Integral Equations, Calculus of Variations and their Applications
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Lecture No 1
Definition and classification of linear integral equations

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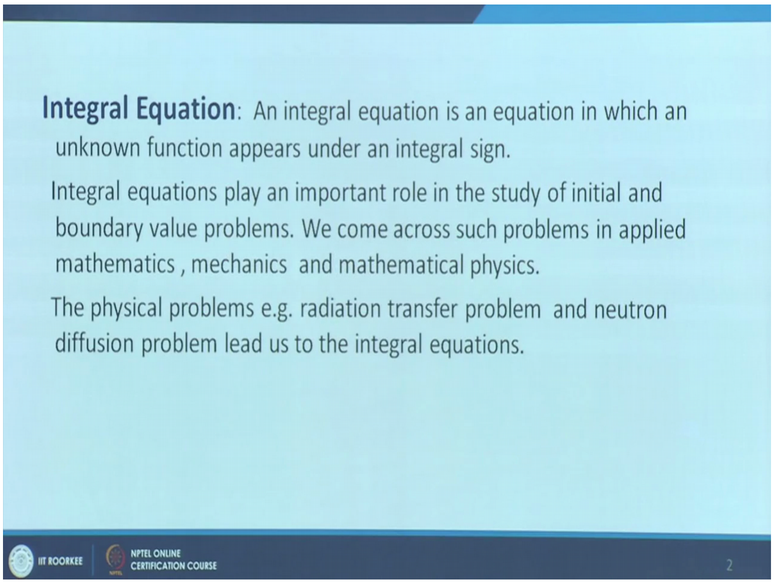

Hello friends I welcome you all to the course on integral equations, calculus of variations and their applications. We shall first discuss the definitions and then we shall discuss the classification of linear integral equation. Let us see what do we mean by an integral equation. An integral equation is an equation in which an unknown function appears under an integral sign. The integral equation play an important role in the study of initial and boundary value problems such problems occur in applied mathematics, mechanics and mathematical physics. The physical problem namely radiation transfer problem and neutron diffusion problems lead us to integral equations. In the present course we shall only the deal with those integral equation in which linear functions of the unknown function are involved.

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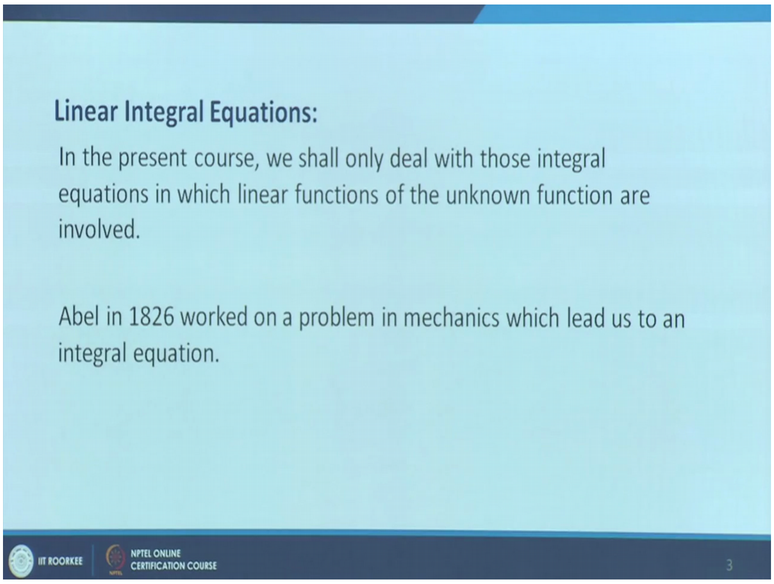

So that means we shall be dealing with only the linear integral equations. Now let us see how the integral equations actually the study of integral equation begin, Abel in 1826 worked on problem in mechanics which lead us to an integral equation.

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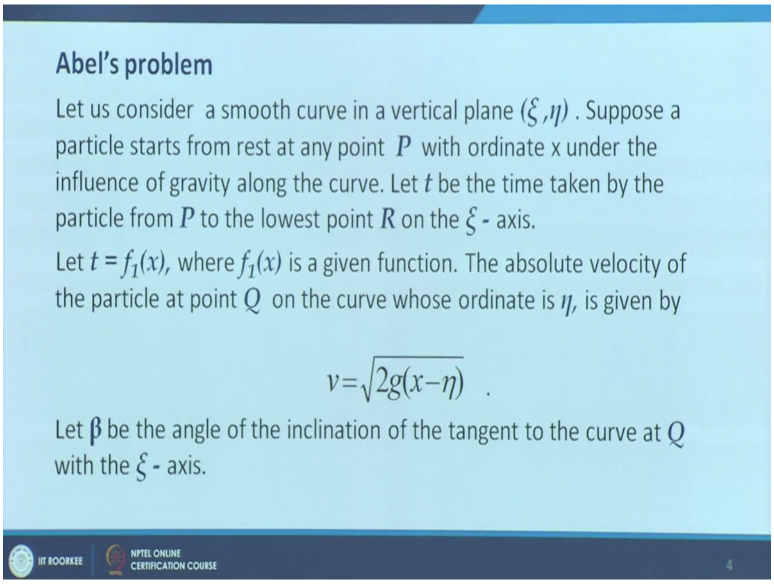

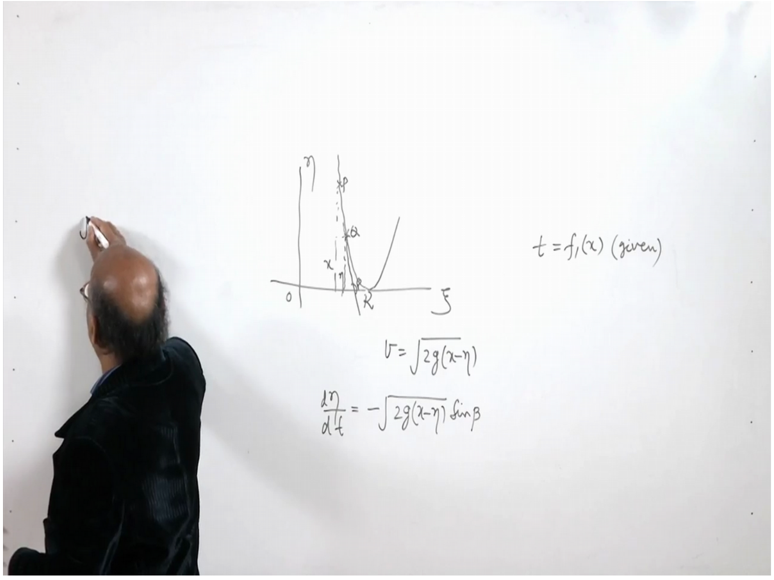

So let us discuss the Abel's problem, let us consider a smooth curve in a vertical plane Xi Eta, let us consider a smooth curve in a vertical plane Xi Eta, suppose a particle starts from rest from any point P with ordinate x, let us say the ordinate of the point P on the curb be x under the influence of gravity along the curve and let t be the time taken by the particle from P to the lowest point R on the Xi axis and let us assume that t is a function of x say f 1 x, where f 1 x is a given function.

The absolute velocity of the particle at point Q on the curve, the absolute velocity of the particle at point Q on the curve whose ordinate is Eta let us say the ordinate of Q point b Eta, then the vertical height PQ will be x minus Eta, so the absolute velocity of the particle at the point Q will then be given by v equal to under root 2 gx that is 2 g times x minus Eta. Now let us say beta be the angle of inclination of the tangent to the curve at the point Q then with the Xi axis, then what do we have? The velocity along the Eta axis d Eta by dt will be minus under root 2g x minus Eta into sin beta.

