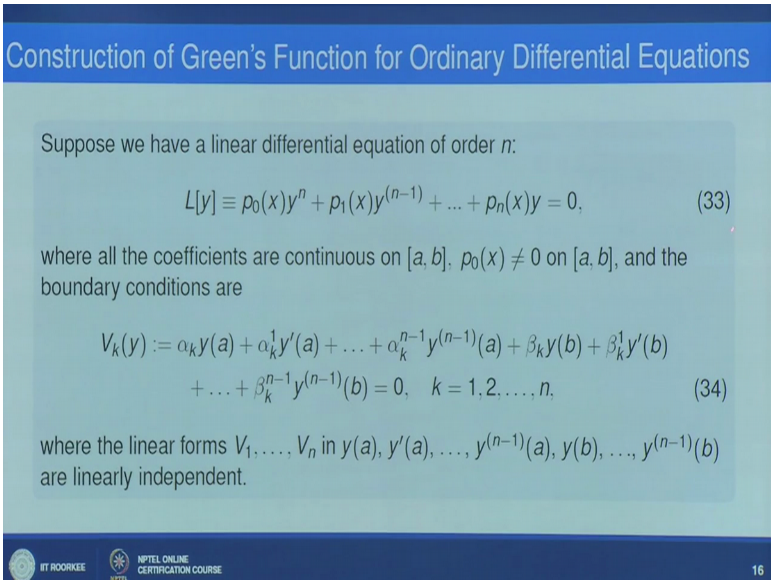

So here we have a theorem which says that if the boundary value problem this boundary value problem which we have defined here by 33 this is a differential equation and 34 is the boundary condition, this if this problem have only a trivial solution then the operator L has one and only one Green function G x, xi so that we are going to define.

So starting let us start the proof of this, so here we assume that let y 1 x, y 2 x, y n x are L linearly independent solution of equation L y equal to 0. Then as such state that we can write down G x, xi as this summation i equal to 1 to n a i y i x for a, xi close interval a open xi and G x, xi as linear combination of these solutions in xi, b because we already known we already know that G is going to be a solution of the operator L in this particular interval.

So it means that G can be written as linear combination of these n solution y 1 to y n. So that is why G x, xi is written as linear combination of y 1 to y n in this interval and similarly here G x, xi as linear combination of these solutions in this xi, b we need to find out our a i and b i so that this will actually solve solve the purpose.

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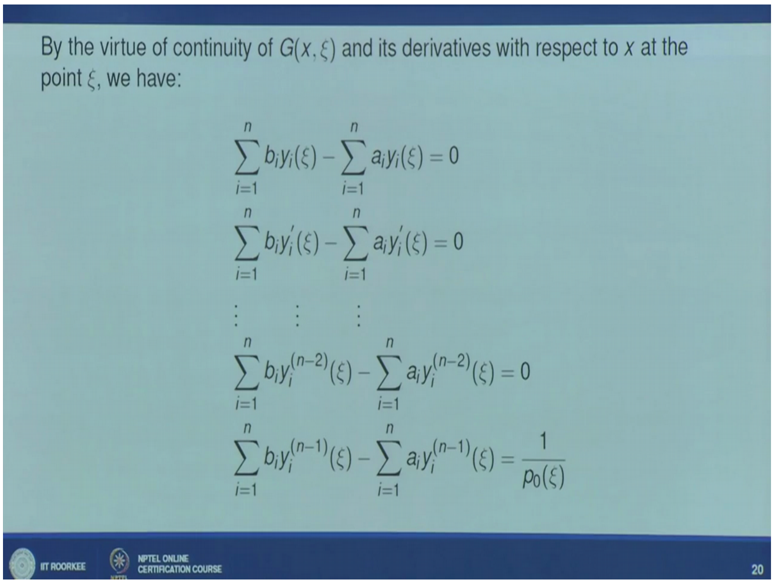

So now here we apply the condition that G x, xi is continuous and its derivative upto order n minus 2 is continuous if you look at the first condition simply says that G x, xi has continuity at x equal to xi and this still here we have assumed that n minus 2th derivative of G has a continuity at x equal to xi.

