Integral Equations, Calculus of Variations and their Applications
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Lecture 13
Fredholm Integral Equations with Symmetric Kernels: Examples

Hello friends welcome to the lecture of Integral Equation and Calculus of Variations with its applications and examples, we were considering the case of symmetric kernel.

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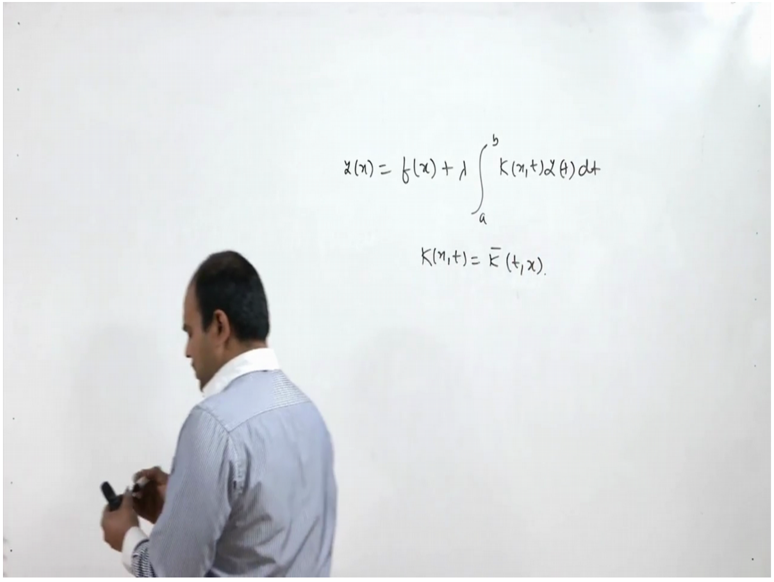

So if you remember we have we were doing this problem y of x equal to f of x plus lambda a to b K of x, t y of t y of t d of t provided that this K of x t is equal to K bar t of x. So here we are assuming that we have a symmetric kernel and then we try to solve the fredholm integral equation of second kind with this thing. So for that we have already discussed theory corresponding to eigenvalues and eigenfunction and how to solve this kind of equation, let us apply that theory to solve our problems.

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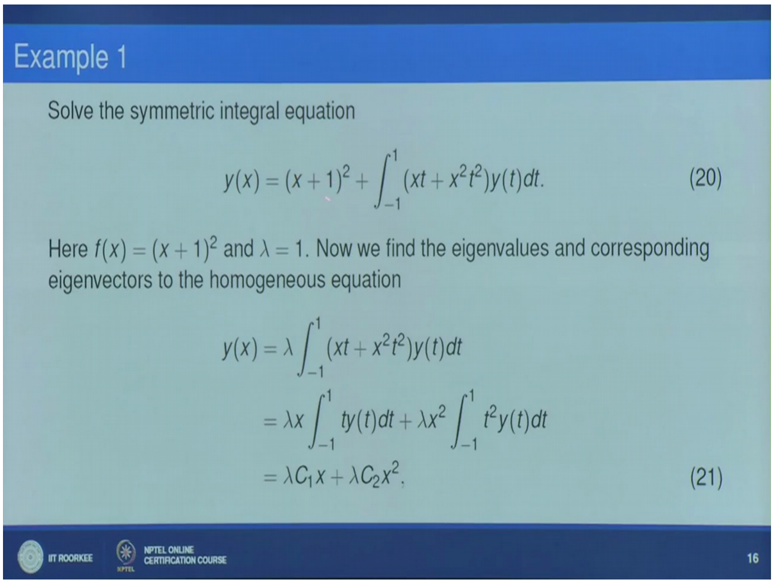

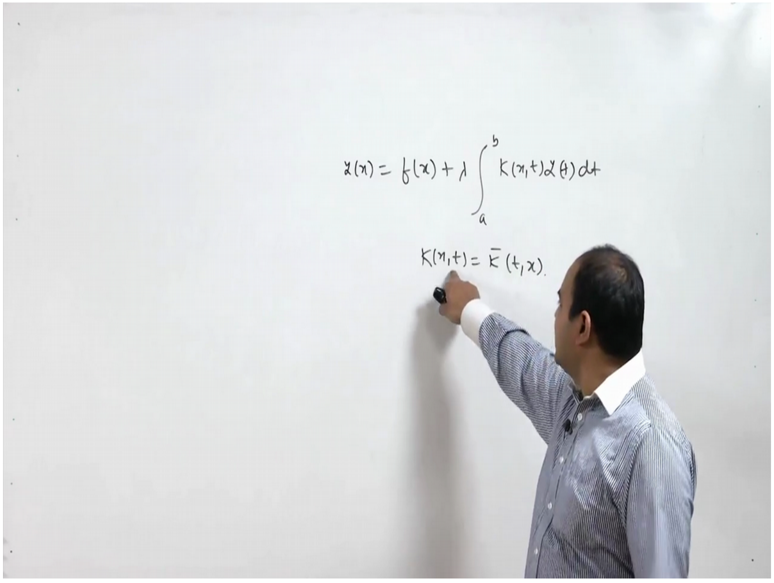

So our first problem is this y of x equal to x plus 1 whole square plus a to b, here a is minus 1 and b is 1 xt plus x square t square y t dt. Now if you look at here your f of x is your x plus 1 whole square and lambda is basically 1 here and if you look at K x, t is this so here K x, t is same as K t, x, here we are considering real so bar is not playing any role. So K x, t is equal to K t, x for this particular example.

If you remember this is of one more type that is separable type, so you can solve with the help of separable type also but we are solving this with the help of theory which we have developed earlier. So here we first we need to find out say eigenvalues and eigenfunction then to find out eigenvalues and eigenfunction let us consider the homogeneous version homogeneous version means this f of x is not there so we are considering this homogeneous problem that is y of x equal to lambda minus 1 to 1 xt plus x square t square y t dt and we want to find out the values of lambda for which we have a nontrivial solution and those that values those values we are saying that it is eigenvalues and corresponding solution eigenfunction.

So we are solving this with the help of method which we have developed for separable kernel, so we are writing this as lambda x minus 1 to 1 ty t dt plus lambda x square minus 1 to 1 t square y t dt. Now this quantity is something which is a constant call it C 1 and this quantity minus 1 to 1 t square y t is another constant which we call as C 2, so y x is written as lambda C 1 x plus lambda C 2 x square, so solution we already know only thing is we need to find out this C 1 and C 2. So for that we use the expression for C 1 and C 2 and use the value of y of x.

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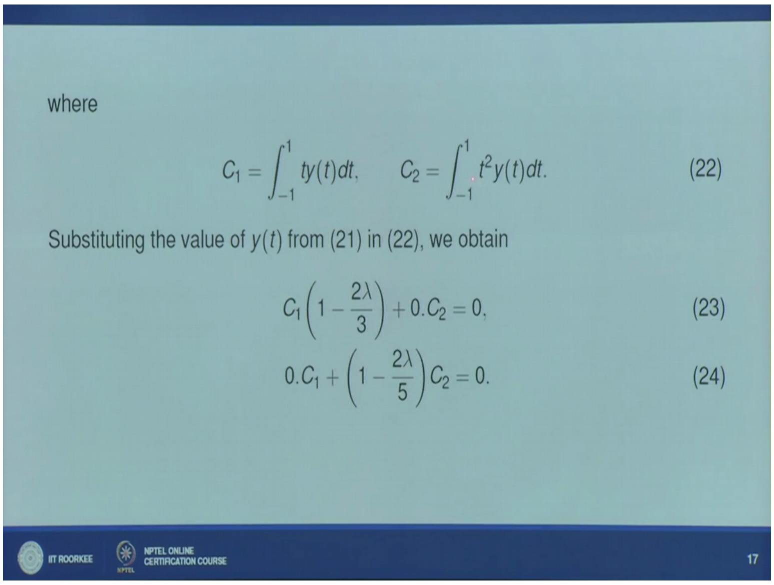