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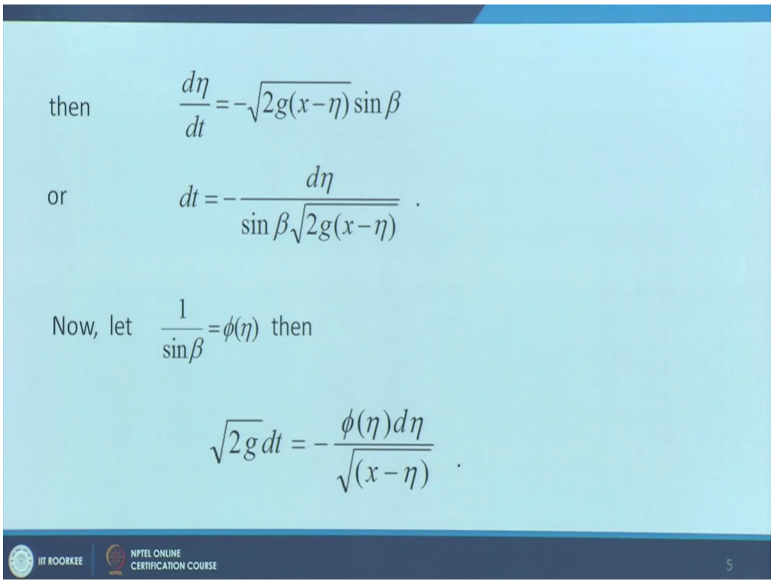

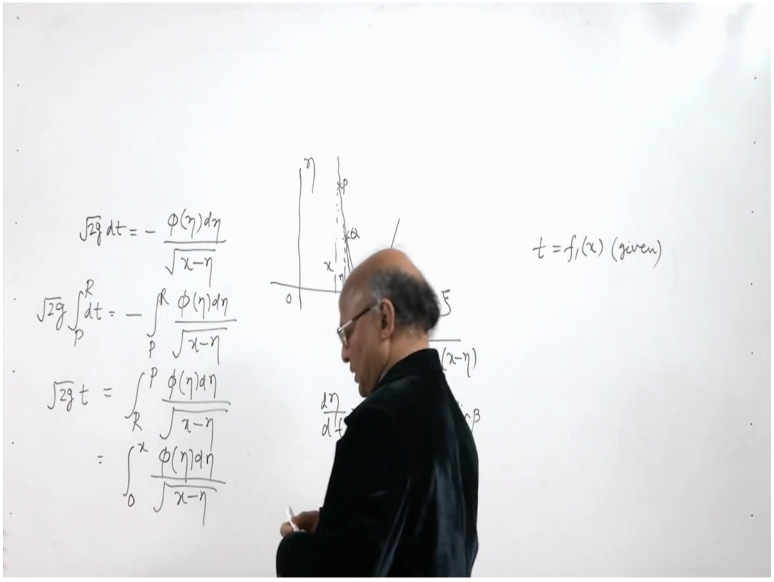

Now we can also write this equation d Eta by dt minus under root 2g x minus Eta into sin beta as dt equal to minus d Eta over sin beta under root 2g x minus Eta. Now let us assume that one over sin beta b equal to phi Eta then we can write under root 2g dt equal to minus phi Eta d eta over under root x minus Eta and integrating okay integrating this equation let us see we have under root 2g dt equal to minus phi Eta d eta divided by under root x minus eta. Now we have assumed that t is the time taken by the particle to slide from the point P to the point R on the Xi axis the lowest point.

So we have to integrate from p to r, from p to r, when we integrate from p to r, we get this okay. Now from p to r the time taken is t so under root 2g into t is equal to. Now because of this negative sign it can also write it from r to p, so we shall have integral r to p, phi eta d eta over under root x minus eta. Now the ordinate of the point r is zero and the ordinate at the point p is x so we have integral 0 to ex, phi eta d eta divided by under root x minus eta.

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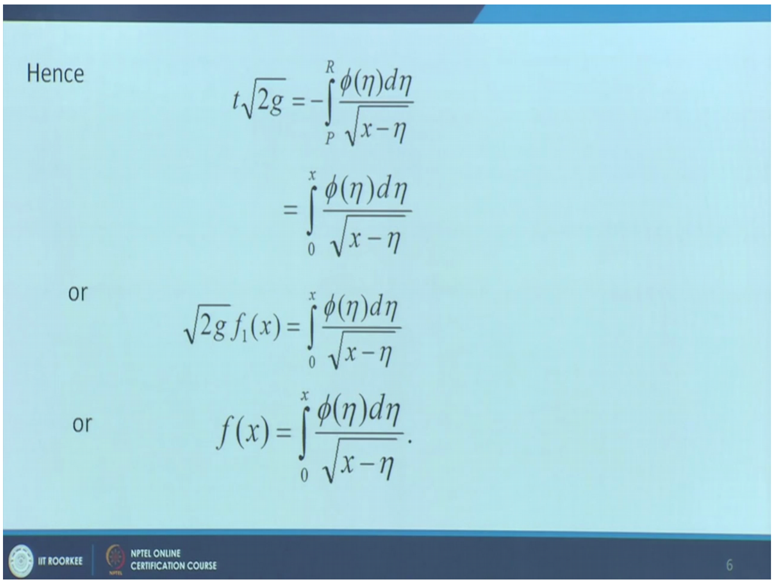

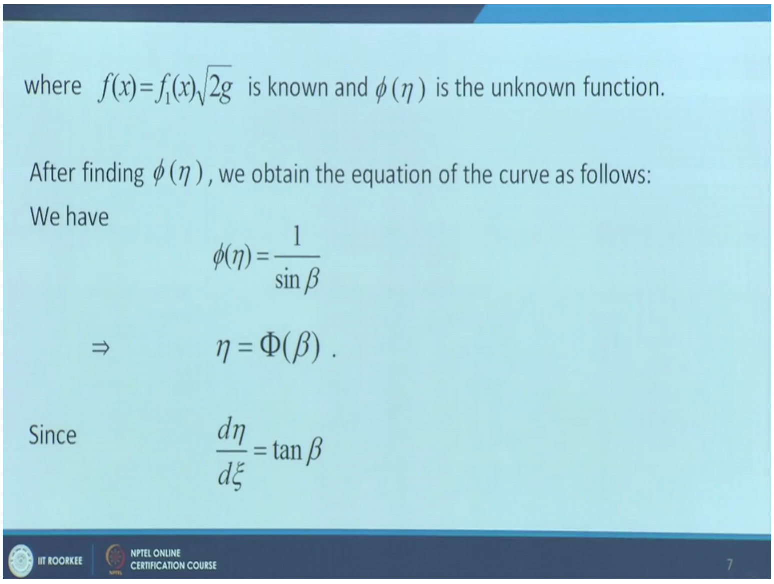

So we get so we get under root t times under root 2g equal to integral 0 to x phi eta, d eta divided by e under root x minus eta. Now t we have assumed to be equal to f 1 x, so we get under root 2g f 1 x equal to integral 0 to x phi eta d eta over under root x minus eta. Let us assumed that under root 2g f 1 x is another function say fx okay so fx we can write as integral 0 to x, phi eta d eta over under root x minus eta and since f 1 x is a known function, so fx equal to f 1 x into root 2 g is also known and phi eta is the unknown function. Now so from the equation fx equal to fx equal to integral 0 to x, phi eta d eta over under root x minus eta.