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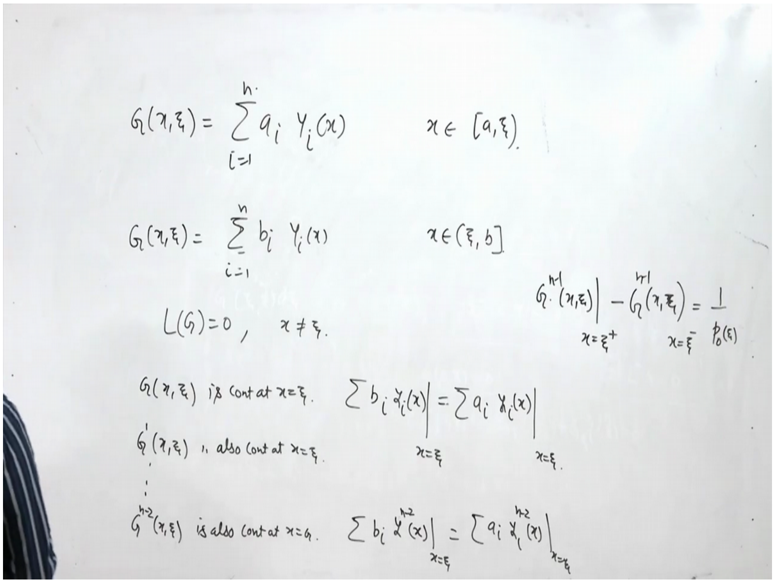

So if you look at here so here we have assumed that G x, xi is equal to summation a i y i x and i equal to 1 to n between x is between your a to xi and G x, xi is equal to summation i equal to 1 to n b i y i x between x is lying between xi to b here. So this we can this we have a (())(18:03) by saying that L of G is equal to 0 for x not equal to xi, right. Now if we use the condition that that G x, xi is continuous at x equal to xi it means that they should match as x equal to xi, so right hand limit is equal to left hand limit so we can write it summation b i y i x is equal to summation a i y i x at x equal to xi.

Similarly we G dash x, xi is also continuous at x equal to xi and so on upto n minus 2th derivative of this, so n minus 2th derivative of G is also continuous at x equal to xi. So this implies that summation b i y n minus 2 x at x equal to xi is equal to summation a i y i n minus 2 derivative of x at x equal to xi so is same here, okay.

Then last condition is that it has a jump discontinuity of order of magnitude 1 upon p not xi. Here it is x, xi at x equal to xi plus minus G n minus 1th derivative of x, xi evaluated at x equal to xi minus is equal to 1 upon p not xi.

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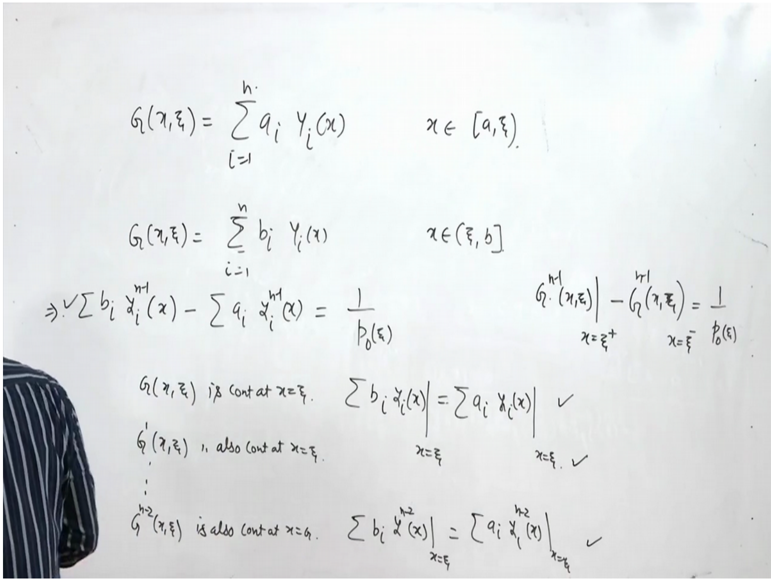

So if we use this then we have one more condition written as this, so here we can write down this condition that summation b i y i n minus 1 x minus summation a i y i n minus 1 x is equal to 1 upon p not xi, right. So here we have one condition and here we have these many condition.