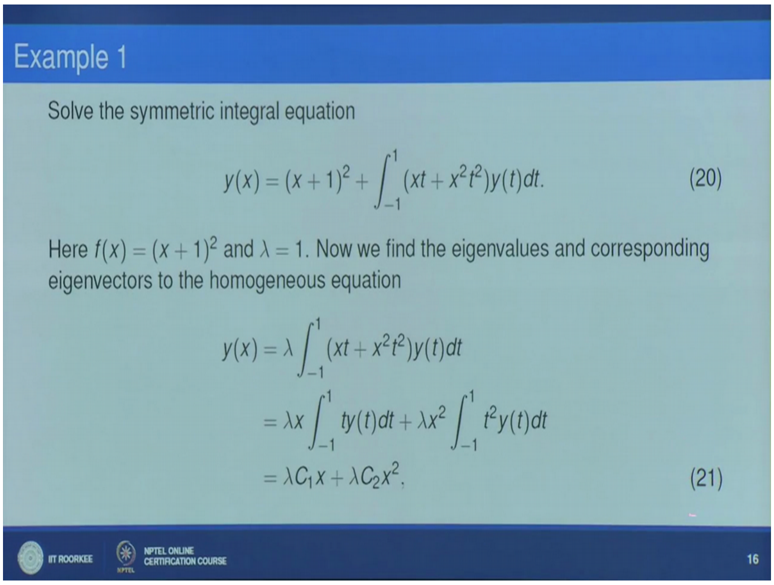

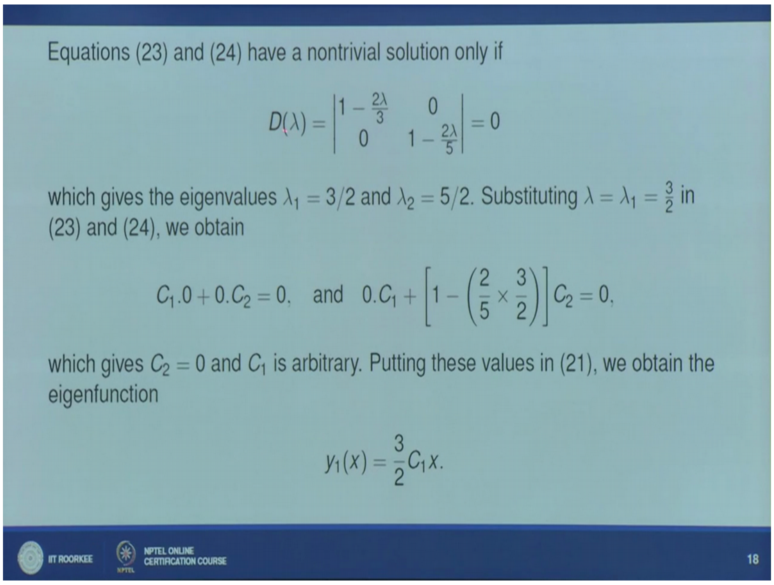

So here we have C 1 is equal to this and C 2 is equal to this, so using y t given by 21 you can get two equation 23 and 24, C 1 1 minus 2 lambda by 3 plus 0 times C 2 equal to 0 and this thing. Now if you remember by solving this is nothing but algebraic equation (D lambda C 1) this D lambda of C 1 and C 2 equal to this thing. So here your D lambda is the coefficient matrix of this algebraic equation.

So D lambda is coming out to be modulus of 1 minus 2 lambda 3, 0, 0, 1 minus 2 lambda by 5 and if you look at this is a simply a triangular matrix and here you can get your eigenvalues very easily and you can say that lambda 1 equal to 3 by 2 and lambda 2 is equal to 5 by 2 is the eigenvalues means for which your D lambda is having 0 determinant. So here eigenvalues we know then we need to find out the corresponding normalized orthonormalized eigenfunction.

So for that let us start with first that is lambda equal to lambda 1 that is 3 by 2 and try to solve this 23 and 24. So when you put lambda 1 equal to 3 by 2 here in 23 and 24 then you look at that the first equation 23 is simply redundant because this quantity is 0, so 0 times C 1 plus 0 times C 2 equal to 0, so this is simply will not give anything. So if you look at here by putting this you can get that C 2 equal to 0 and C 1 is completely arbitrary.

So when we have C 1 is completely arbitrary then you can write down our solution y 1 x here we can put the value of C 2, so C 2 is 0 here so y x is lambda C 1 x. So I can say that lambda is basically what lambda is 3 by 2 here, so I can write y 1 x is equal to 3 by 2 C 1 x. So corresponding to this eigenvalue lambda 1 equal to 3 by 2, we have a nontrivial solution that is 3 by 2 C 1 x.

Now with C 1 is some arbitrary constant so we can always take C 1 in a way such that this 3 by 2 C 1 is 1.

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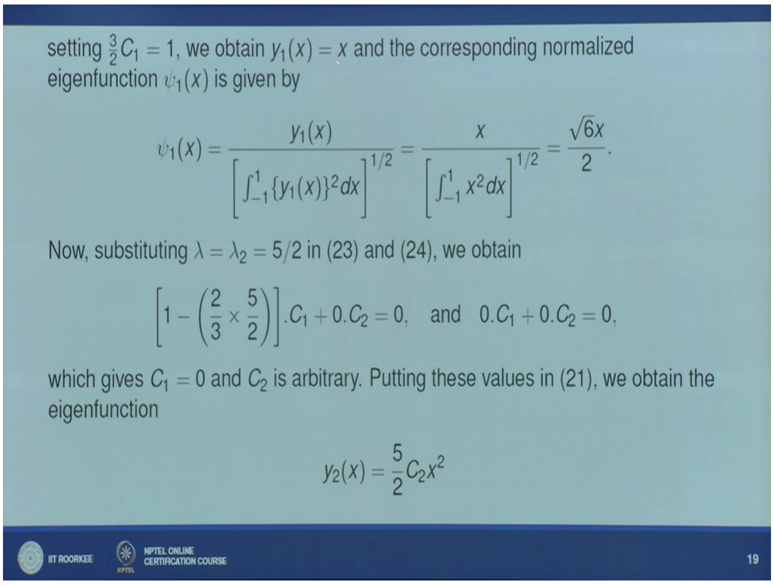

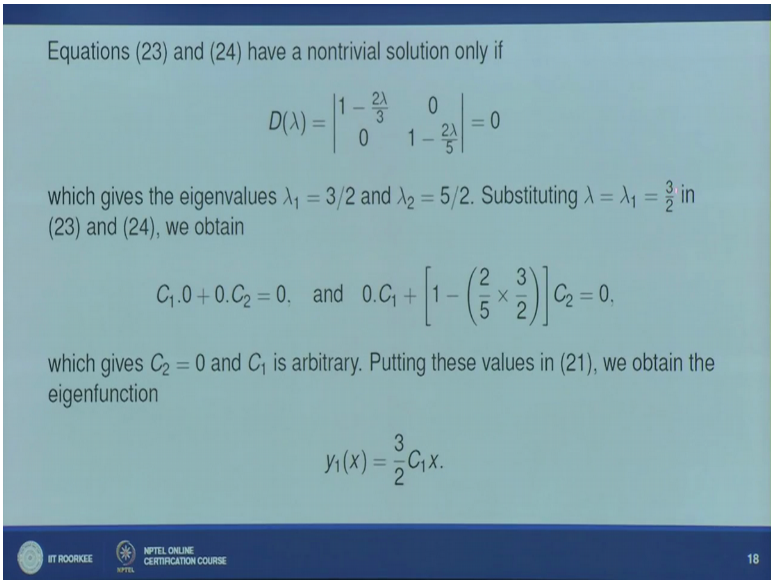

So we can say that your y 1 x equal to x is the eigenfunction corresponding to lambda 1 equal to 3 by 2. Similarly we can find out (eigen) eigenfunction corresponding to 5 by 2. But we need normalized eigenfunction, so first let us normalize this y 1 x, so normalize means you just divide by norm of this, so what is the norm here so norm here is minus 1 to 1 y 1 x square d of x.

Now if you calculate y 1 x is x here, if you calculate this quantity is coming out to be this 2 by root 6, so you just calculate and divide by this, so Psi 1 x is given by this root 6 by 2 x. So this is very easy to solve, okay. So now so lambda 1 is known to you, Psi 1 is known to you, now repeat the same procedure for lambda 2, so lambda 2 is 5 by 2. So first equation gives you this and second is again the same way redundant.

So here you can solve and you can say that C 1 is coming out to be 0 and C 2 is arbitrary.

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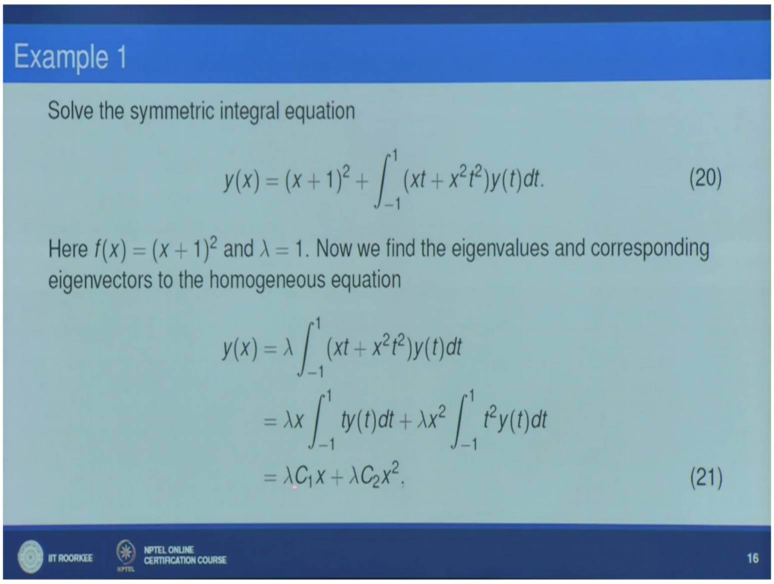