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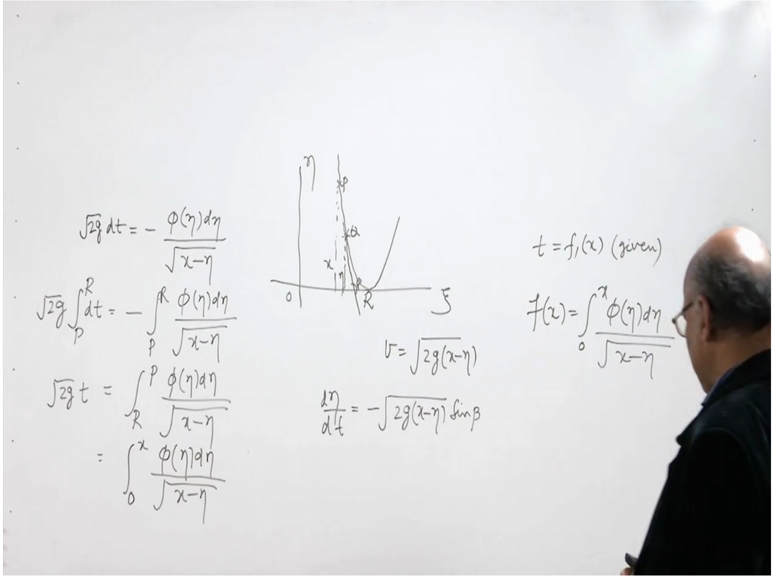

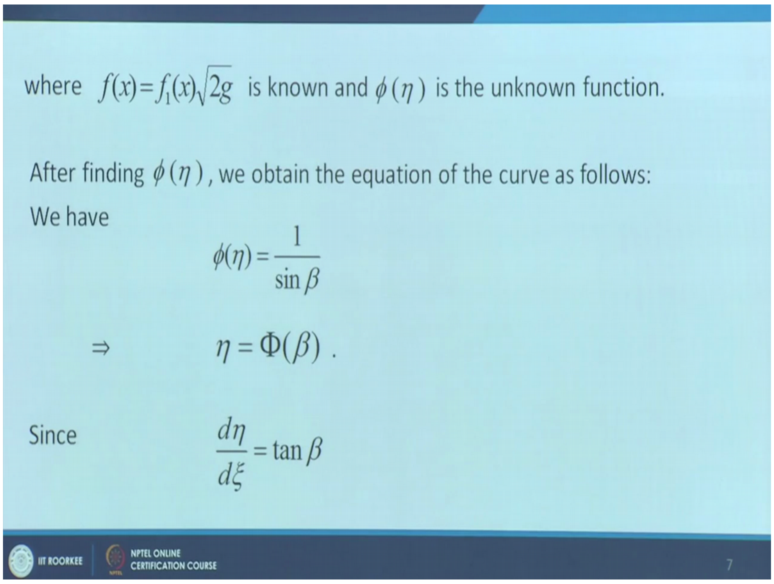

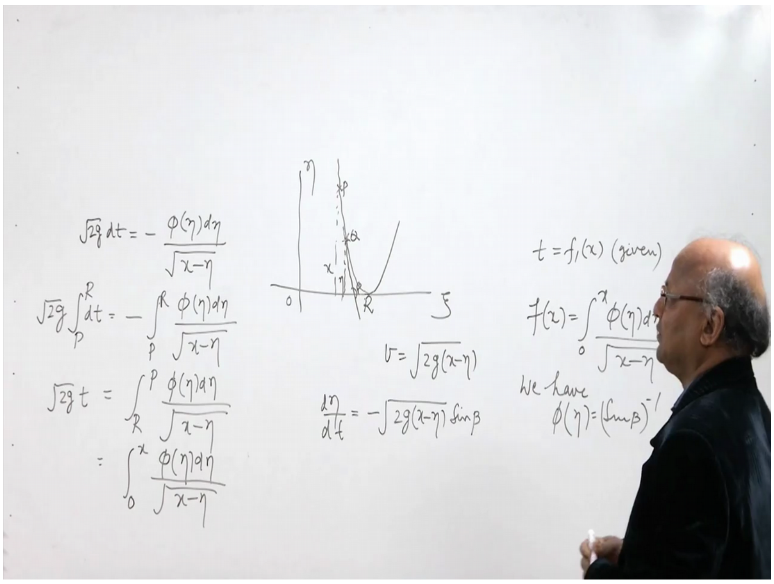

We notice that fx is the known function while phi eta is an unknown function, so it is an integral equation and we shall see later on when we discuss the classification of integral equations, we shall see that this is that volterra integral equation of first kind. So after finding this phi eta this unknown function phi eta, we can write the equation of the curve in the parametric form we have phi eta equal to 1 over sin beta, so phi eta equal to…we have assumed phi eta to be equal to 1 over sin beta, so we can say that eta is a function of beta.

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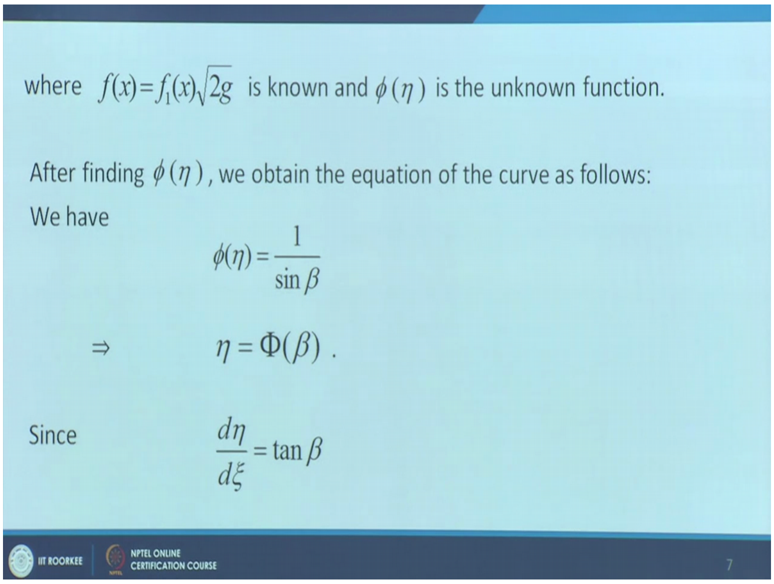

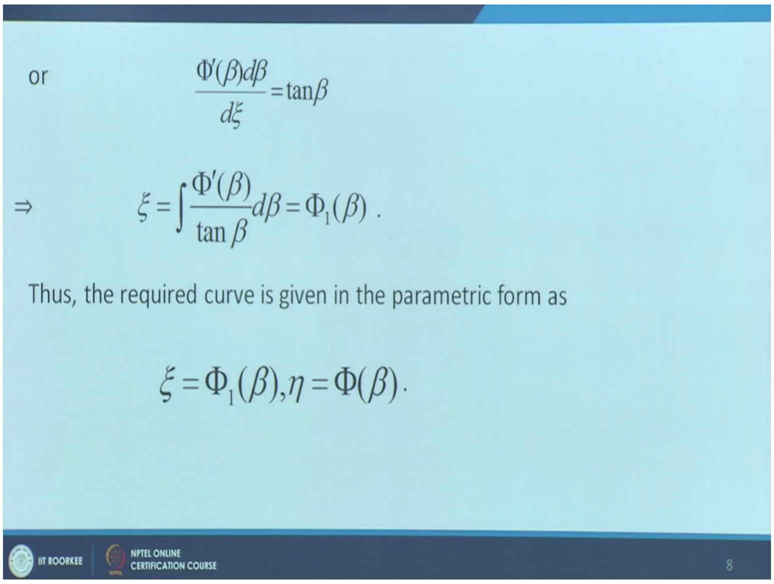

And now beta is the angle made by the tangent at the point Q with the Xi axis, so we can say that d eta over d Xi, d eta over d Xi is equal to tan beta and r we can say that d eta from the equation eta equal to phi beta you can say that d eta is phi dash beta d beta, so we can write phi dash beta d beta over d Xi equal to tan beta are integrating this equation we shall be getting is Xi equal to integral of phi dash beta d beta upon tan beta. Now this will be another function of beta which we can write as capital phi 1 beta so we can thus we get the required curve, the path followed by the particle we can write in the parametric form which is Xi equal to phi 1 beta and eta equal to eta equal to phi eta.

