PROBABILITY THEORY FOR DATA SCIENCE

Prof. Ishapathik Das

Department of Mathematics and Statistics

Indian Institute of Technology Tirupati

Week - 05

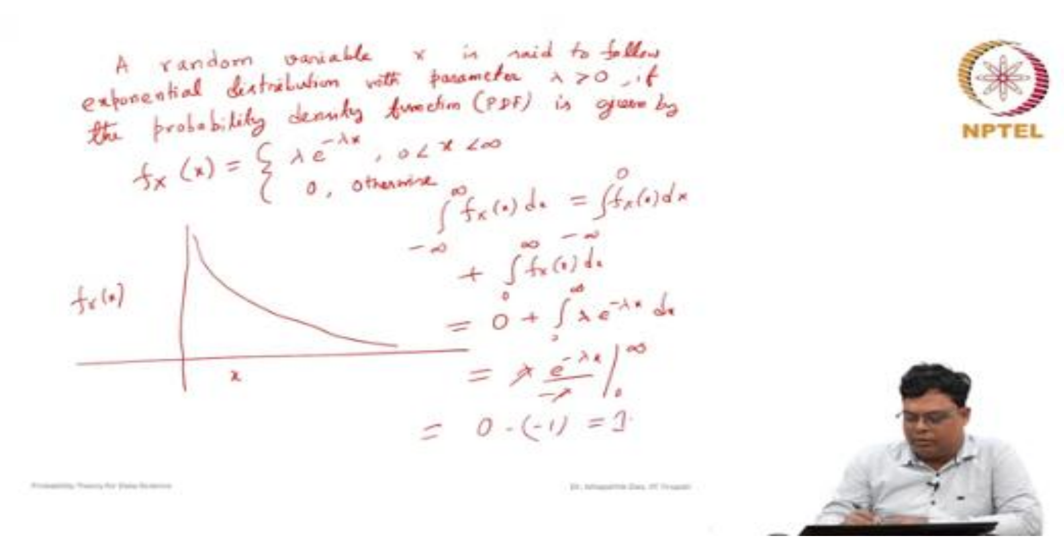
Lecture - 24

Applications of Exponential Distribution

Note that this density function satisfies all the properties of a probability density function. Remember that a probability density function must always be greater than or equal to 0, and the integral from minus infinity to plus infinity of the function should equal 1. Let's check if this holds here. We can split the function as follows: from minus infinity to 0, it's 0, and from 0 to infinity, it is given by $\lambda \times e^{(-\lambda \times x)}$. So, the integral becomes:

$\int$(from $-\infty$ to ∞) f(x) dx = $\int$(from 0 to ∞) $\lambda \times e^{(-\lambda \times x)}$ dx.

Here, we're not directly substituting infinity for x; instead, we treat this as an improper integral by integrating up to a real number x and then taking the limit as $x \to \infty$. As $x \to \infty$, the expression tends to 0, and at x = 0, it evaluates to 1. This confirms that the integral is indeed equal to 1. So, that is why—one minute—no, this is equal to 1 because it is already negative. So, at 0, this is 0, so it's -0, and then we get -(-1), which is equal to 1.



So, now this is a density function; we have found that this is indeed a density function. Now we will find the mean and variance of this density function. So, let us find the mean. Mu one dash is the mean, as we have computed earlier. This is nothing but the expected value of X.

Now, I think you understand how we can find the mean and variances of a discrete random variable. Also, we are dealing with a continuous random variable here because this distribution is very important, so we need to know its mean and variance. Later, we'll use this information and won't need to derive it each time; we can simply say the mean, since if it is an exponential random variable, the mean is already known. If it's a Poisson random variable, then the mean will be this, and the variance will be this. Now, we are computing all those.

So, now, the mean of this exponential random variable is, by definition, the integral from minus infinity to plus infinity of $x * f(x)$ dx. This is equal to the integral from 0 to infinity, since it is non-zero only in that range. So, we can write it as:

$\int$(from 0 to ∞) $x * \lambda * e^{(-\lambda * x)}$ dx.

Here, λ is a constant. Now we have to perform the integration from 0 to ∞ of $x * e^{(-\lambda * x)}$ dx.