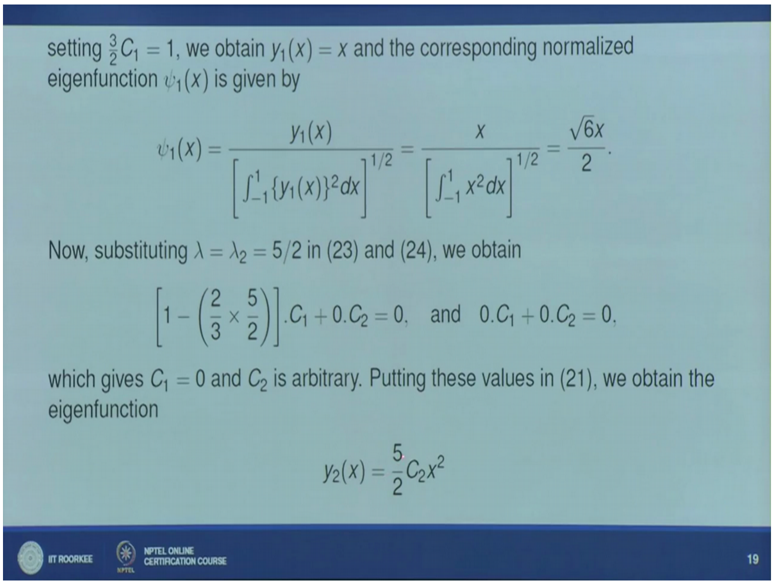

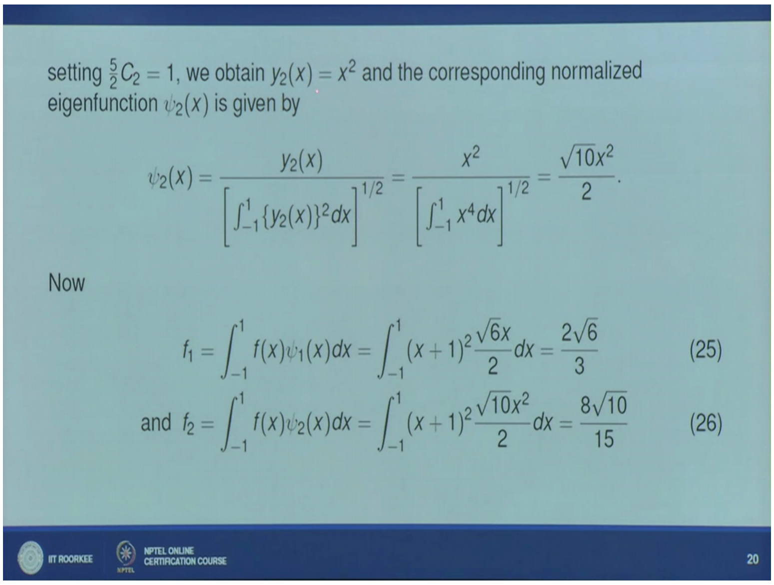

Again using the solution this 21 C 1 is simply 0, so y x is nothing but lambda C 2 x square. So here lambda is 5 by 2, so it is y 2 x is 5 by 2 C 2 x square. Now again in a same way C 2 is completely arbitrary, so we can always consider C 2 as 1 and so y 2 x is your x square is the eigenfunction corresponding to eigenvalue 5 by 2. Now you normalize it so when you normalize it you divide by norm of this, so norm is minus 1 to 1 y 2 square dx, so y 2 x is x square simply put it here and then you can find out the Psi 2 x under root 10 by 2 x square.

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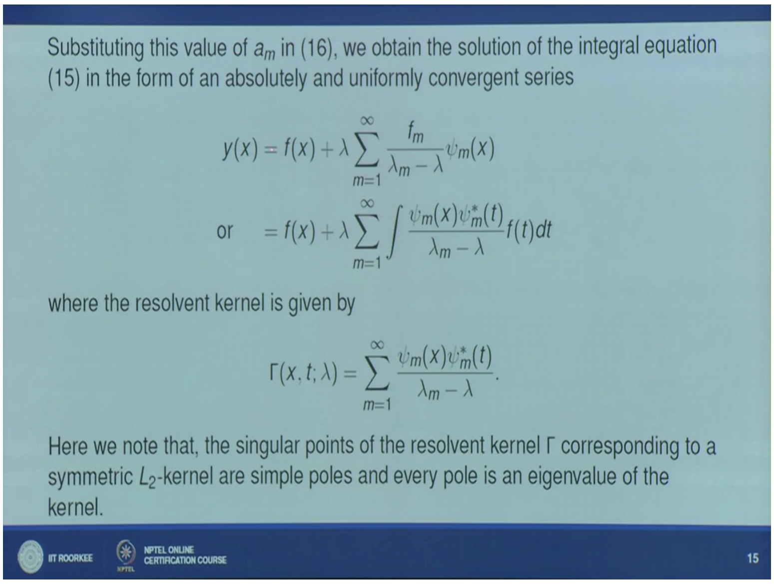

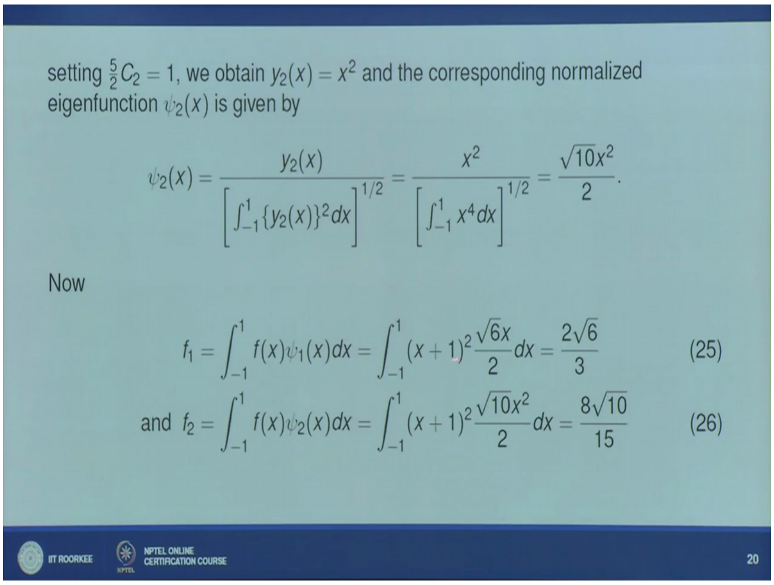

So once we have Psi 1 and Psi 2 then you can use our formula which says that our solution I am just going back to this and here we say that the solution is written as y x equal to f of x plus lambda m equal to 1 to infinity f m upon lambda m minus lambda Psi m x, here. Here m is from 1 to infinity but here in our particular example we have only two eigenvalue so this infinite sum is reduced to only finite sum that is m equal to 1 and 2.

So what we want? We already have lambda 1 and lambda 2 and Psi 1 and Psi 2 we have, only thing we have to calculate is f 1 and f 2. So to calculate f 1 and f 2, f 1 and f 2 are fourier coefficient corresponding to f, so (f of) so f 1 is basically what? Minus 1 to 1 f of x Psi 1 x dx, so Psi 1 x is already known to you, you put it here and f x is already known to you so you can get the value of f 1 that is 2 root 6 by 3, this quite simple calculation you can just plug in and then you can find out f 1.

Similarly in a same way you can find out f of 2, f of 2 is minus 1 to 1 f of x Psi 2 x, so this is the fourier coefficient corresponding to this Psi 2. So again you can put it all values already you know f, you know Psi 2, you can get your f 2, f 2 is coming out to be 8 root 10 by 15.

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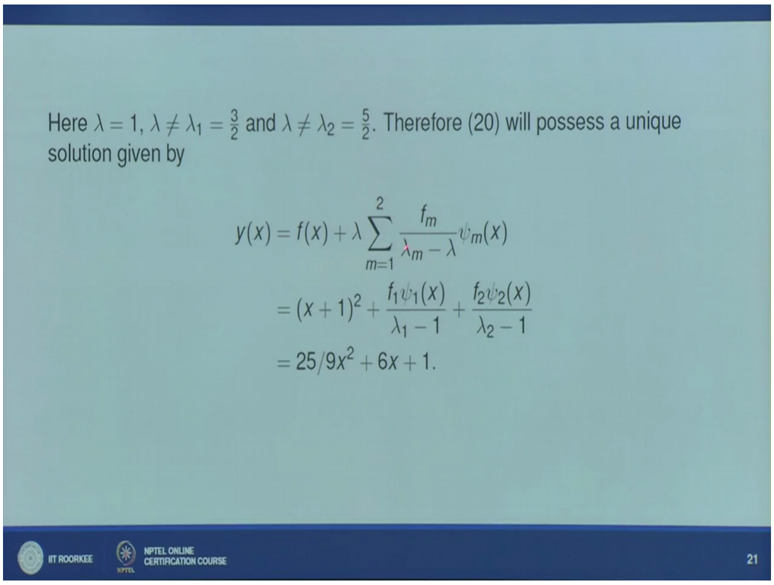

So now you know lambda 1, lambda 2, Psi 1 and Psi 2 and f 1 and f 2, you just write down your solution as y x equal to f x plus lambda m equal to 1 to 2 f m upon lambda m minus lambda Psi m x, so here lambda is 1 we have taken as 1, so f x is this, f 1 is this f 1 Psi 1 x upon lambda 1 minus 1 plus f 2 Psi 2 x upon lambda 2 minus 1, so putting all the values we have y x is equal to 25 by 9 x square plus 6 x plus 1.