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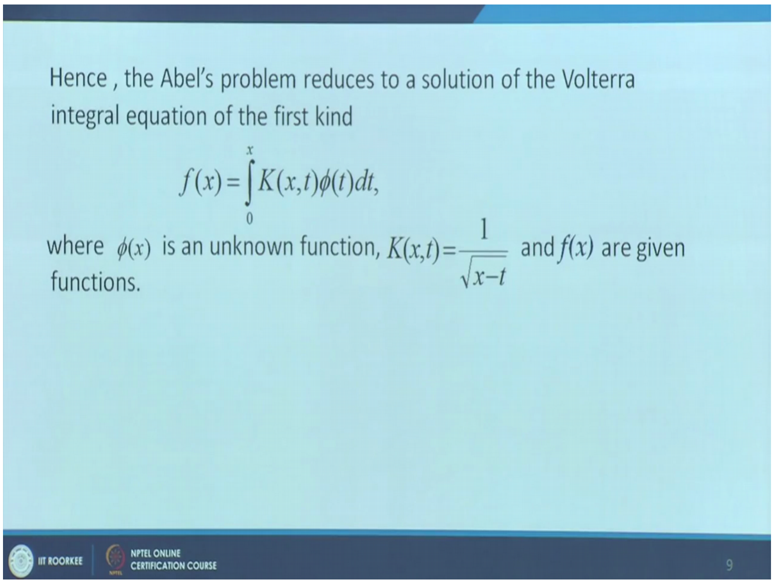

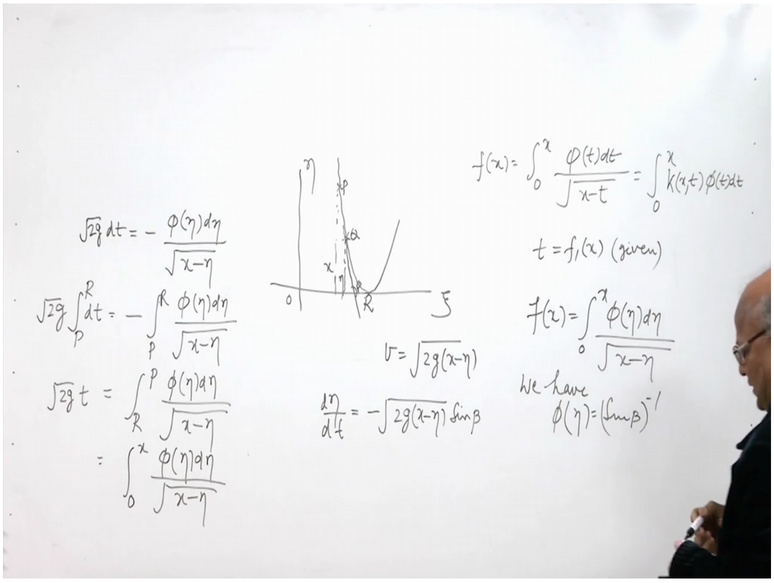

Now this so Abel's problem this Abel's problem which where we are finding the curve followed by the particle can be expressed in the standard form where if you define one over under root x minus t we can write it as also as fx equal to integral 0 to x phi t dt in the parameter t let us write so under root x minus t then this is nothing but integral 0 to x K xt into phi t dt, where K xt is equal to 1 over under root x minus t, so K xt is known to us and fx is also known to us and phi x is an unknown function, so we get a Volterra integral equation of the first kind where K xt and fx are known and phi x is an unknown function. So this is how the integral equations occurred while solving problem in mechanics.

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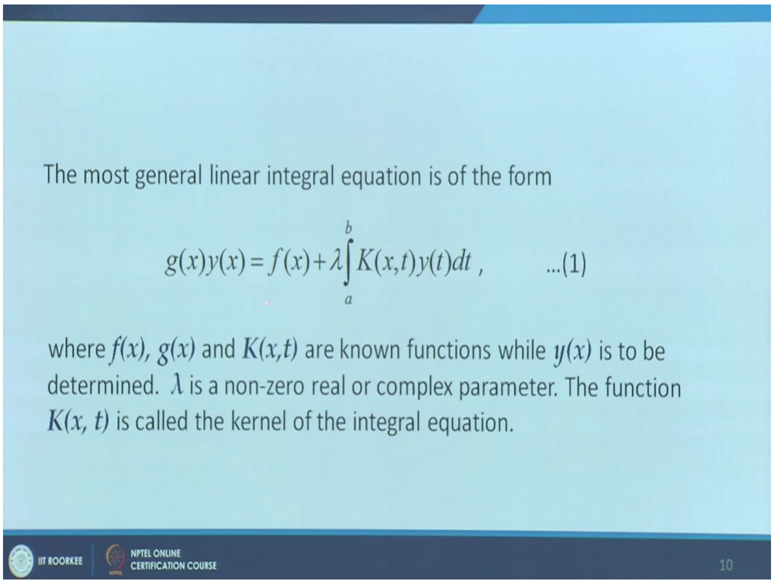

Now let us discuss the most general linear integral equation, the most general linear integral equation is dx into yx equal to fx plus lambda times integral a to b K xt yt dt where fx gx and K xt are known functions of x and t while yx is to be determined. So yx is an unknown function, lambda here is a non-zero real or complex parameter. The function K xt which occurs inside the integral is called the kernel of integral equation. now if it so happens that the limits of integration, in the limits of integration a is a constant while b is equal to x, so then the integral equation will be called as Volterra integral equation.

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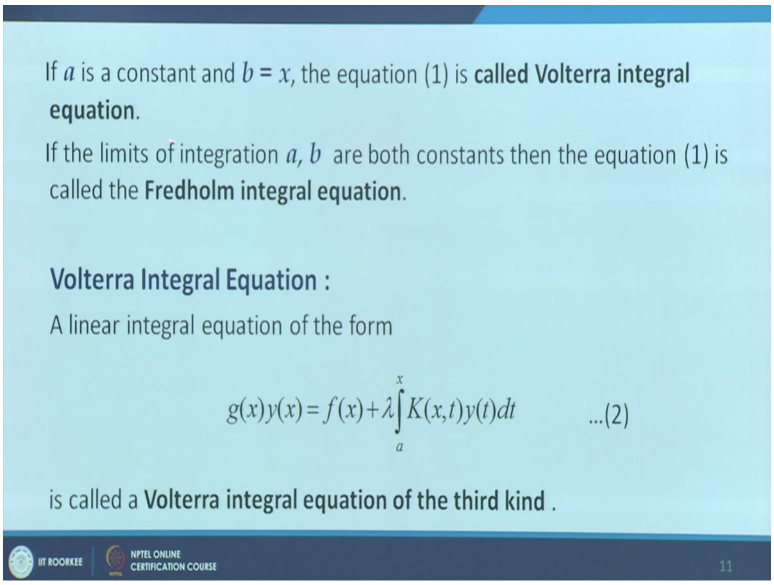

If both the limits of integration a and b are constants then the equation will be called as Fredholm integral equation, so there are 2 kinds of integral equations one is Volterra integral equation another one is Fredholm integral equation. In the case of Volterra integral equation the lower limit is constant while the upper limit is a variable x and in the case of Fredholm integral equation both the limits of integration are constants. Now we will discuss the various kind of Volterra integral equation, so Volterra a linear integral equation of the form gx yx equal to fx plus lambda a to x K xt yt is called as a Volterra integral equation of the third kind.

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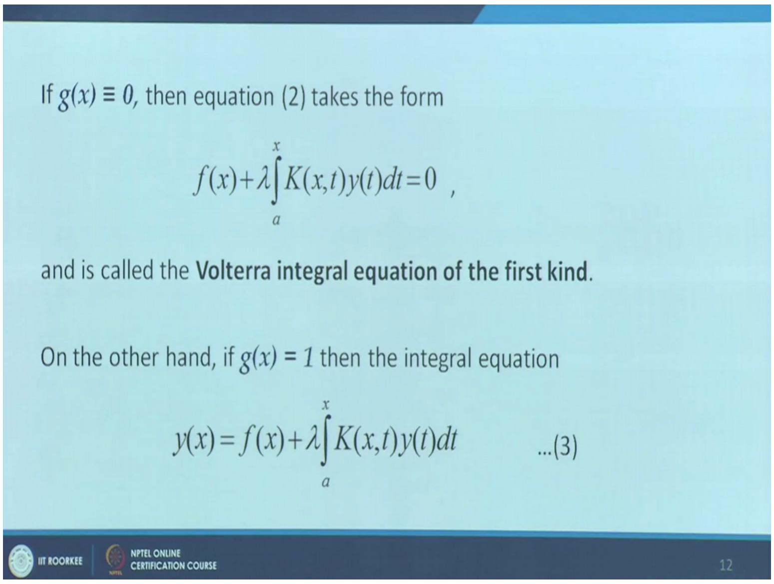