How do we do this integration? One way is to use integration by parts, but when we need higher-order moments such as x^2 or x^3, this method requires additional computations and steps because you have to apply integration by parts multiple times. For this reason, we'll use a different approach. We'll apply the concept of a gamma function, an improper integral. You may have studied it in your calculus courses, as it's an important function.

In fact, we'll discuss one distribution that also uses the gamma function; it's called the gamma distribution. So, we're introducing the gamma function here. This gamma function, for any $\alpha > 0$, is defined by:

$\Gamma(\alpha) = \int$(from 0 to ∞) $x^{(\alpha - 1)} * e^{(-x)}$ dx.

In calculus, you may have learned that this is an improper integral, but it is convergent and has a specific value, represented by $\Gamma(\alpha)$. Now, for any real number $\alpha > 0$, we denote this value as $\Gamma(\alpha)$.

But when α is an integer, there are additional properties. For example, $\Gamma(\alpha + 1)$ can be represented by $\alpha * \Gamma(\alpha)$. So, for any real number $\alpha > 0$, $\Gamma(\alpha)$ is defined from 0 to ∞. When α is an integer, say, a natural number ($\alpha = 1, 2, ...$), by using this property, we can say that:

$\Gamma(\alpha + 1) = \alpha * \Gamma(\alpha)$.

This is true for any integer α; we can write:

$\Gamma(\alpha) = \alpha * (\alpha - 1) * \Gamma(\alpha - 1)$,

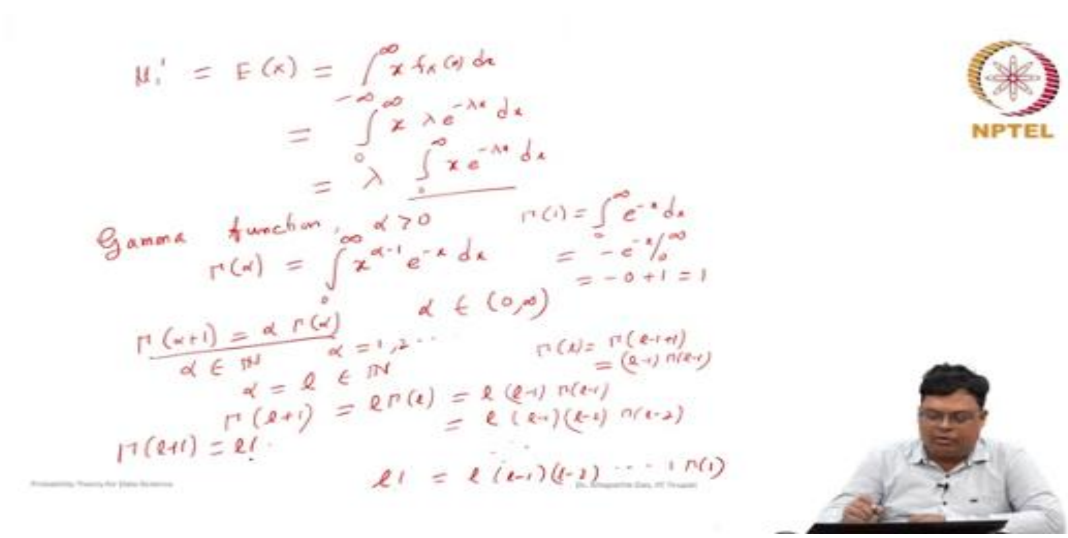
and we can continue in this way until we reach $\Gamma(1)$. $\Gamma(1)$ is simply:

$\Gamma(1) = \int(\text{from } 0 \text{ to } \infty)\ e^{(-x)}\ dx = 1$.

As the limit approaches infinity, $e^{(-x)}$ goes to 0, so $\Gamma(1) = 1$. Thus, $\Gamma(\alpha + 1)$ is nothing but $\alpha!$ (alpha factorial).

Whenever α is an integer, $\Gamma(\alpha + 1) = \alpha!$.