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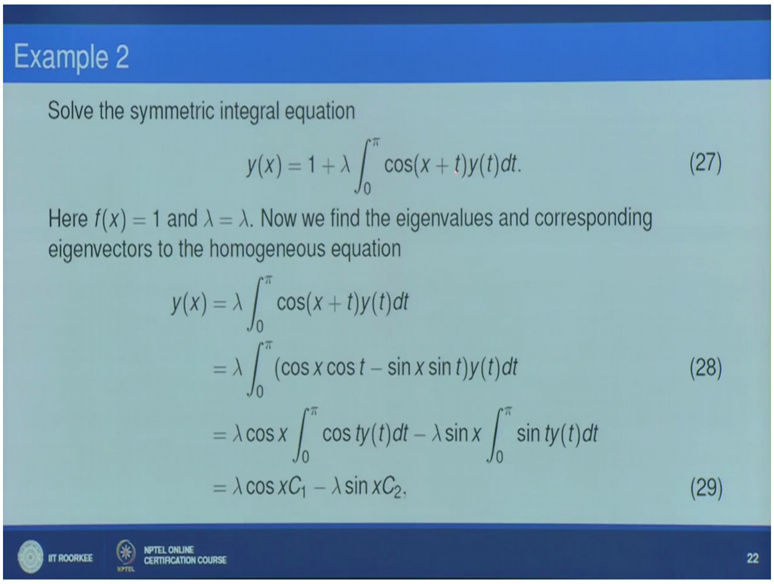

So let us consider one more example to have a better understanding I hope this is not very difficult. Let us look at this problem y x equal to 1 plus lambda 0 to pi cos of x plus t y t dt. Now here again you look at cos of x plus t is a symmetric kernel, so it means that if you interchange it is first of all real then if you interchange the role of x and t then cos of x plus t is same as cos of t plus x. So it is case of symmetric kernel and then we try to find out.

So here f of x is simply 1 and since lambda is not given any constant value so we can say that lambda is same as lambda. So to find out the solution in terms of resolvent kernel or using our theory we need to calculate our eigenvalues and eigenfunction. So consider the homogeneous equation that is you take simply f x as 0, so consider this y of x equal to lambda 0 to pi cos of x plus t y t dt, now we want to find out say eigenvalues and eigenvectors.

So this is this you can solve using the method of separable kernel, so you can find out eigenvalues and eigenfunction with the help of method we have discussed for separable kernel. So here you write down cos of x plus t as cos of cos x cot t minus sin x sin t y t dt. Now again you simplify then it is 0 to pi cos t y t dt minus lambda sin x 0 to pi sin t y t dt. Now this quantity is again constant call it C 1, this quantity is your constant call it C 2.

So it means that your y x is already known to you, so this is lambda cos x C 1 minus lambda sin x C 2. So only you need to find out this C 1 and C 2.

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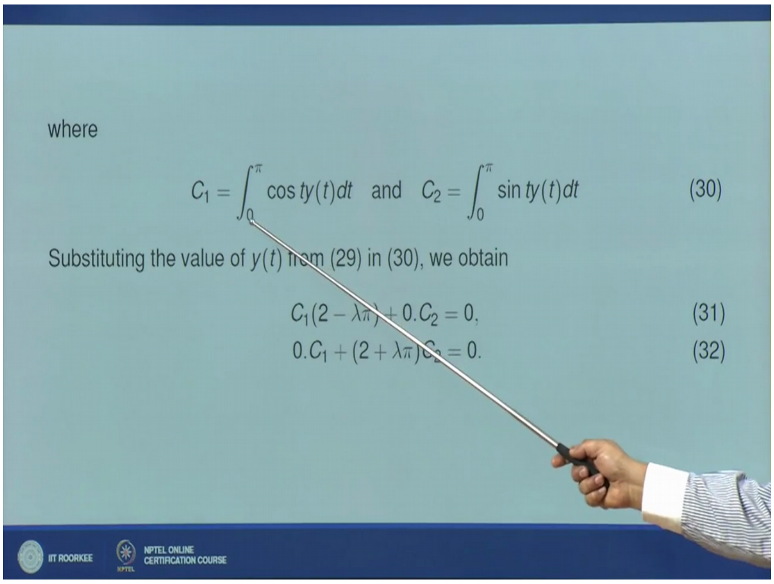

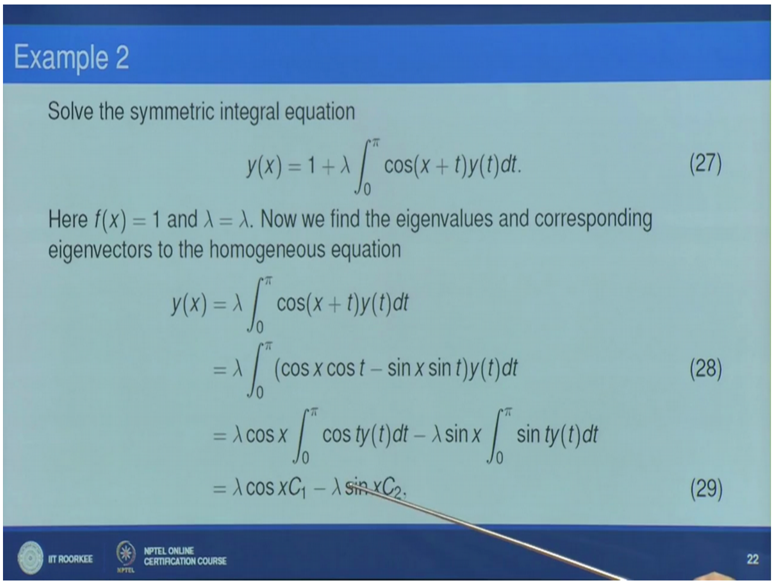

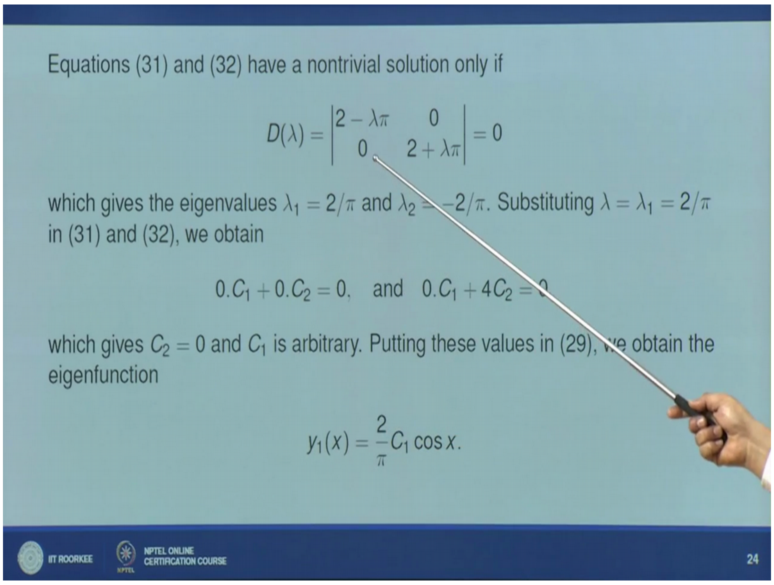

So so here using this we already know that C 1 is 0 to pi cos t y t dt and C 2 is 0 to pi sin t y t dt. So C 1 and C 2 and you already know the solution y like this, so from this you can calculate y t put it back to C 1 and C 2 you can have these two equation C 1 2 minus lambda pi plus 0 times C 2 equal to 0 and this 32 equation number 32 and this is again if you look at this is again a triangular equation to solve and here your eigenvalue can be find out by say coefficient matrix determinant of coefficient matrix and determinant of coefficient matrix is nothing but D lambda and when you solve this you will get that we have two eigenvalues that is lambda 1 is equal to 2 by pi and lambda 2 equal to minus 2 by pi.

So eigenvalues is with us, now we try to find out the corresponding eigenfunction. So let us start with lambda equal to lambda 1 that is 2 by pi, so when you put lambda equal to lambda 1 that is 2 by pi then first equation is simply redundant and you can solve this and you can say that C 2 is coming out to be 0 and C 1 is arbitrary as we have considered in the previous case.

So here we say that C 2 is 0 and C 1 is arbitrary and which says that look at equation number 29. So y x is already given so using C 2 equal to 0 and C 1 arbitrary y 1 x is equal to 2 by pi C 1 cos of x. Again in a same way which we have already discussed you can take 2 by pi C 1 as 1, C 1 is arbitrary.

