PROBABILITY THEORY FOR DATA SCIENCE

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Week - 09

Lecture - 46

Numerical Examples on Conditional Mean and Variance

We have already worked on similar calculations. Let us compute this again. We are focusing on the region where x is between 1/2 and 0, and y is also between 1/2 and 0. Essentially, this corresponds to a specific region on the graph. Let us identify this region.

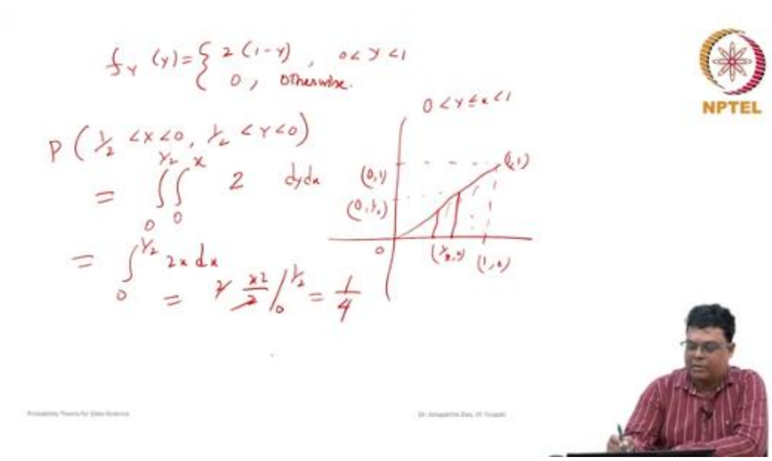
Suppose this is the area under consideration. We need to work with the probability density function and compute it using dx and dy. It is important to note that the function is not non-zero everywhere. To clarify, we define some key points: one point is at (1, 0), another at (0, 1), a third at (0, 1) again, and finally at (1, 1). Additionally, there is a line where $y = x$.

The region where the function is non-zero is bounded by these constraints. Next, we need to calculate the probability in this non-zero region. Specifically, the function is non-zero in the interval where $y \geq 0$, $y \leq x$, and $x \leq 1$. This defines the area we are working with. As you can observe, the boundaries meet here, forming the region where $y \geq 0$, $y \leq x$, and $x \leq 1$.

To compute this, we integrate over the region. First, we integrate with respect to y, followed by x. The order of integration can be switched because the final result will exist within the defined region. In this case, the limits for y are from 0 to x, and x ranges from 0 to 1/2. With these limits established, the computation becomes straightforward. The range for y is determined by the line $y = x$.

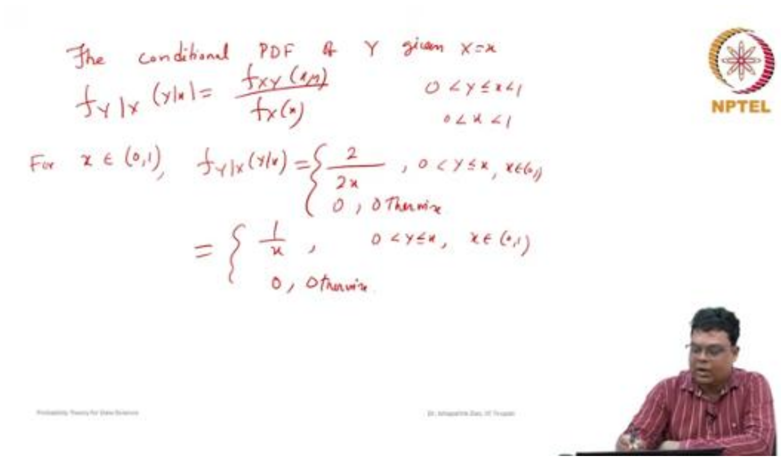
After calculating the integral, the constant value of 2 is factored in, and the computation simplifies further. Evaluating over the specified range, the final probability is found to be 1/4. So, this probability of 1/4 was something we computed earlier as well. I remember that the value came out to be 1/4. To find the conditional probability density function of Y given

X, we first need to compute the conditional mean. For this, we must derive the conditional probability density function.