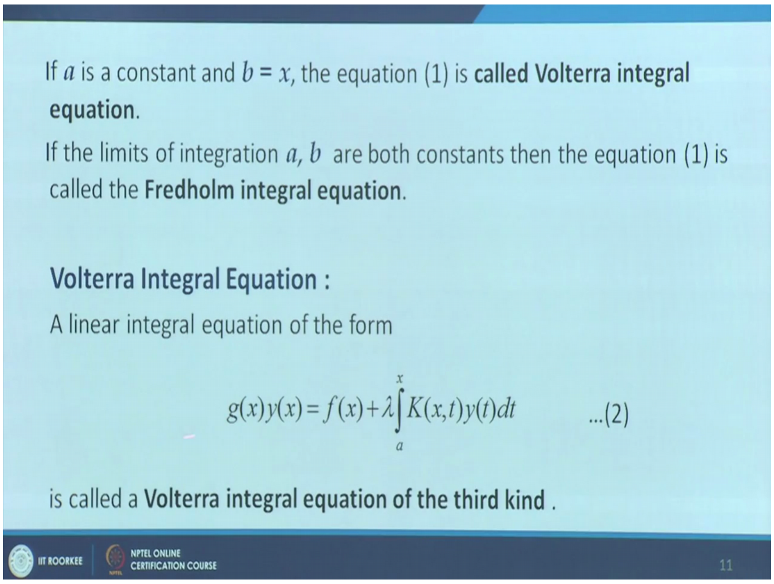

Now in particular suppose gx is equal to 0, gx is identically 0 that is gx is 0 over the domain of g then the equation this equation 2 will take the form fx plus lambda integral a to x K xt yt dt equal to 0. Such an equation is called as the Volterra integral equation of the first kind. fx plus lambda you can also write it as fx equal to minus lambda integral a to x K xt yt dt. The avail problem is of this type and where you can say that lambda is equal to minus 1. So this is Volterra integral equation of the first kind.

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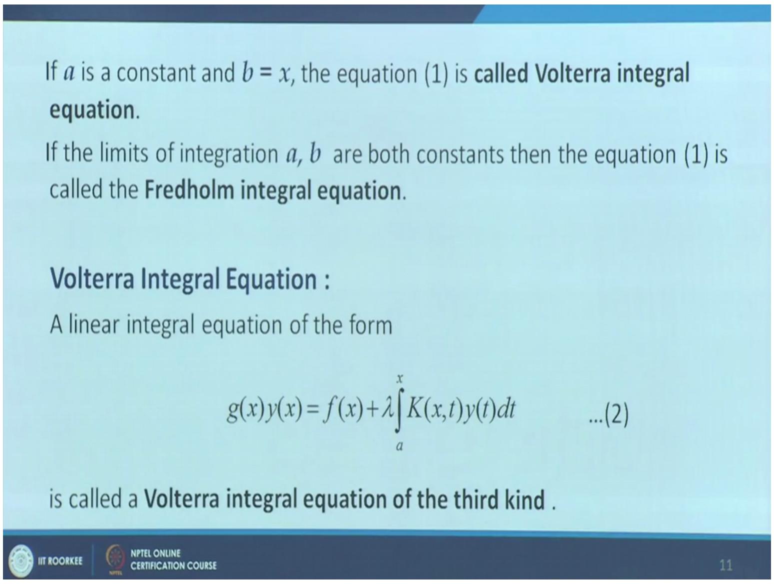

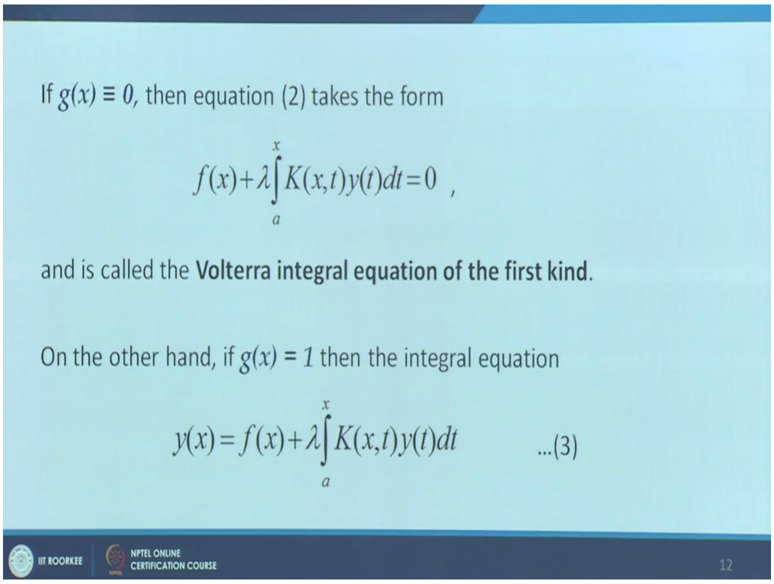

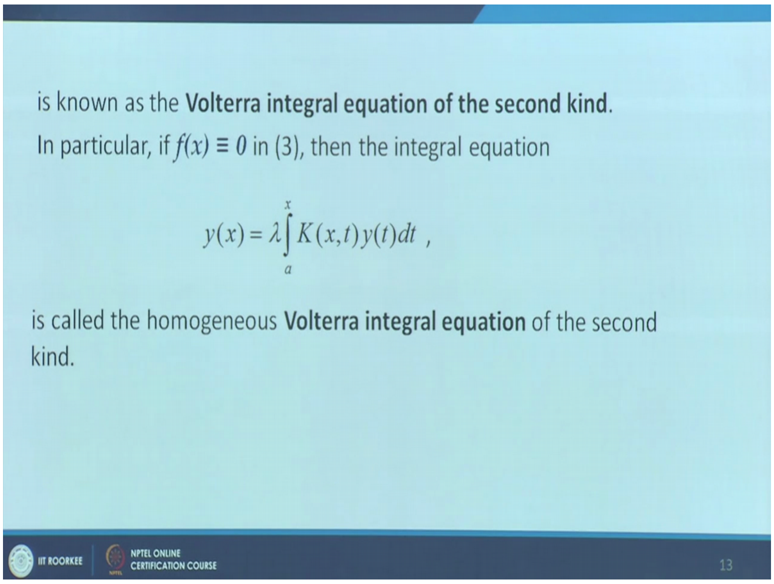

Now on the other hand if gx is equal to 1, if gx is equal to 1 here then we will get YX equal to fx plus lambda integral a to x K xt yt dt. Such an equation is called as Volterra integral equation of the second kind. If fx is identically 0 fx is 0 for all the values of x in the domain of f then what we will get yx is equal to lambda integral a to x K xt yt dt. Such an equation is called as homogeneous Volterra integration equation of the second kind.

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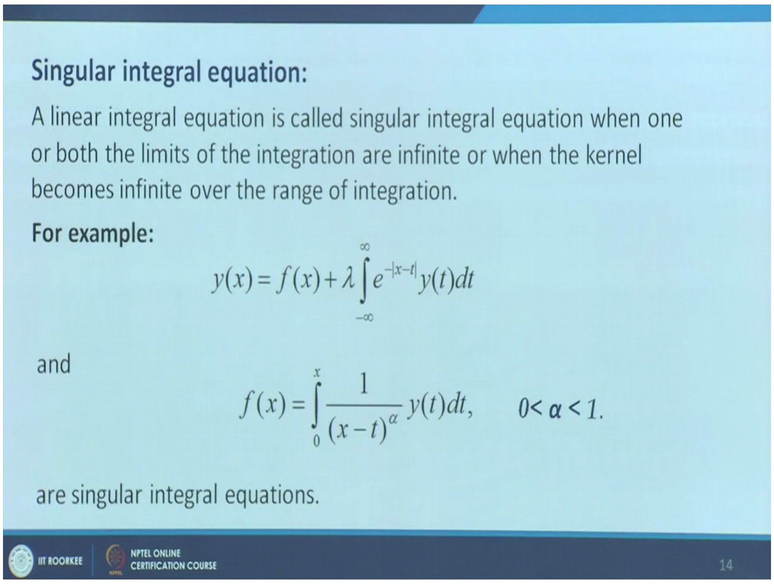

Okay now there occurred similar integral equation also in the literature. A linear integral equation will be called a singular integral equation when 1r both the limits of integration a and b are infinite are when the kernel K xt becomes infinite over the range of integration for example let us consider yx equal to fx plus lambda integral over minus infinity into infinity E to the power minus mode of x minus t by tdt.