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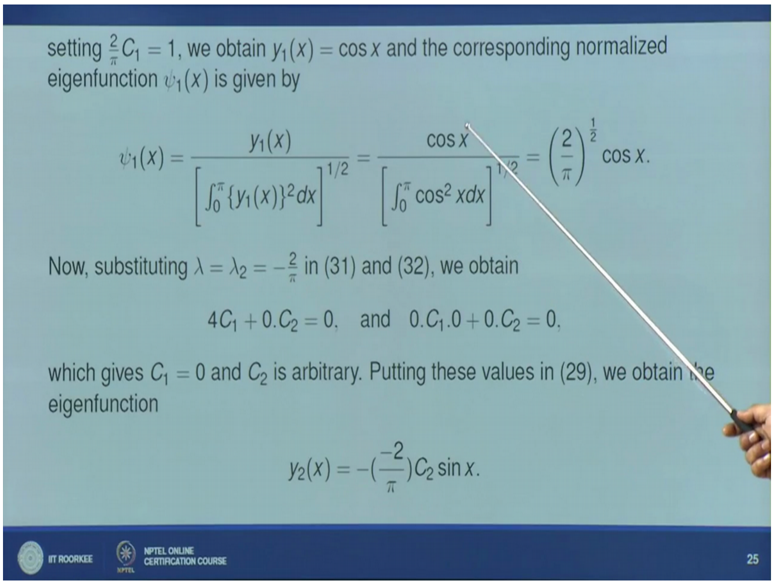

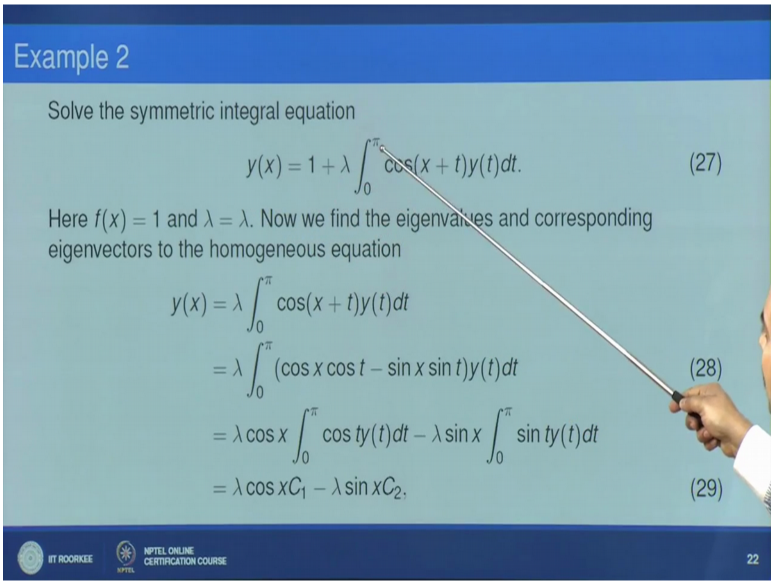

So y 1 x is your cos of x is the corresponding eigenfunction, so just normalize it. So how to normalize it you divide it by norm norm here is a to b, a is what 0 and b is pi, so I think it is already cleared here a is 0 and b is pi, so 0 to pi y 1 x square d of x half so this is a norm of y 1 so divide by this and this quantity is simply pi by 2, so this is what 2 by pi root of 2 by pi cos of x.

So this is the eigenvalue normalized eigenfunction corresponding to lambda equal to pi by 2. Now let us do the same procedure for lambda 2 equal to minus 2 by pi and you can say that in a same way you can get C 1 equal to 0 and C 2 is arbitrary and you can write down the corresponding nontrivial solution as y 2 x and it is minus of minus 2 by pi C 2 sin x, again you can take this constant as 1 and y 2 x is nothing but sin of x is the corresponding eigenfunction.

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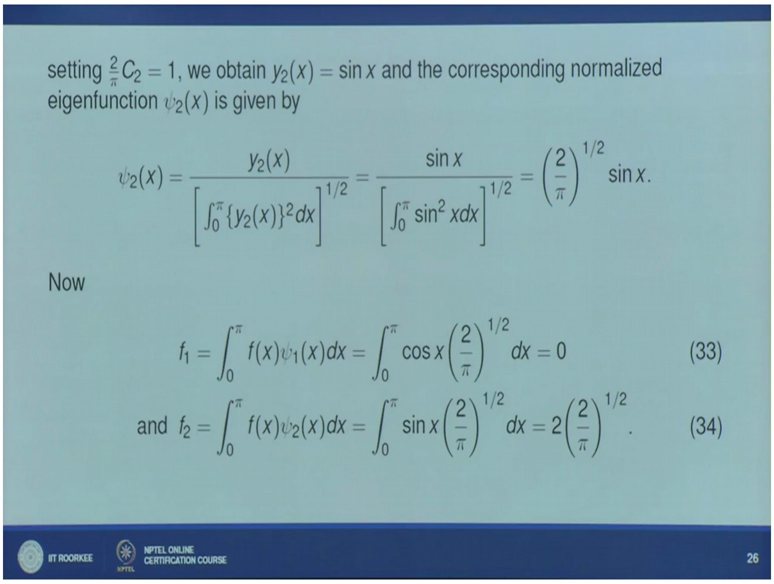

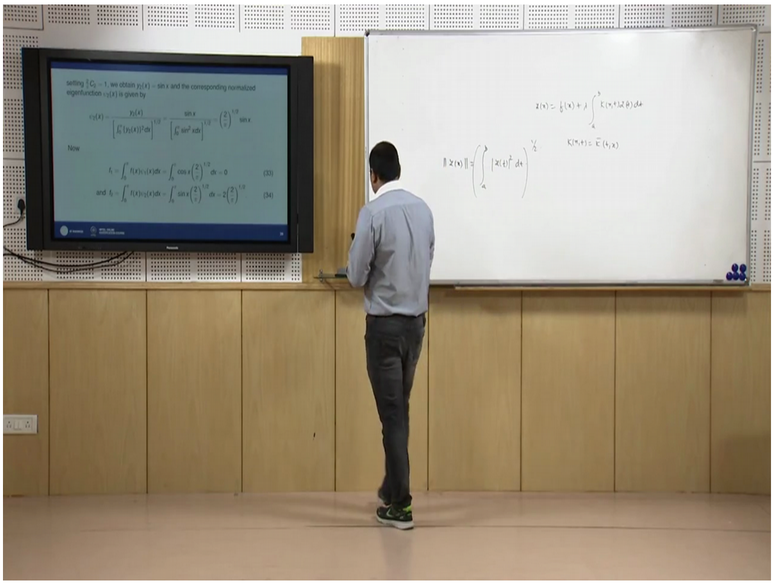

You can again normalize in a same way so Psi 2 x is y 2 x divided by norm of y 2 x which is nothing but this quantity. So please I hope we already know norm of y of x is equal to a to b y of x square d of x 1 by 2, okay by the way this is this is a dummy variable so you can use t or anything, I am using t because I am using x here, okay. So this is norm of y of x so that is why we are dividing by norm of this. So here we have Psi 2 x is given to you.

Now in a similar way we can find out f 1 and f 2 which is fourier coefficient of f corresponding to these eigenfunction Psi 1 and Psi 2, so f 1 is 0 to pi a to b, so a to b means 0 to pi f x Psi 1 x dx and you can say that it is coming out to be 0 and f 2 is 0 to pi f of x Psi 2 x dx which you can calculate it is coming out to be this quantity 2 upon 2 by pi square root of this 2 into square root of 2 by pi. So f 1, f 2 is known to us.

Now you simply put it back to your solution and now problem is that this lambda is not given any constant value. So your solution will also depend on the value of lambda because lambda we can take any constant, so right now we have a choice that lambda is none of the eigenvalues, one of the eigenvalues you can take this thing.

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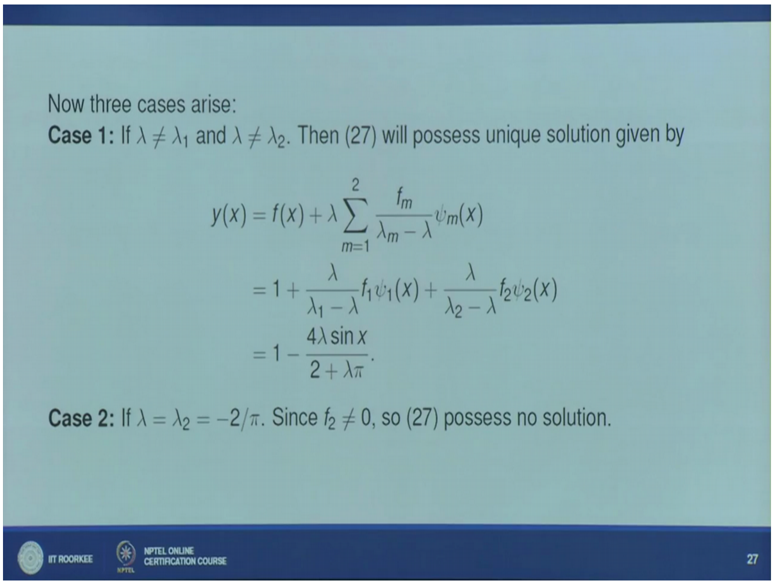

So let us consider the case where lambda is say taking no values taken as lambda 1 and lambda 2. So lambda is neither lambda 1 or lambda 2. So in this case you have y x equal to f of x plus lambda m equal to 1 to 2 f m lambda m minus lambda Psi m x. So writing f 1 and f 2 you can get the solution like this. Now if you look at here look at this 2 plus lambda pi, so if so in case when lambda is neither of these eigenvalues you can write y of x as this.