The conditional probability density function of Y given X is defined as the joint probability density function divided by the marginal probability density function of X. The joint probability density function is non-zero in the region where y is between 0 and x, and x is between 0 and 1. Similarly, the marginal probability density function of X, denoted as f(X), is non-zero when x is between 0 and 1. For a given value of x in the open interval (0, 1), the conditional probability density function of Y given X, denoted as f(Y | X), can be expressed as the joint probability density function divided by f(X). Here, the joint probability density function is equal to 2, and the marginal probability density function f(X) has been found to be 2x.

Thus, f(Y | X) = 2 / 2x, which simplifies to 1 / x. This is valid for the range of y, which is between 0 and x, and only when x is between 0 and 1. Otherwise, the function is zero. In summary, the conditional probability density function of Y given X is uniform over the interval where y is between 0 and x, for a given value of x in the range from 0 to 1. Outside of these bounds, the value is zero. Outside of these bounds, the value is zero.



Hence, the conditional mean of Y given X = x. The definition of the conditional mean of Y given X is the expected value of Y given X, which is expressed as:

E(Y | X = x) = integral from 0 to x of [y * f(Y | X)] dy

For a particular value of x, the distribution of Y given X = x is uniform over the interval 0 ≤ y ≤ x, with the density function f(Y | X) = 1/x. Since the function is non-zero only in the range 0 ≤ y ≤ x, we evaluate the integral over this range:

E(Y | X = x) = integral from 0 to x of [y * (1/x)] dy

Since 1/x is constant for a given x, it can be factored out of the integral:

E(Y | X = x) = (1/x) * integral from 0 to x of y dy

The integral of y is y^2 / 2, so we have:

E(Y | X = x) = (1/x) * [y^2 / 2] from 0 to x

Substituting the limits:

E(Y | X = x) = (1/x) * [(x^2 / 2) - 0]

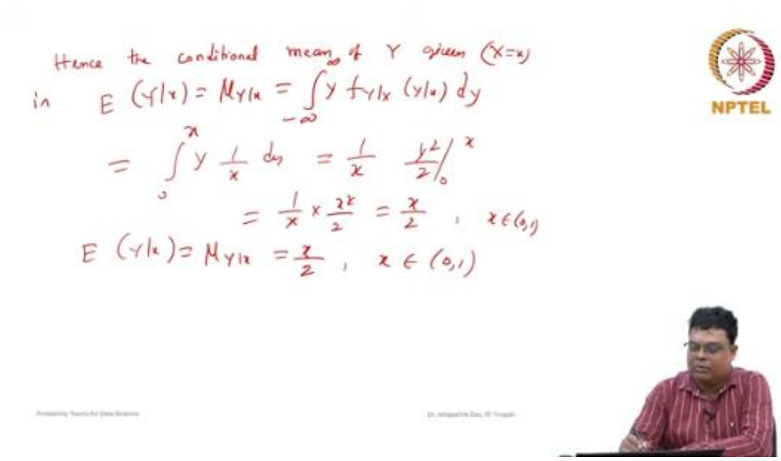
Simplifying:

E(Y | X = x) = x / 2

Therefore, the conditional mean of Y given X, denoted as μ_Y | X, is:

μ_Y | X = x / 2

This is a function of x and depends on the value of x in the range $0 \leq x \leq 1$. Similarly, we can proceed to find the conditional mean of X given Y.



By definition, the conditional mean of X given Y is the expected value of X given Y. This is denoted as E(X | Y) and is defined as:

E(X | Y = y) = integral from y to 1 of [x * f(X | Y)] dx

First, we need to determine the conditional probability density function of X given Y. This is defined as:

f(X | Y = y) = f(X, Y) / f(Y)

The joint probability density function f(X, Y) is non-zero in the interval where $x \geq y$ and $x \leq 1$, while y is between 0 and 1. The marginal probability density function of Y is given as:

f(Y) = 2(1 - y), for $0 \leq y \leq 1$.

Thus, the conditional probability density function of X given Y, denoted as f(X | Y), is:

f(X | Y = y) = 2 / [2(1 - y)] = 1 / (1 - y)

This holds for x in the range $y \leq x \leq 1$, and for y being a specific value between 0 and 1. Otherwise, the conditional probability density function is zero.

In this case, the conditional probability density function of X given Y is uniform over the interval from y to 1 for a specific value of y, where $0 \leq y \leq 1$.