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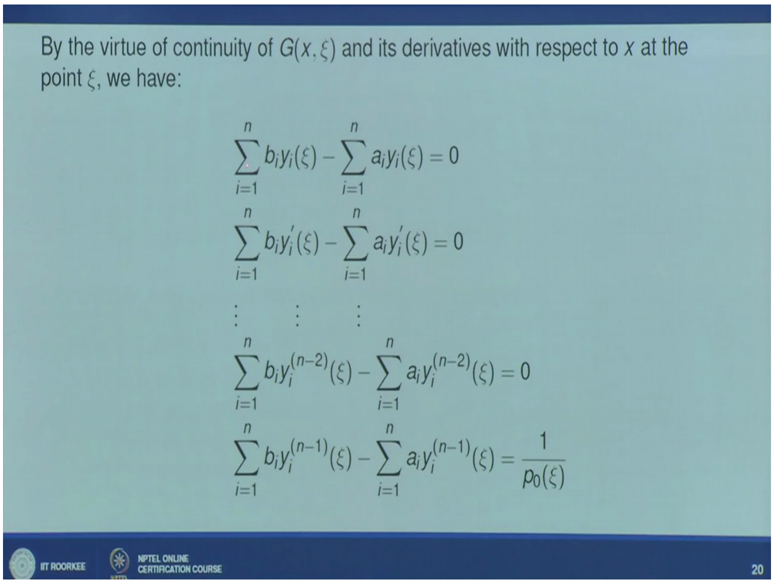

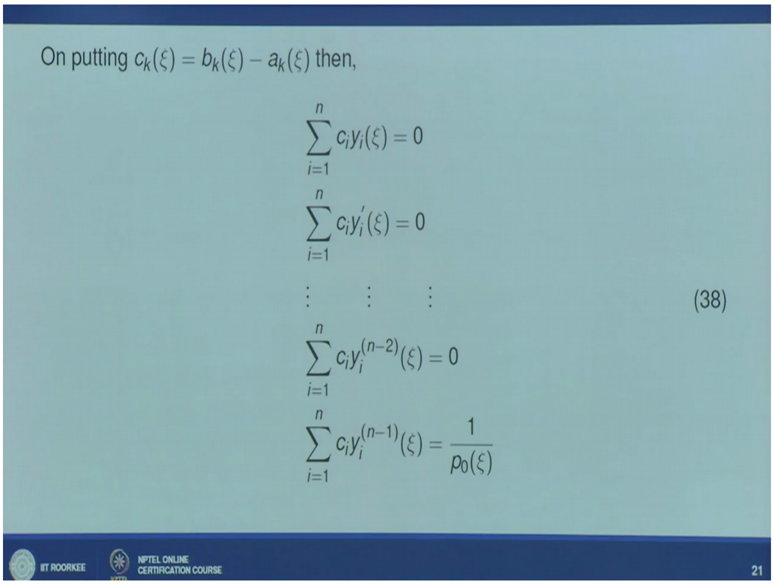

So all these condition we can write it in this particular form and look at here here we have all such condition. So first these still here we have assumed that G has continuity upto order n minus 2th derivative and this last represent the that n minus 1th derivative of G has jump of size 1 upon p not xi. So here this can be made simplify by writing b i minus c i, b i minus c I as some other variable that is c i.

So here we assume that your c k xi equal to b k xi minus a k xi and if we assume this then this can be written in a this simpler form, okay. Now what we try to have is that we try to find out all these constant a i and b i, so first what we try to do is we first find out this c i and with the help of c i we try to find out both b i and c i. So if you look at this represent linear differential equation a x equal to b, so if you look at equation system of equation given as 38.

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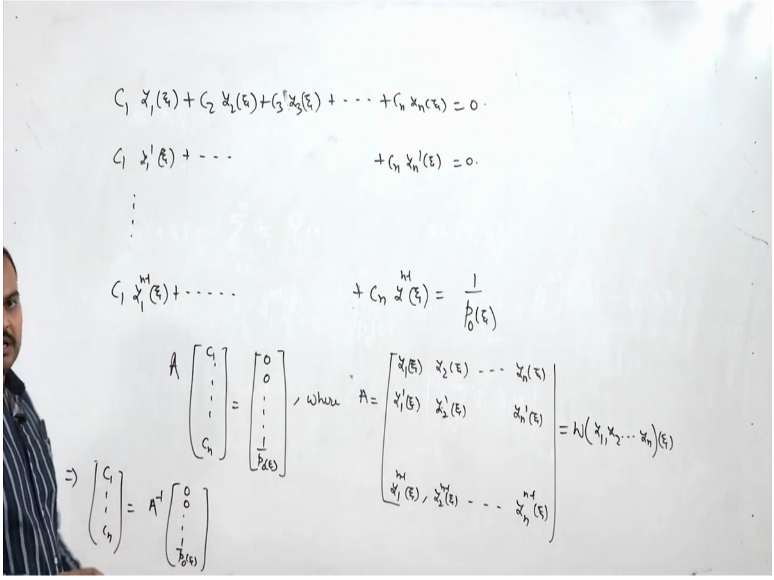