Now in a second case when lambda is equal to lambda 2, so if you look at lambda equal to lambda 2 here then this is something which creates problem. Now here lambda is lambda 2 means denominator is 0 and if you look at numerator lambda is some fixed quantity, now look at this Psi 2 is nonzero so only thing depend on this f 2 norm. Now if f 2 is again 0, if f 2 is coming out to be 0 then it is kind of 0 by 0 form, right.

And in this case this can take any value. So right now we say that f 2 is nonzero which we have calculated by equation number 34 that f 2 is nonzero. So in this case it is (0) nonzero divided by 0, so this cannot happen. So it means that in this case when lambda is equal to lambda 2 that is minus 2 by pi we do not have any solution. So it means that no solution exists if lambda has value minus 2 by pi.

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Case 3: If $\lambda = \lambda_1 = 2/\pi$. Since $f_1 = 0$, there exists infinitely many solutions given by

$$y(x) = f(x) + A\psi_1(x) + \frac{\lambda}{\lambda_m - \lambda} f_m \psi_m(x)$$
$$= 1 + A\left(\frac{2}{\pi}\right)^{1/2} \cos x + \frac{(2/\pi)}{-(2/\pi) - (2/\pi)} \times 2\left(\frac{2}{\pi}\right) \sin x$$
$$= 1 + C\cos x - \frac{2}{\pi} \sin x.$$

where $C = A\left(\frac{2}{\pi}\right)^{1/2}$ is an arbitrary constant.

setting $\frac{2}{\pi}C_2 = 1$, we obtain $y_2(x) = \sin x$ and the corresponding normalized eigenfunction $\psi_2(x)$ is given by

$$\psi_2(x) = \frac{y_2(x)}{\left[\int_0^\pi \{y_2(x)\}^2 dx\right]^{1/2}} = \frac{\sin x}{\left[\int_0^\pi \sin^2 x dx\right]^{1/2}} = \left(\frac{2}{\pi}\right)^{1/2} \sin x.$$

Now

$$f_1 = \int_0^\pi f(x)\psi_1(x)dx = \int_0^\pi \cos x \left(\frac{2}{\pi}\right)^{1/2} dx = 0 \quad (33)$$

$$\text{and } f_2 = \int_0^\pi f(x)\psi_2(x)dx = \int_0^\pi \sin x \left(\frac{2}{\pi}\right)^{1/2} dx = 2\left(\frac{2}{\pi}\right)^{1/2}. \quad (34)$$

Now three cases arise:

Case 1: If $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$. Then (27) will possess unique solution given by

$$y(x) = f(x) + \lambda \sum_{m=1}^{2} \frac{f_m}{\lambda_m - \lambda} \psi_m(x)$$
$$= 1 + \frac{\lambda}{\lambda_1 - \lambda} f_1 \psi_1(x) + \frac{\lambda}{\lambda_2 - \lambda} f_2 \psi_2(x)$$
$$= 1 - \frac{4\lambda \sin x}{2 + \lambda\pi}.$$

Case 2: If $\lambda = \lambda_2 = -2/\pi$. Since $f_2 \neq 0$, so (27) possess no solution.

Now let us take lambda equal to lambda 1 that is 2 by pi, now in this case f 1 is coming out to be 0 that is calculated here 33, so it means that here your f 1 is 0 denominator is also 0 so it is kind of 0 by 0 form. So this I can take any value A so if this limit exist we call it any value 0 by 0 is you can take any value A you can take any value so here we have infinitely many solution so I can write it A Psi 1 x plus lambda upon lambda m minus lambda f m Psi m x. So here lambda m is your 2, okay.

So here I can write A is any arbitrary constant and this lambda m is lambda 2 so you can put it the value here and you can say that it is 1 plus C cos x minus 2 by pi sin of x where C is this and since A is arbitrary, C is also an arbitrary thing. So in this particular case if lambda is one of the eigenvalue and if the corresponding fourier coefficient is nonzero then we have no solution as we have discussed in case 2.

But if lambda is one of the eigenvalue and the corresponding fourier coefficient f 1 say f 5 whatever is 0, so it means that when lambda takes eigenvalues and the f nonhomogeneous term is orthogonal to the corresponding eigenfunctions in that case we have infinitely many solution and this we can see in a this we have already discussed in the case when we have separable kernel. So in case of separable kernel this case we have already discussed for the example when we have taken kernel K x, t as 1 minus 3 x t.

So here we are considering the same case that lambda equal to lambda 1, so lambda is taking the first eigenvalue and f 1 is 0 means that f is orthogonal to the first eigenfunction corresponding to the eigenvalue here. So if it is the case then we have infinitely many solution and if that lambda is lambda 2 and f is not orthogonal to the second eigenfunction then we do not have any solution. So we have these many cases.

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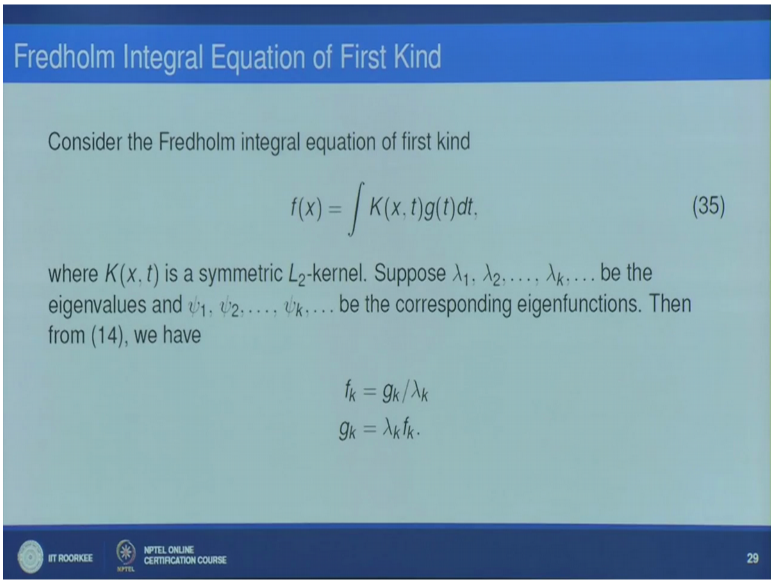

Now with the help of this now let us consider this fredholm integral equation of the first kind. Here again we use Hilbert Schmidt theorem which says that f we have f x equal to K x, t is g t dt then where K and g both are L 2 functions. Then f x can be written as your linear combination of eigenfunctions corresponding to K x, t so that we already so Hilbert Schmidt theorem we are assuming.

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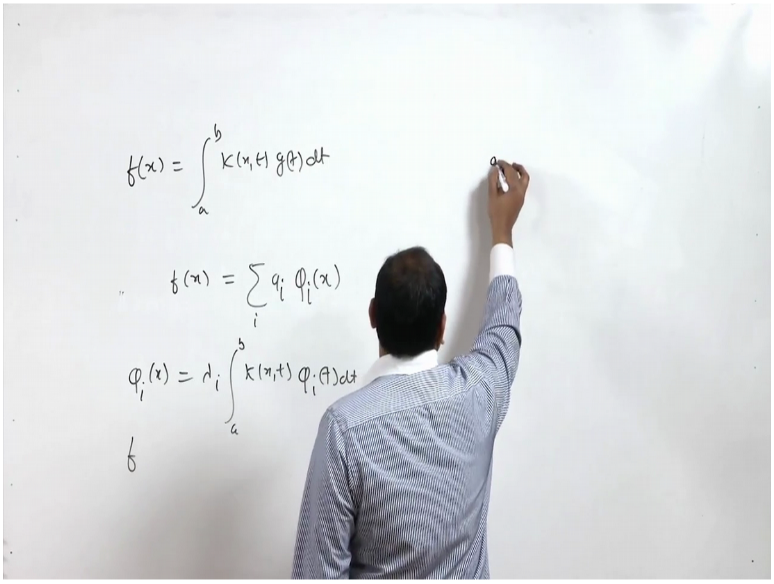

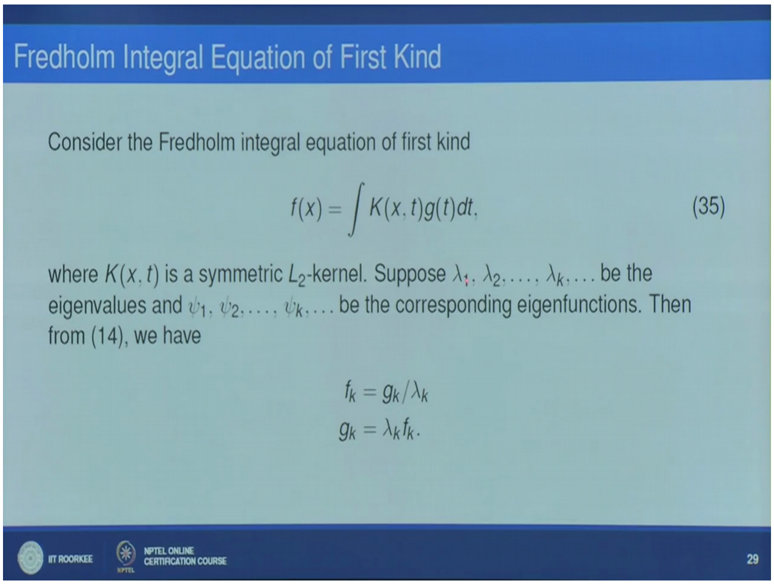

So here let me write it here so here we are using this is the Hilbert Schmidt theorem that f of x is equal to a to b K of x, t g t dt and if K and g both are L 2 function then f can be written as summation of ai ai phi i x and summation over i and where phi x is normalized eigenfunction corresponding to this kernel K. So it means that we are solving this y of x is equal to lambda a to b K of x, t and y x, so here phi i x is the solution of this, so let me write it here phi x lambda i (())(21:39) okay.