Here you can see that both the limits of integration are infinite, so it is singular integral equation and in this case the other example we can consider where the limits of integration are finite 0 and x but the integrant one over x minus t to the power alpha okay this becomes in finite over the range of integration kernel one over x minus t to the power alpha becomes in finite when t becomes equal to x. So fx equal to integral 0 to x 1 over x minus t to the power alpha yt dt where alpha is between 0 and 1 are both singular integral equations.

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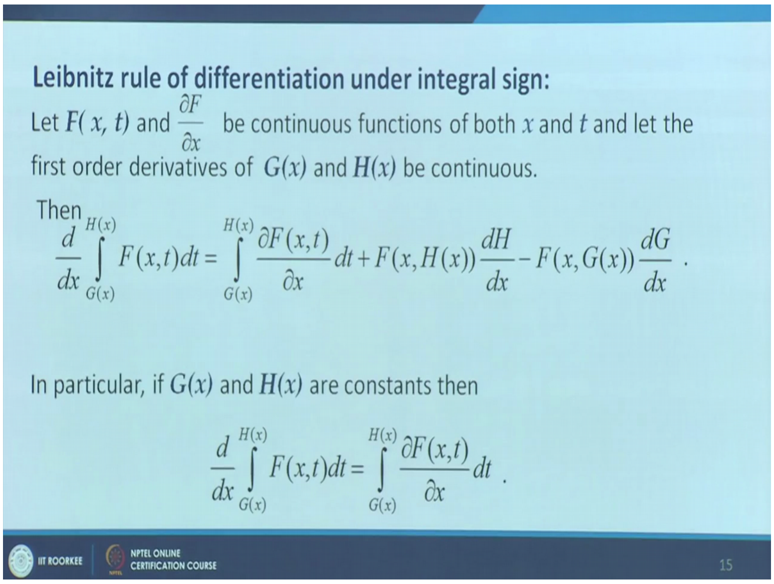

So now let us discuss Leibnitz rule of differentiation under integral sign, we shall need this result when we discuss the when we convert the integral equation to differential equation. So let us say let Fxt and its partial derivative with respect to x be continuous function of both x and t and the first order derivatives of Gx and Hx are continuous functions. then d over dx of integral Gx to Hx Fxt dt is equal to integral Gx to Hx partial derivative of Fxt with respect to x into dt and then Fx Hx.

So in Fxt we put t as Hx the upper limit and differentiate the upper limit with respect to x to get dH by dx and then subtract fxt at the lower limit t is equal to Gx in to the derivative of g with respect to x. So this is a well-known formula the Leibnitz rule it is called Leibnitz rule of differentiation under integral sign. Now in particular if Gx and Hx are constant functions so then what will happen dH by dx and dG by dx will be equal to 0 so this formula will then reduce to the derivative of integral Gx to Hx fxt dt with respect to x is equal to integral Gx to Hx partial derivative of fxt with respect to x into dt.

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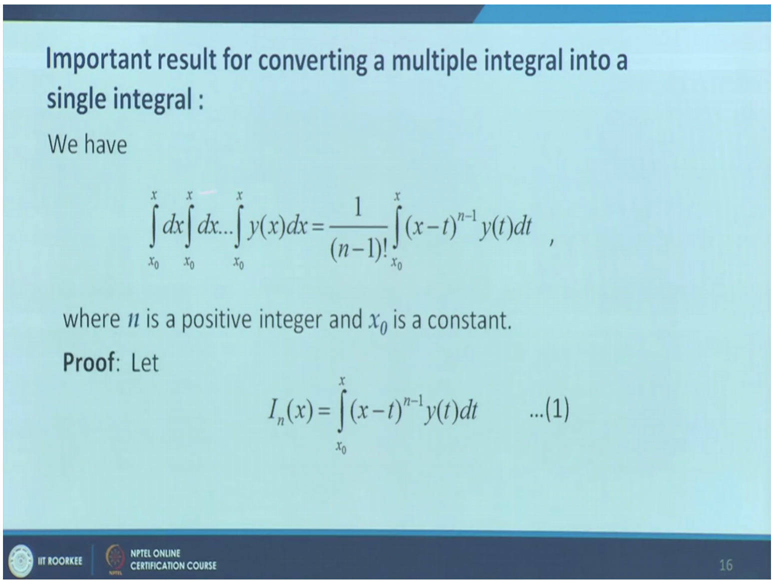

Okay so now we are going to discuss an important result of by using this result we can convert a multiple integral into a single integrals so you can see here that we have n integrals here, the limits of integration are x naught 2x and we are integrating a function of x which is y x, so when you integrate y x end times with respect to x what you do is you get a 1 over n minus 1 factorial integral x naught 2x x minus t raise to the power n minus 1 yt dt. So the n integrals can be converted into a single integral. Now here n is a positive integer and x naught is a constant. So what you do we let us discuss the proof of this let us consider I n x equal to integral x naught 2x x minus t to the power n minus 1 yt dt.

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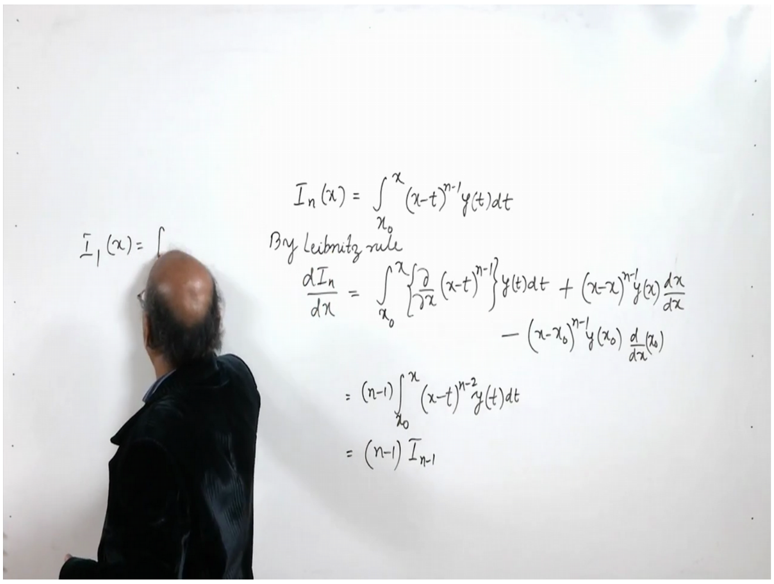

So let us define I n x equal to integral x naught 2x x minus t raise to the power n minus 1 yt dt. Now let us use Leibnitz rule to differentiate this, so by Leibnitz rule dI n over dx that is differentiated with respect to x, so here lower limit is x naught which is constant upper limit is variable x, so we have we had differentiating with respect to x, so we get x naught 2x partial derivative of x minus t race to the power n minus 1partial derivative with respect to x into yt dt plus the value of the integrant at the upper limit so we get x minus x raise the power n minus 1 by x into derivative of the upper limit with respect to x minus the value of the integrant at the lower limit.

So x minus x naught race to the power n minus 1 y x naught into d over dx of x naught. Now what we get is then when you differentiate x minus t race to the power n minus 1 with respect to x partially you get n minus 1 into x minus t race to the power n minus 2. So n minus 1 being a constant can be written outside and we get x naught 2x x minus t race to the power n minus 2 into yt dt. Now here x minus x is 0, so this is 0 into yx and dx by dx is one, so we get 0 and here we see that x naught is a constant so it is derivative with respect to x is 0, so these 2 terms vanish and we get n minus 1 x naught integral over x naught 2x x minus t race to the power n minus 2 into yt dt. So we can write it as n minus 1 times In minus 1 where n is strictly greater than 1. Now from this equation, from the equation I n x equal to integral x naught 2x x minus t raised to power n minus 1 into yt dt.