So here we can write it like this so here we have c 1 y 1 xi plus c 2 y 2 xi plus c 3 y 3 xi and so on c n y n xi is equal to 0. And second equation is c 1 y 1 dash xi plus so on plus c n y n dash xi equal to 0 and so on last one is c 1 y 1 n minus 1th derivative evaluated at xi and it is written as c n y n minus 1th derivative at x equal to xi equal to 1 upon p not xi. So this I can write it as A into this is your c 1 to c n equal to this 0, 0 upto 1 upon p not xi, where this A is given as this y 1 xi, y 2 xi and y n xi and y 1 dash xi, y 2 dash xi and y n dash xi and so on, last one is y 1 n minus 1th xi, y 2 n minus 1 xi and so on.

And if you remember that this is nothing but the (())(23:58) of y 1, y 2 and so on upto y n evaluated at xi.

Now here since in the beginning itself we have assumed that y 1 to y n are all linearly independent solution. So this means that this determinant of this coefficient matrix A is (going to be nonzero) going to be 0. So it means that this implies that this will have a unique solution. So I can get our solution c 1 to c n by saying A inverse operating on 0 to p not xi, so it means that with the help of these conditions we can actually find out the solution for this c 1 to c n.

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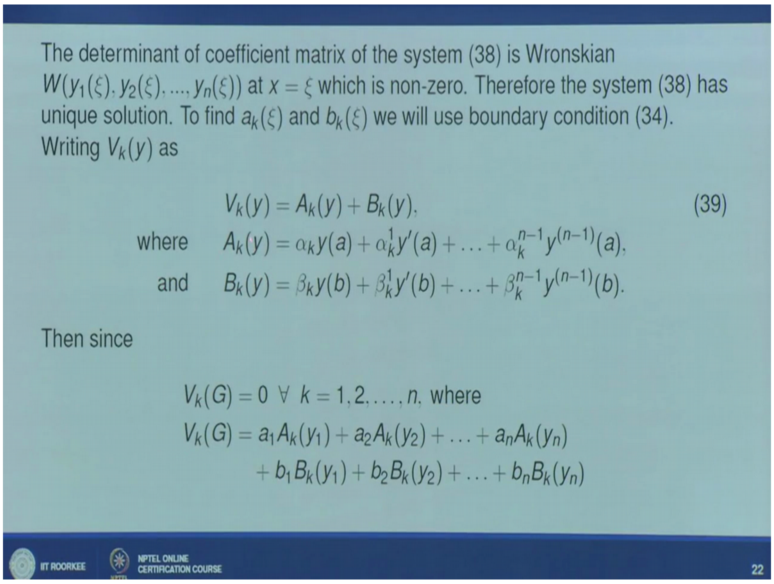

But we want to find out a 1 to a n and b 1 to b n not c 1 to c n. So what we do is we need to find out our coefficient a i and because i with the help of c i. For this we assume here that we know that this V k y can be written as A k y plus B k y, where we have separated the boundary condition at A and B, so this A k y represent the boundary condition given at the end point A and B k y represent the boundary condition given at the point B.

So we already know that Green function is going to satisfy the boundary condition, so here V k G is equal to 0, so if we simplify I can write this V k G as a 1 A k y 1 plus a 2 A k y 2 and so on plus b 1 B k y 1 plus b 2 B k y 2 and this thing and this is going to be 0.