And this fourier coefficient here f i is related to fourier coefficient g i like this, so here you can say that your f k is written as g k divided by lambda k, this we discussed earlier. So here I can write g k as lambda k f k. Now if you look at unknown function we are assuming in the first kind we are assuming that this g is the unknown function which we are discussing. So how to construct this unknown function from the fourier coefficient of this unknown function g.

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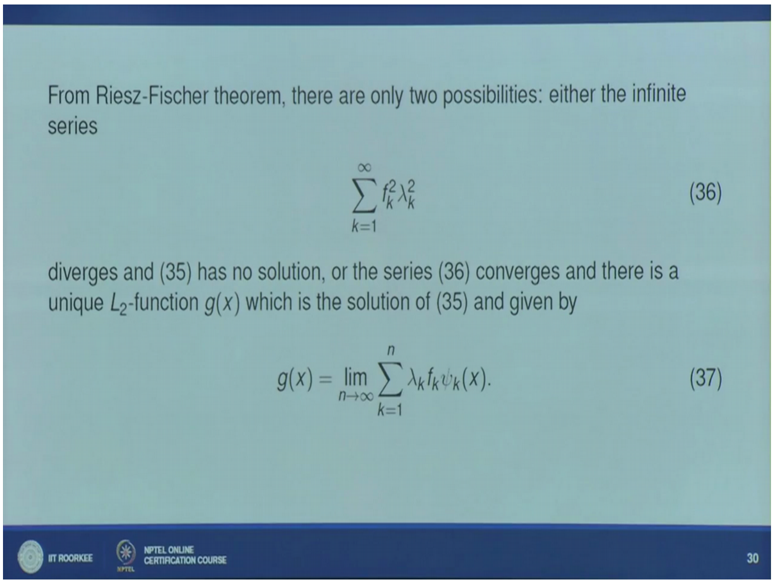

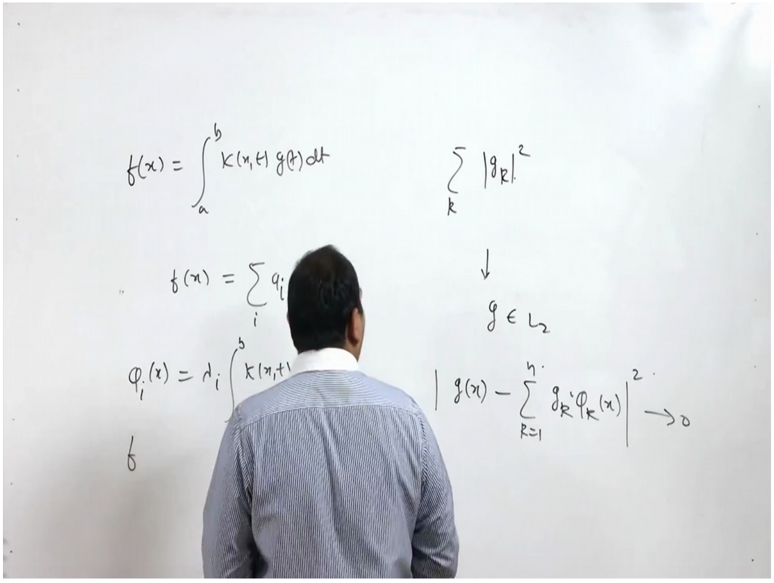

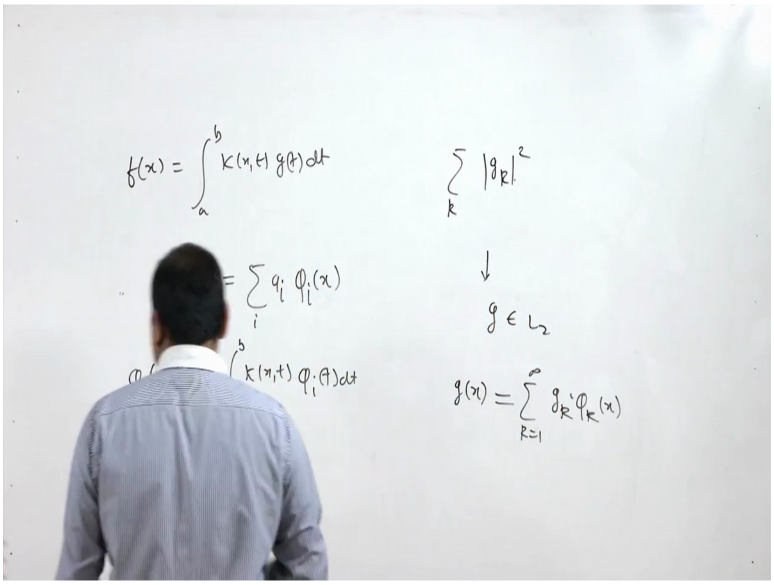

So here we have to recall the Riesz-Fischer theorem, which says that if we have say this g k, right. So if this series is absolutely and uniformly convergent then we have a function g ofcourse in L 2 such that this g has this representation g x is minus the summation this a k sorry this has this representation g k and phi k x K is equal to 1 to n is tending to 0, okay or you can say that that g has this representation, okay.

So so you can write that if this infinite series if this infinite series if it diverges then we do not have any solution, but if it converges then Riesz-Fischer theorem says that your solution g x can be written as this limit of this infinite series, okay and this limit this Riesz-Fischer theorem says so. So it means that everything depend on this infinite series.

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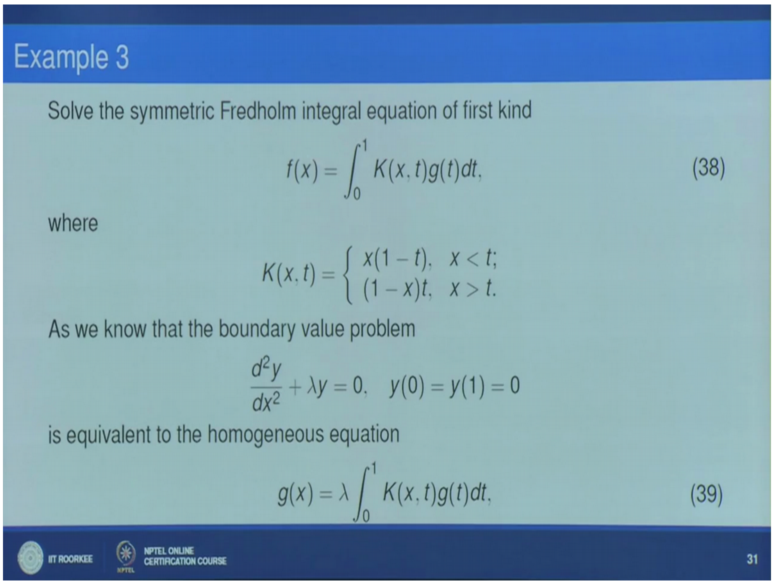

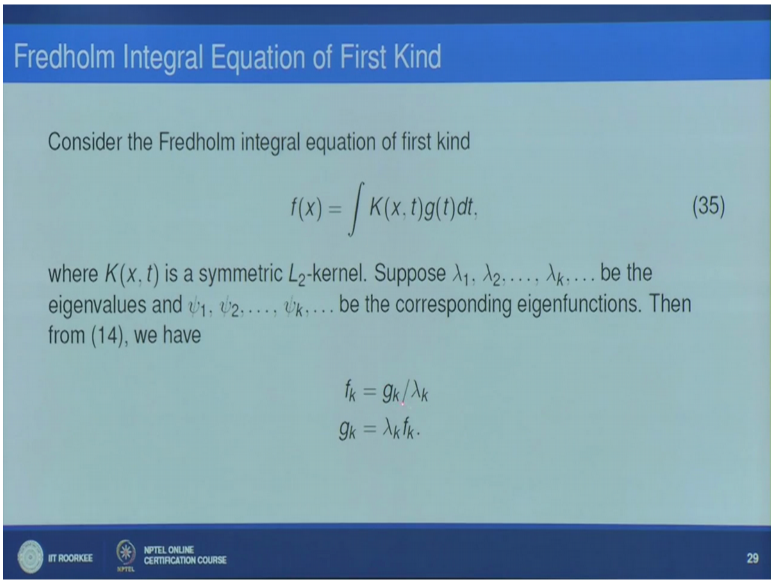

So let us take one simple example here, so let us take this example here f of x equal to 0 to 1 K x, t g t dt where K x, t is given by this, so here it is the fredholm integral equation of the first kind. Then we need to find out that this is falling in this category I need to find out f k and g k all these thing. So eigenvalues we need to find out and eigenfunction we need to find out.

So for that corresponding to this we need to find out the eigenvalue and eigenproblem. So here for that we need to find out we need to solve this where K x, t is given by this this thing.