To compute the conditional mean of X given Y, we integrate:

E(X | Y = y) = integral from y to 1 of [x * (1 / (1 - y))] dx

Factoring out the constant (1 / (1 - y)):

E(X | Y = y) = (1 / (1 - y)) * integral from y to 1 of x dx

The integral of x is x^2 / 2, so:

E(X | Y = y) = (1 / (1 - y)) * [(x^2 / 2)] from y to 1

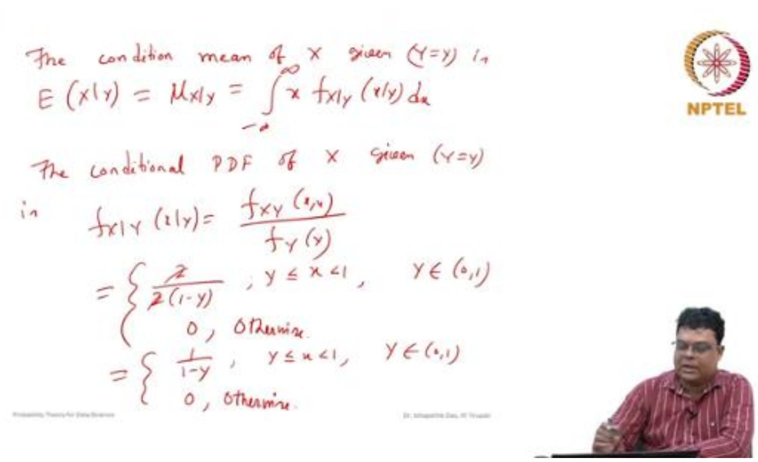
Substituting the limits:

E(X | Y = y) = (1 / (1 - y)) * [(1^2 / 2) - (y^2 / 2)]

Simplifying:

E(X | Y = y) = (1 / (1 - y)) * [(1 / 2) - (y^2 / 2)]

Thus, the conditional mean of X given Y is a function of y, and it depends on its value in the range $0 \leq y \leq 1$.



The conditional mean, or expected value of X given Y, is denoted as μ(X | Y). The conditional mean is calculated as the integral of x multiplied by the conditional probability density function of X given Y, integrated over the possible values of x. For this specific case, the conditional probability density function of X given Y is non-zero for x between y and 1, and zero otherwise, for a particular value of y between 0 and 1. Therefore, the integral limits for x are from y to 1. The conditional probability density function is 1 / (1 - y), so the integral becomes the product of x and 1 / (1 - y), integrated over the range from y to 1. Since 1 / (1 - y) is a constant with respect to x, it can be factored out.

The integral then becomes:

(1 / (1 - y)) * integral from y to 1 of x dx

The integral of x is $x^2 / 2$, so evaluating the integral:

(1 / (1 - y)) * [(x^2 / 2)] from y to 1

Substituting the limits:

(1 / (1 - y)) * [(1^2 / 2) - (y^2 / 2)]

Simplifying:

(1 / (1 - y)) * [(1 / 2) - (y^2 / 2)]

Now, simplifying the expression further:

= (1 / (1 - y)) * [(1 - y^2) / 2]

Distribute (1 / (1 - y)):

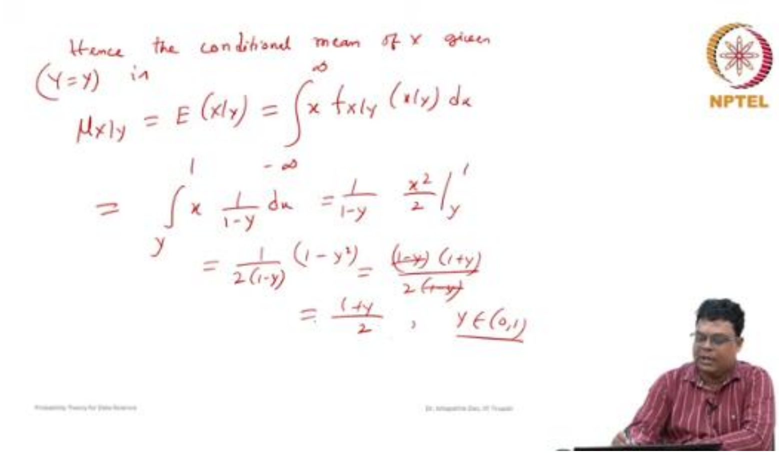
= (1 - y^2) / 2(1 - y)

Now, canceling out (1 - y), the final result is:

(1 + y) / 2

Thus, the conditional mean of X given Y, denoted as $\mu(X \mid Y)$, is a function of y and is equal to (1 + y) / 2 for $0 \leq y \leq 1$. This shows that the conditional mean is indeed a function of y.