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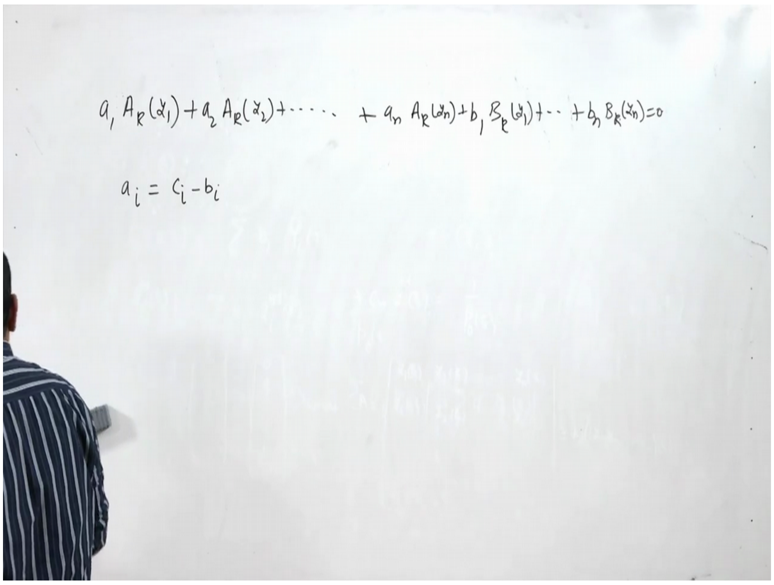

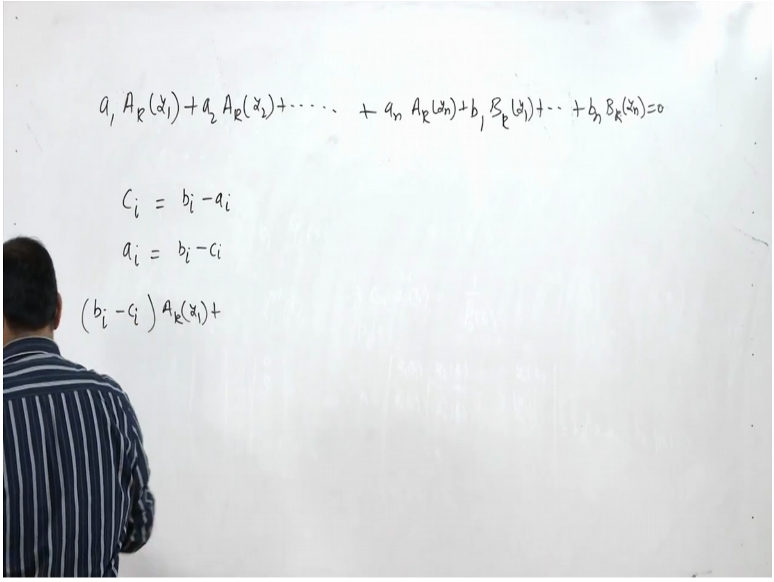

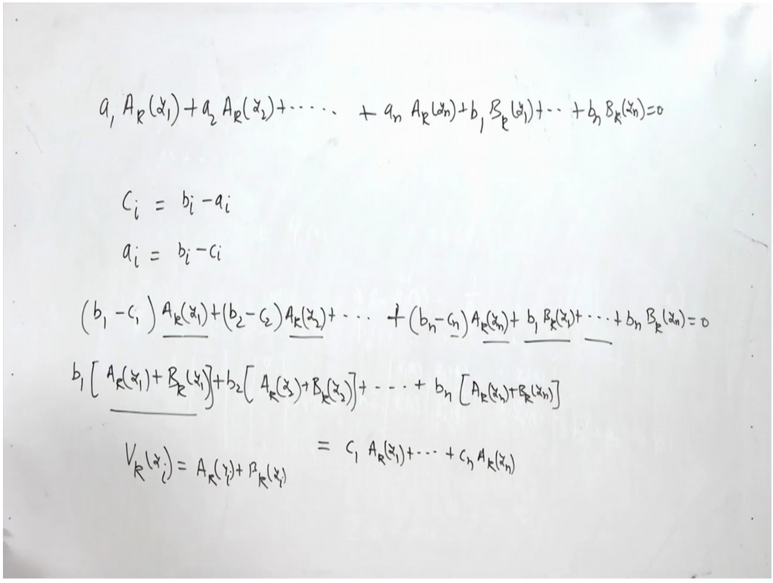

So here if you look at this here we have this a 1 A k y 1 plus a 2 A k y 2 and so on a n A k y n is equal to sorry plus b 1 B k y 1 and so on b n B k y n is equal to 0. So what we know here that I can write (a k) a i as c i minus b i, okay (no) c i we have assumed here c i as your b i minus a i, so I can write it here your a i as b i minus c i. So if we use this then I can write it here b i minus c i A k y 1 plus sorry this is I am writing for a 1 so it is b 1 minus c 1, so it is b 1 minus c 1 A k y 1 plus b 2 minus c 2 A k y 2 and so on it is b n minus c n this A k y n plus b 1 B k y 1 and so on.