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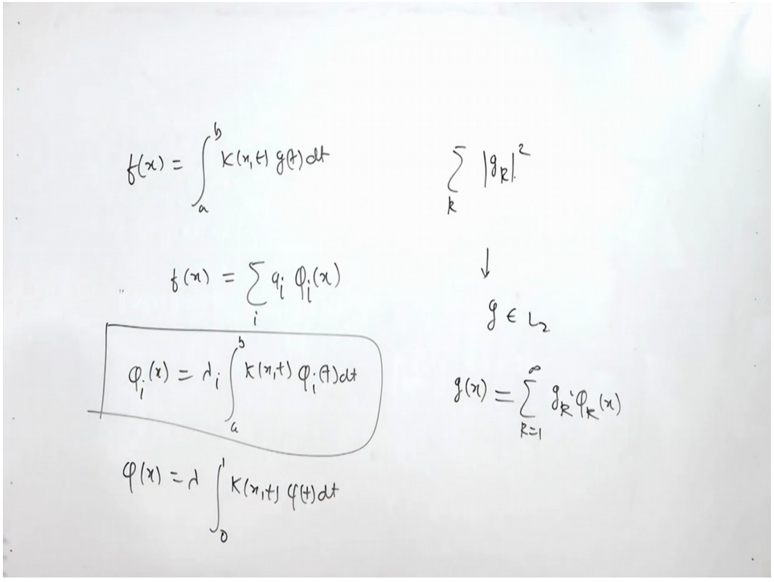

So to find out this we can let me write it here phi of x equal to lambda here it is 0 to 1 K x, t and your phi t dt, okay. So from this you can easily say that by conversion of this integral equation into corresponding boundary value problem we can say that the corresponding boundary value problem is this d 2 y by dx square plus lambda y equal to 0 with the boundary condition that y of 0 equal to y of 1 equal to 0, so that is clear from this equation that y of 0 when you put it here so y of 0 equal to y 1 equal to 0.

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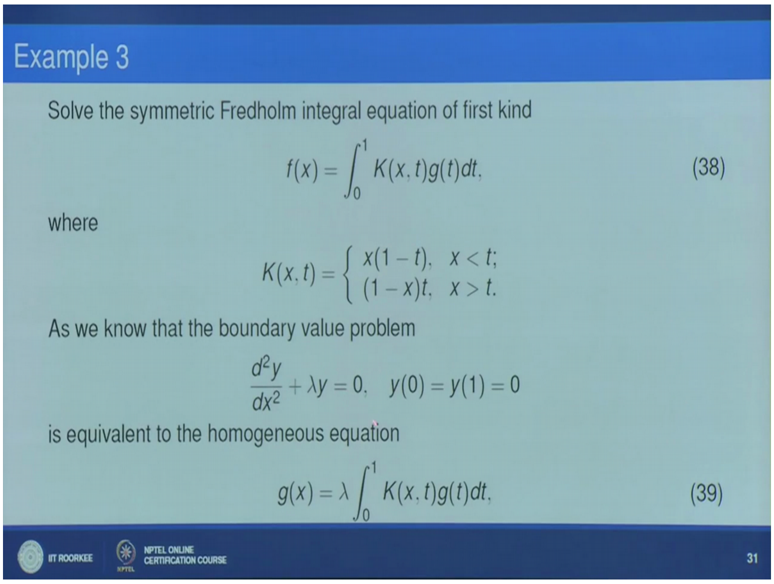

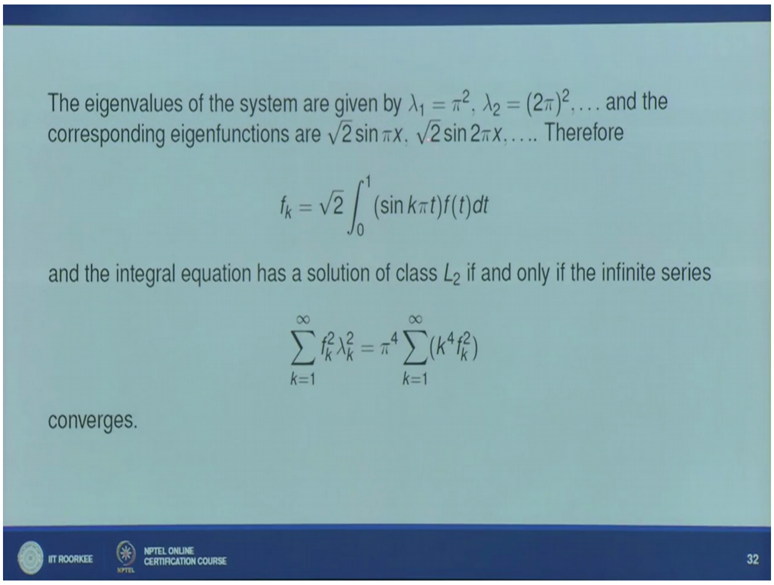

So to solve this we have to solve this particular problem, so I think this is quite simple problem and you can say that in this case your lambda i are simply what pi square and basically it is n pi square n pi whole square and the corresponding eigenfunctions are root 2 sin pi x root 2 sin 2 pi x and you can say that root 2 sin n pi x are your eigenfunction. So it means that corresponding to this kernel look at the eigenvalue problem that is g of x equal lambda 0 to 1 K x, t g t dt.

And you look at that this is equal and to the boundary value problem here, so find out the solution here. So whatever be the solutions here we have this solution here and the values for which we are getting a nontrivial solution here the same will be nontrivial solution here. So it means that solving this problem is equivalent to solving this problem. So here we are finding the lambda for which we have a nontrivial solution.

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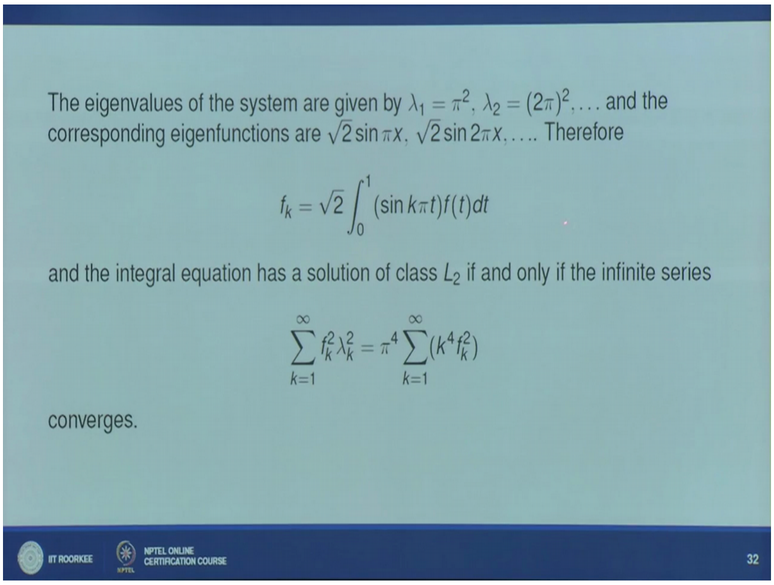

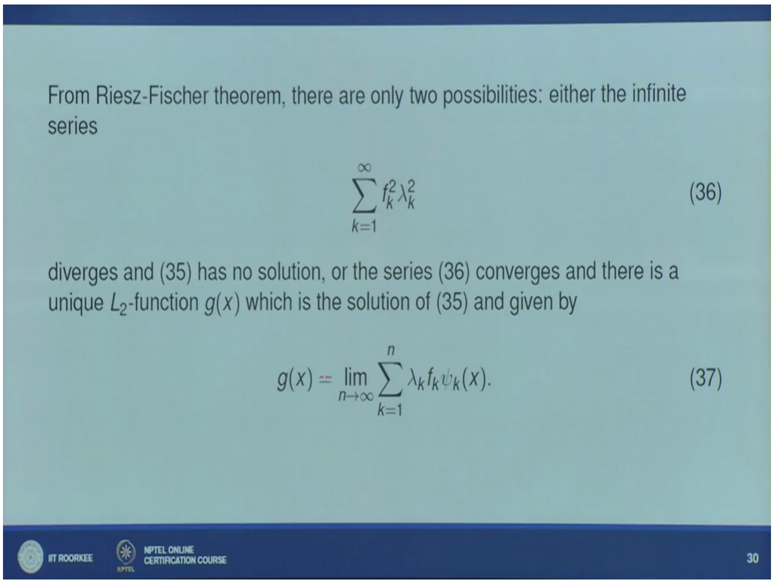

So lambda is coming out to be n pi whole square and the corresponding eigenfunctions are this. So now you can calculate your fourier coefficient f k like this root 2 0 to 1 sin k pi t f t dt. Now once we have f k you can calculate f k and lambda k square. Now this is this quantity pi to power 4 k equal to 1 to infinity k 4 f k square. Now if this series converges then you can write down your g as this limit, so once this lambda f k this series is convergent then you can write yours g x as limit of this series.

So it means that we need to find out f k such that this series is converges one this series is converges you can find out your solution g, is it okay. So we will discuss some more problem in our exercise so thank you for listening us we will meet in next lecture, thank you.