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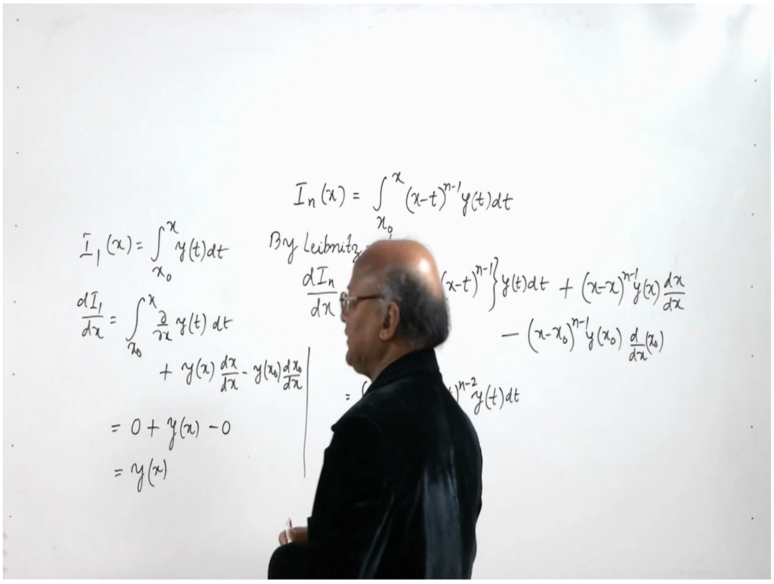

If you put n equal to 1 what we get? I 1 x equal to integral x naught 2x yt dt. So if we again applied the Leibnitz rule here then we will get dI 1 by DX equal to integral x naught 2x delta 1 partial derivative of yt with respect to x into DT plus yx into dx by dx minus y x naught into dx naught over dx. Derivative of x with respect to x is 1 so we get yx here, here y is a function of t only, so when you differentiate it with respect to x partially it is 0, so what we get is 0 plus here we get yx and here the derivative of x naught with respect to x is 0, so we get 0 so we get yx here so when we differentiate when we differentiate even with respect to x what we get is yx.

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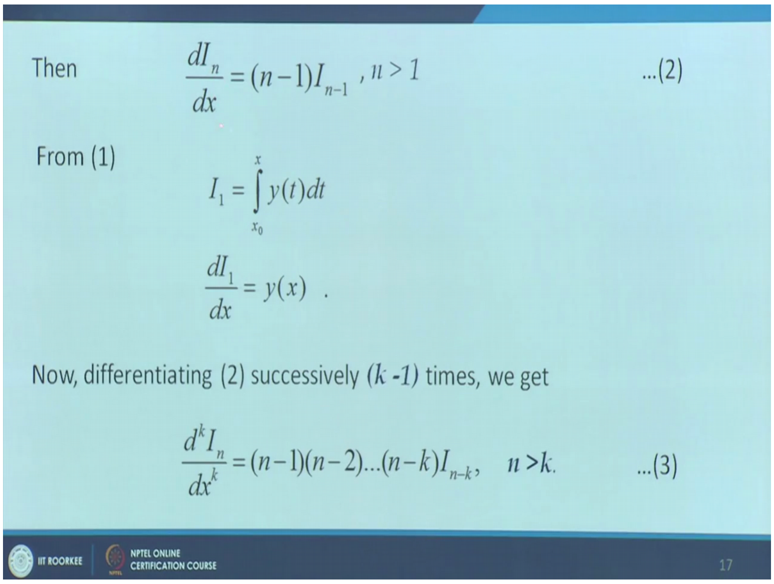

Now let us differentiate this equation dI n over dx again with respect to x, so we will get d square I n over dx square equal to n minus 1 dI n minus 1 and dI n minus 1 if you use it repeatedly dI n minus 1 is n minus 2 I n minus 2 and so on so when we differentiated k minus 1 times what we will get is d k I n over dx k equal to n minus 1 into n minus 2 into n minus k into I n minus k where n is greater than k. Now in particular if we put k equal to n minus 1 here we will get dn minus 1I n over d xn minus 1 equal to n minus 1 factorial into I 1. So this is what we get when we put k equal to n minus 1.

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We get

$$\frac{d^{n-1}I_n}{dx^{n-1}} = (n-1)!\,I_1 \qquad \ldots(4)$$

Again, differentiating

$$\frac{d^n I_n}{dx^n} = (n-1)!\,y(x) \qquad \ldots(5)$$

From (1), (3) and (4), it follows that $I_n(x)$ and its first $(n-1)$ derivatives vanish at $x = x_0$.

Important result for converting a multiple integral into a single integral :

We have

$$\int_{x_0}^{x} dx \int_{x_0}^{x} dx \ldots \int_{x_0}^{x} y(x)dx = \frac{1}{(n-1)!}\int_{x_0}^{x} (x-t)^{n-1} y(t)dt \;,$$

where n is a positive integer and x_0 is a constant.

Proof: Let

$$I_n(x) = \int_{x_0}^{x} (x-t)^{n-1} y(t)dt \qquad \ldots(1)$$

Then

$$\frac{dI_n}{dx} = (n-1)I_{n-1} \;,\; n > 1 \qquad \ldots(2)$$

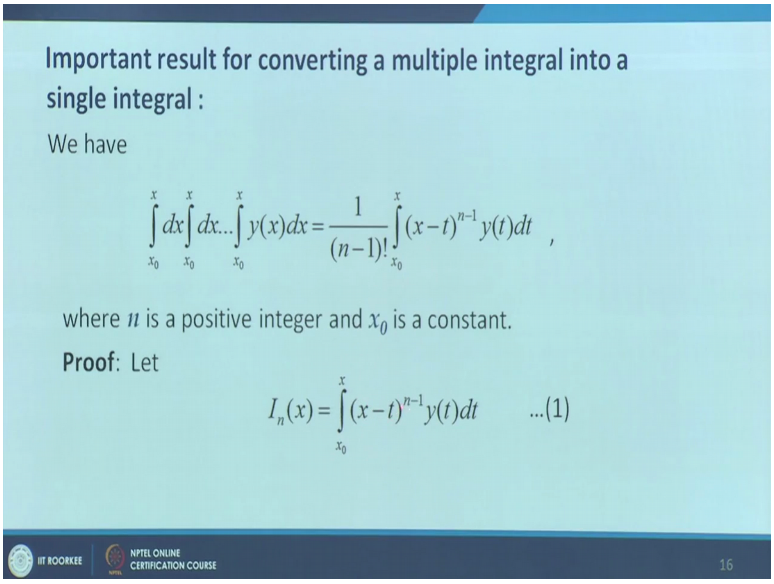
From (1)

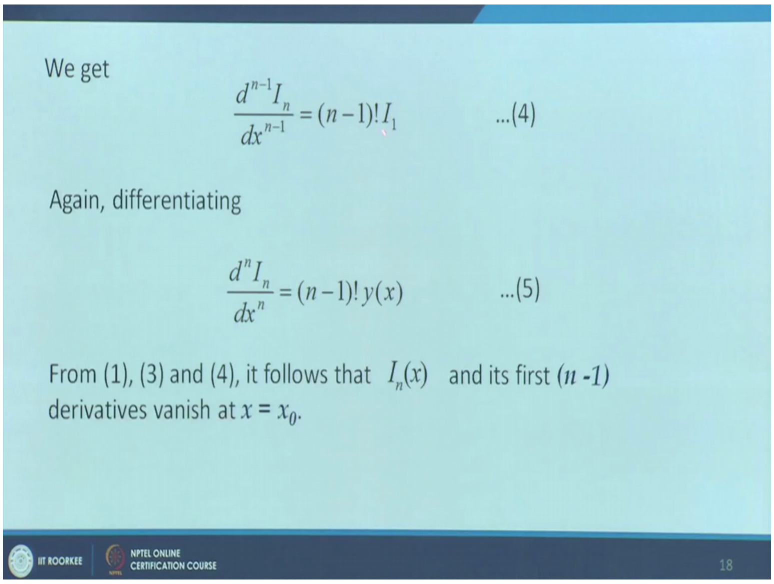
$$I_1 = \int_{x_0}^{x} y(t)dt$$

$$\frac{dI_1}{dx} = y(x) \;.$$

Now, differentiating (2) successively $(k-1)$ times, we get

$$\frac{d^k I_n}{dx^k} = (n-1)(n-2)\ldots(n-k)I_{n-k}, \quad n > k. \qquad \ldots(3)$$