Now here if we look at this I can simplify as b 1 and I can write it here A k y 1 plus b k this is your A k y 1 and I can write it here B k y 1, so I can write b 1 as b 1 A k y 1 plus B k y 1. Similarly coefficient of b 2 I can write it here A k y 2 and I can write it here the B k y 2 and so on till b n equal b n coefficient of b n as A k y n plus B k y n is equal to so this we are going to other side so it is c 1 A k y 1 and till c n A k y n.

Now this is what this A k y 1 plus B k y 1 we have assumed that this is nothing but your V k y 1, so here V k y is defined as A k y plus B k y. So V k y is defined as A k y plus B k y, so I can say that V k y i is defined as A k y i and B k y i.

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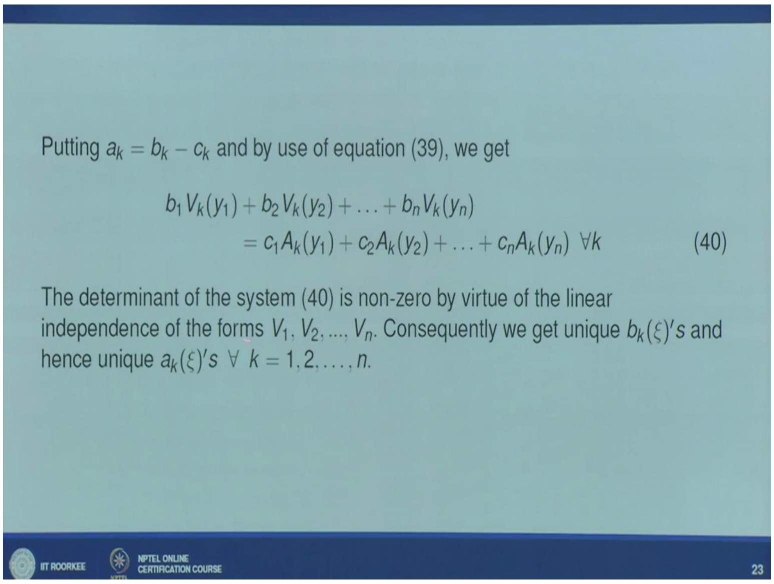

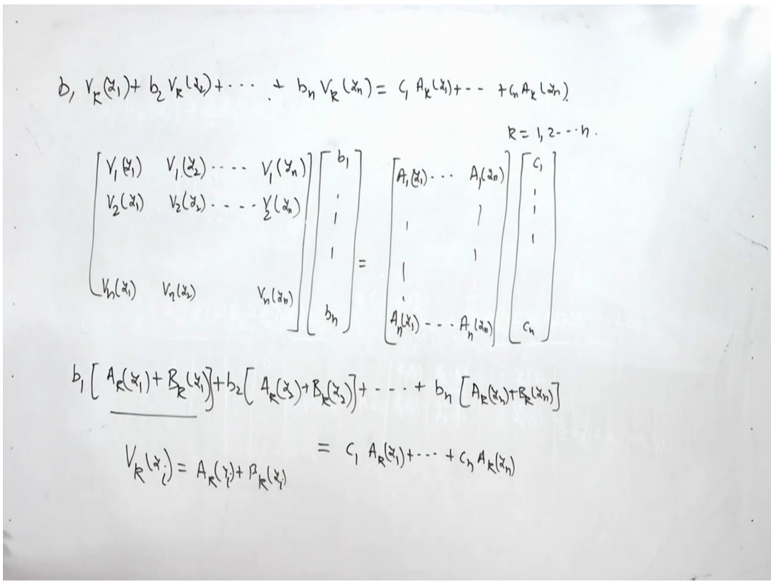

So I can write this as this relation that b 1 V k y 1 plus b 2 V k y 2 and so on. Now if you look at here this is again forming a linear differential equation in terms of b 1 to b n, right and coefficient matrix is what coefficient matrix is given as let me write it here. So here I am just writing here. So here this is what this is b 1 V k y 1 plus b 2 V k y 2 and so on b n V k y n is equal to your c 1 A k y 1 and so on c n A k y n, so if we and here k is from 1 to n.