In this particular question, the variance was not asked.



Similarly, we can also compute the conditional variance. For example, if you are asked to find the conditional variance of Y given X = 0.5, we can apply the definition of conditional variance. The conditional variance of Y given X = 0.5 is the difference between the expected value of Y and the conditional mean of Y given X = 0.5, squared.

We already computed the conditional mean of Y given X = 0.5. From the earlier results, we know that the conditional mean is 1/2 when X = 0.5.

This means the conditional mean of Y given X = 0.5 is 0.25. To find the conditional variance, we need to calculate the expected value of the squared difference between Y and its conditional mean, which is 0.25. This involves subtracting the conditional mean from Y and squaring the result.

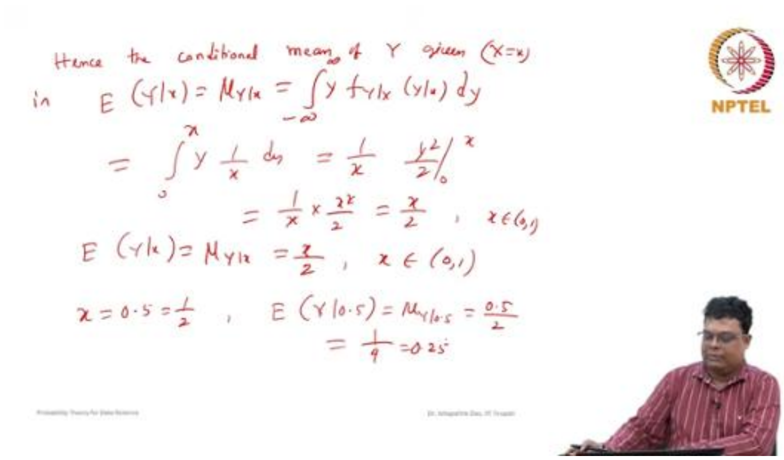
To compute the conditional variance when X = 0.5, we use the definition of variance. The formula for conditional variance is:

$$Var(Y \mid X = 0.5) = E[(Y - \mu(Y \mid X = 0.5))^2]$$

This is the expected value of $(Y - 0.25)^2$ when X = 0.5. To calculate this, you need to compute the integral of $(Y - 0.25)^2$ multiplied by the conditional probability density function of Y given X = 0.5, integrated over the possible values of Y.

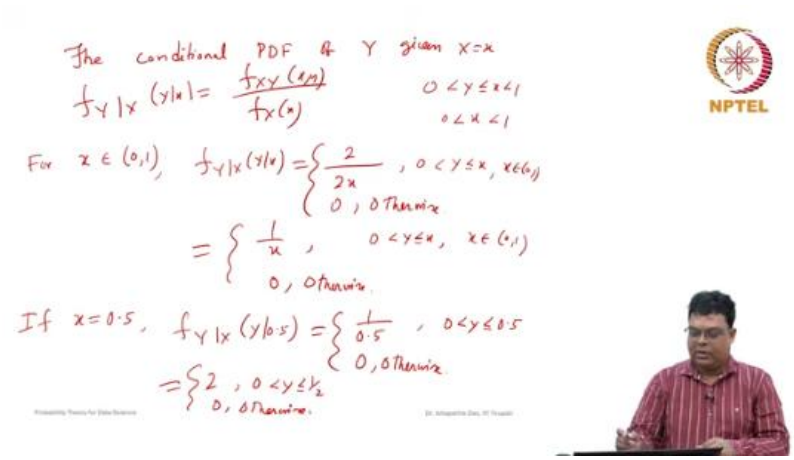
So the conditional variance of Y given X = 0.5 can be computed by integrating:

$$E[(Y - 0.25)^2] = \int (Y - 0.25)^2 * f(Y \mid X = 0.5)\, dY$$

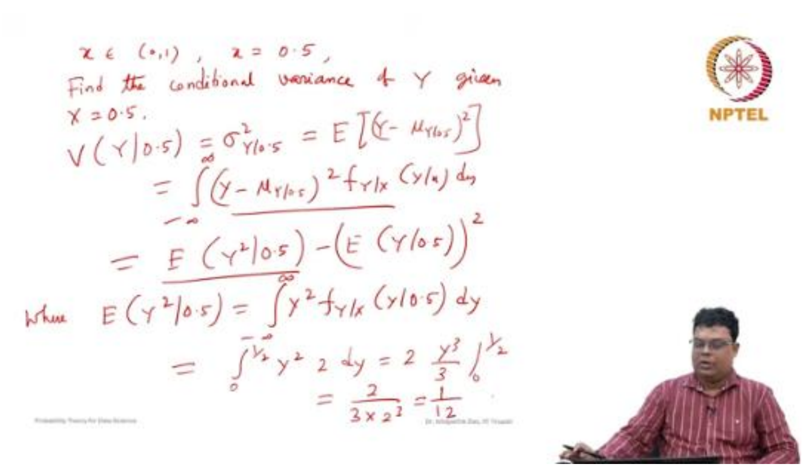


We already know that the conditional mean μ(Y | X = 0.5) is 0.25, so now we need to compute the expected value of Y^2 given X = 0.5. To do this, we need to integrate Y^2 multiplied by the conditional probability density function of Y given X = 0.5 over the range of possible Y values. From previous calculations, we know that the conditional probability density function of Y given X = 0.5 is 2 when Y is between 0 and 0.5, and it is 0 otherwise. This means the probability density function is 2 in this range and 0 outside it. We will use this probability density function to calculate the expected value of Y^2 by integrating it over the interval from 0 to 0.5.

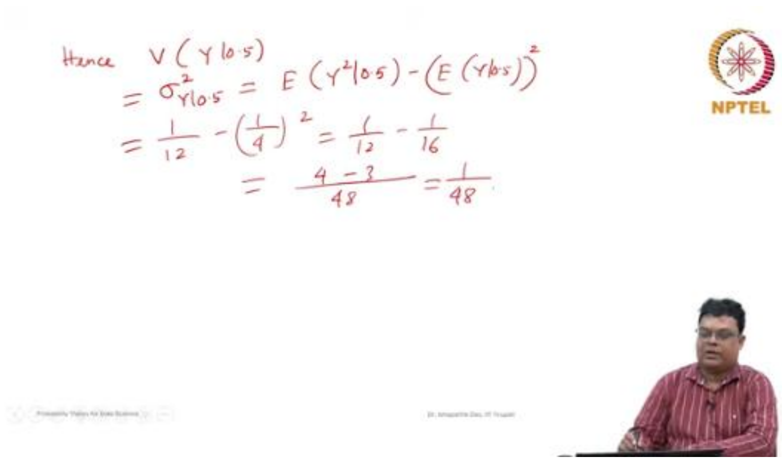
This will give us the necessary value to compute the conditional variance. So, the probability density function is non-zero whenever Y is between 0 and 0.5. Therefore, we integrate Y^2 over this range, with the probability density function being 2 in this interval. We now need to compute the integral of Y^2 multiplied by the probability density function. The result of the integration gives us Y^3 / 3, with the limits of integration from 0 to 0.5.



After performing the integration, we substitute the upper limit of 0.5 into the result, which simplifies to 1/12. Hence, the variance of Y given X = 0.5 is 1/12. So, the variance of Y given X = 0.5 is calculated by subtracting the square of the expected value of Y given X = 0.5 from the expected value of Y^2 given X = 0.5. From earlier, we found that the expected value of Y^2 given X = 0.5 is 1/12, and the expected value of Y given X = 0.5 is 0.25 (which is 1/4). We square 1/4, which gives us 1/16, and subtract that from 1/12. After performing the calculation, we find that the variance of Y given X = 0.5 is 1/48.



This is how we would approach solving such problems. This is how we would approach solving such problems. To find the conditional mean and conditional variance for a continuous random variable, we consider two random variables, X and Y, with given values. The conditional mean of X given Y = y can be calculated, as well as the conditional mean of Y given X = x. Similarly, we can find the respective variances for these conditional probabilities.



In general, for any given value of X or Y, we can calculate the conditional mean and variance of the other variable. These calculations involve using the conditional probability density or mass functions, depending on whether the random variables are continuous or discrete. We have covered several important topics related to bivariate random variables, including the joint probability mass and density functions, conditions for independence, as well as the conditional probability functions. Additionally, we explored how to compute conditional means and variances for both discrete and continuous cases.