Now if you differentiate this equation again what you get is the n I n over dx n equal to n minus 1 factorial into dI 1 over dx and dI 1 over dx is yx so we will get nth derivative of I n with x as n minus 1 factorial into yx. Now let us look at the questions 1, 3 and 4 if you look at equation 1 here okay when you put x equal to x naught, the limits of integration are same lower end upper limits so at x equal to x naught I n x is 0 and if you look at equation 3 here, if you look at equation 3 here so this equation dk I n over dx k okay is n minus 1 I n minus 2, n minus k I n minus k.

I n I n minus k will also be 0 by our notation In minus k is integral x naught 2x x minus t race to the power n minus k minus 1 yt dt, so I n minus k and where n is greater than k I n minus k is also 0 at x equal x naught. So the dk I n over dx k will be 0 when n is greater than k at x equal to x naught. This is 3 and then 4 here also I1 0x equal to x naught so dn minus 1 I n over dx n minus 1 will be 0. So from 1, 3 and 4 it follows that I n x and its first and minus 1 derivatives vanish at x equal to x naught.

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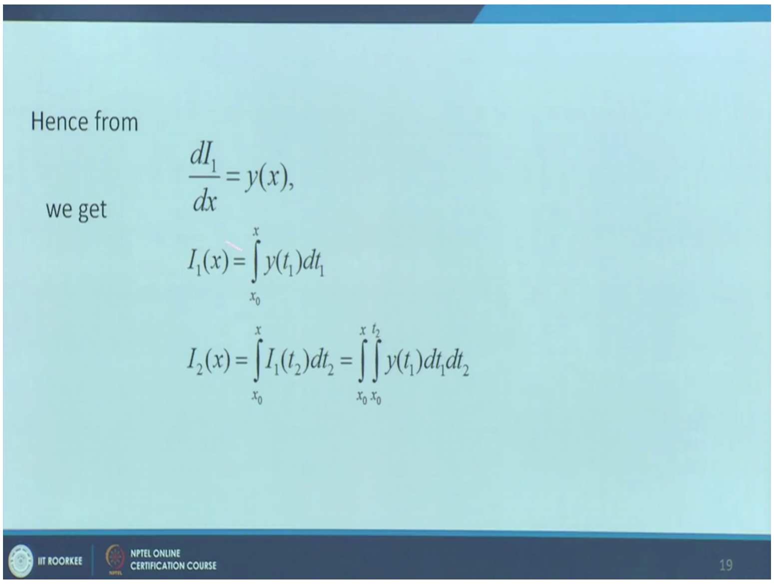

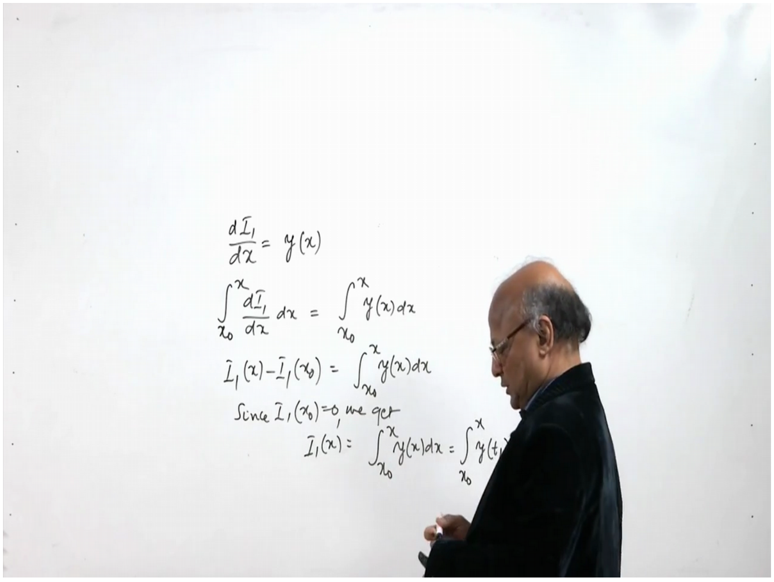

So let us now go back to the question dI 1 over dx, so dI 1 over dx we can see when we integrate it with respect to x we get I 1 x equal to integral over x naught 2x yt1 dt 1 this is because at x equal to x naught I 1 x is 0, so we can also see it like this. We have dI 1 over dx equal to yx, so when we integrate over x naught 2x what we get is this okay and dI1 over dx when we integrate we get xI 1 so I 1 x minus I 1 x naught equal to integral x naught 2x yx dx but I 1 at x naught is equal to 0, so since I 1 x naught is equal to 0 we get I 1 x equal to integral x naught 2x yx dx. This can also be written since this is a definite integral we can also write it as x naught 2x y t 1 dt 1. So I 1 x is integral x naught 2x y t 1 d t 1.

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We get

$$\frac{d^{n-1}I_n}{dx^{n-1}} = (n-1)!\,I_1 \qquad \ldots(4)$$

Again, differentiating

$$\frac{d^nI_n}{dx^n} = (n-1)!\,y(x) \qquad \ldots(5)$$

From (1), (3) and (4), it follows that $I_n(x)$ and its first $(n-1)$ derivatives vanish at $x = x_0$.

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Hence from

$$\frac{dI_1}{dx} = y(x),$$

we get

$$I_1(x) = \int_{x_0}^{x} y(t_1)\,dt_1$$

$$I_2(x) = \int_{x_0}^{x} I_1(t_2)\,dt_2 = \int_{x_0}^{x}\int_{x_0}^{t_2} y(t_1)\,dt_1 dt_2$$

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Proceeding in this manner, we get

$$I_n(x) = (n-1)!\int_{x_0}^{x}\int_{x_0}^{t_n}\cdots\int_{x_0}^{t_3}\int_{x_0}^{t_2} y(t_1)\,dt_1 dt_2 \ldots dt_n$$

$$\int_{x_0}^{x}\int_{x_0}^{t_n}\cdots\int_{x_0}^{t_3}\int_{x_0}^{t_2} y(t_1)\,dt_1 dt_2 \ldots dt_n = \frac{1}{(n-1)!}\int_{x_0}^{x}(x-t)^{n-1}\,y(t)\,dt.$$

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Similarly we can get I 2 x equal to because we see if you look at this we have d square, let us put an equal to 2 here so we will get d 2 I 2 over dx 2 equal to n minus 1 by x, so when we integrate it twice we will get I 2 and integrating twice means when we integrate once we get I 1 x when we integrate again we get I 2 x so I 2 x equal to I x naught 2x I 1 t 2 d 2 which can be written as when you put the value of I 1 t 2 from here you get x naught 2x x naught to t 2 yt 1 dt 1 dt 2 we can proceed in this manner and integrated n times integrate and time so we will get I n x equal to n minus 1 factorial x naught 2 x x naught 2 t n x naught 2 t 3 x naught t 2 y t 1 dt 1 dt 2 dt n so this is equal to 1 over n minus 1 factorial I n x we have assumed as x naught 2x x minus t raise the power n minus 1 yt dt. So this is how we prove that multiple integral and integral of the function yx can be converted into a single integral so this result we shall be using when we deal with initial value problems, so this is what I have to say in this lecture thank you very much for your attention.