So if we write it for each one I can write it this as (V 1 y 1 plus V k) sorry V 1 y 1 I am writing for the same equation for k equal to 1. So it is V 1 y 1, V 2 y 2 and so on V 1 y n, so this is V 1 here so for k equal to 1, I am writing it in a system of linear equation format. So here first row is V 1 y 1, V 1 y 2 and so on. Second is V 2 y 1, V 2 y 2 and V 2 y n and last is V n y 1, V n y 2 and V n y n and here your A 1 y 1 to A 1 y n and here it is under dash here we have A n y 1 to A n y n.

Now we already assumed here that these V k y 1 all these linear forms are linearly independent. So it means that this coefficient matrix has nonzero determinant. So if it has a nonzero determinant then we can find out a unique solution by inverting this coefficient matrix here.

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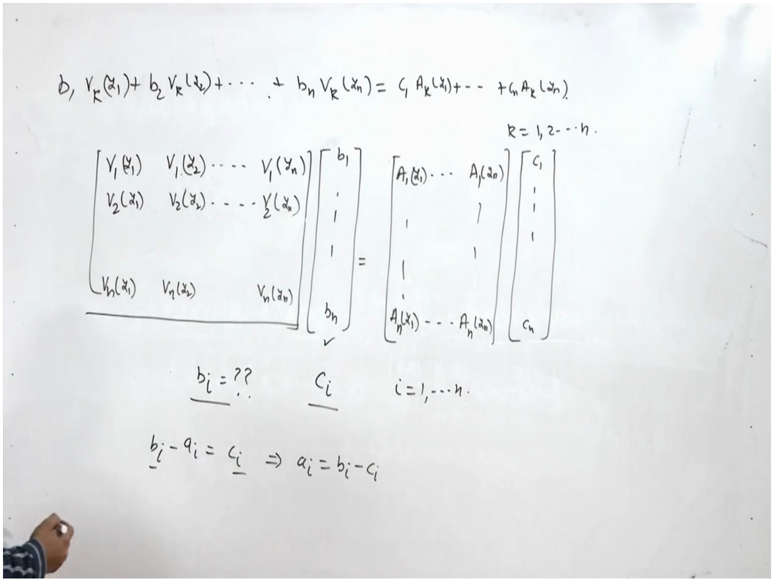

So it means we can find out say unique solution for these coefficient b 1 to b n. So with this system of linear equation we can find out b i we can calculate this and we already know what is c i for i equal to 1 to n. So it means that once we know b i and c i you can already get a i as b i minus a i is equal to c i, so we know c I, we know b i so we can get our a i as b i minus c i and hence our Green function is known for this particular problem.

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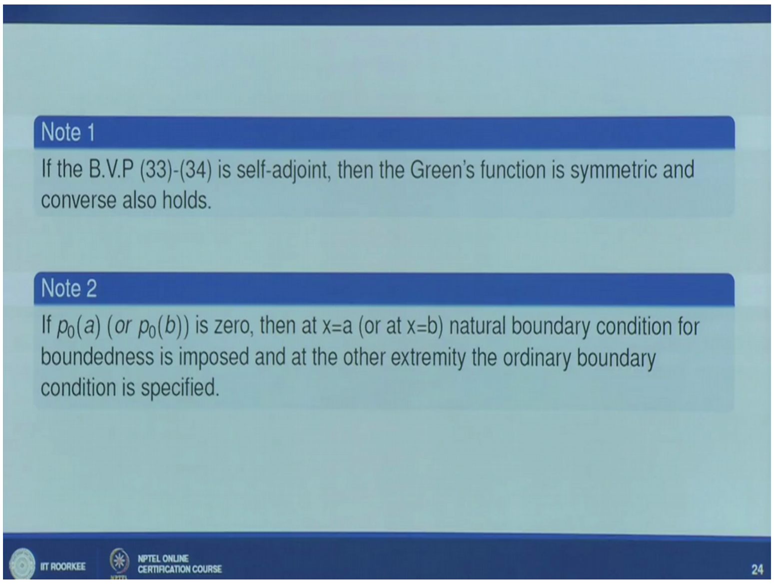

So we can get all coefficient with all these conditions, right. So now next is that if this boundary value problem is self adjoint then your Green function is going to be symmetric and it is converses also true and here it may happen that the highest order coefficient of highest order derivative is going to vanish at say one of the end point say A or B. In that particular case we will assume one natural condition that your solution is going to be bounded at that particular end.

For example if I assume that p not A is 0 then we assume that our solution is going to be bounded at x equal to A. So what we have done here in this particular lecture we have generalized the concept of construction of Green function for nth order linear differential equation. So thanks for listening us.