So, this is some information about the gamma function that we'll use in our computation. There are other known values, but we're not going to compute them, just remember them. For example, when $\alpha = 1/2$, $\Gamma(1/2) = \sqrt{\pi}$, which we'll use here. You might have learned this in your integral calculus courses, and we'll use this information as we move forward.



So, you can remember this, and later we'll mention it again. Now, let's find the expected value of x, which is μ_1, the expected value of x. This is just the integral from 0 to ∞ of x *

$\lambda * e^{(-\lambda * x)}$ dx. Since λ is a constant, it's independent of x, so this becomes the integral from 0 to ∞ of $x * e^{(-\lambda * x)}$ dx.

Now, we know some of the values for this type of integration, so what we'll do is take a transformation to make the integral look like a standard form. We can transform this into an $e^{(-x)}$ form and x raised to some power. Since we have $e^{(-\lambda * x)}$, we'll set $z = \lambda * x$. This gives $dz = \lambda\, dx$, so $dx = dz / \lambda$. Therefore, x becomes z / λ, and the integral becomes:

$(1 / \lambda) * \int(\text{from } 0 \text{ to } \infty)\ z * e^{(-z)}\, dz$.

The limits remain the same because as $x \to 0$, $z \to 0$, and as $x \to \infty$, $z \to \infty$.

This now matches the gamma function form, which is:

$\int(\text{from } 0 \text{ to } \infty)\ x^{(\alpha - 1)} * e^{(-x)}\, dx$.

In our case, the power of z is 1, so we need to rewrite it as $z^{(2 - 1)}$. This gives us:

$(1 / \lambda) * \Gamma(2)$.

Since $\Gamma(\alpha + 1) = \alpha!$ for integers, $\Gamma(2) = 1! = 1$. So, the expected value of an exponential distribution with parameter λ is:

$\mu_1 = 1 / \lambda$.

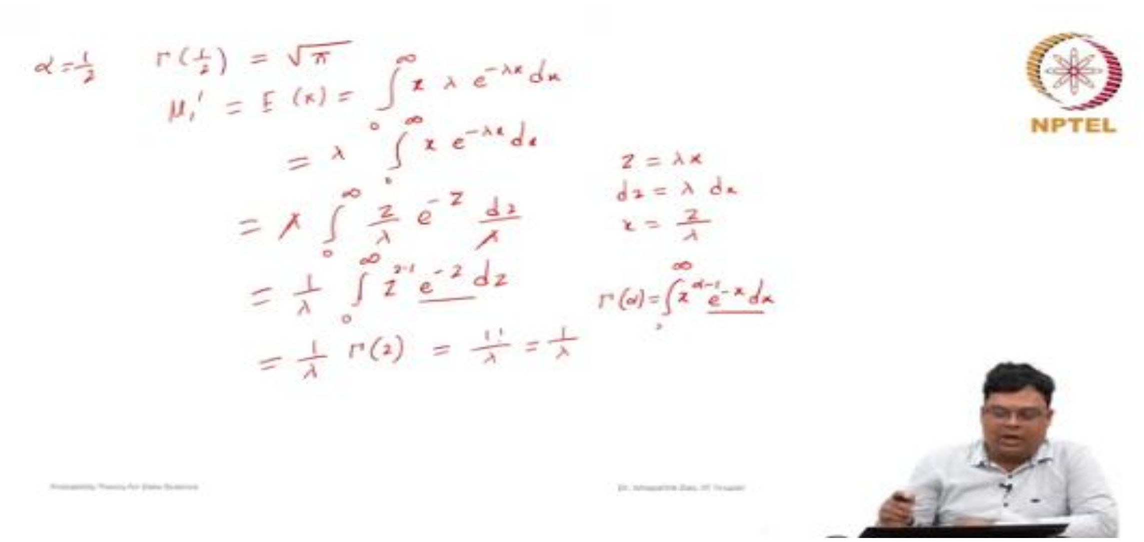
Now, we want to find the variance of this random variable, which is the variance of x, denoted as σ^2_x. To find this, by definition, the variance is:

$\sigma^2_x = E[x^2] - (E[x])^2$.

However, we will use the simplified formula, which is:

$\sigma^2_x = \mu_2 - (\mu_1)^2$.

So, how do we find μ_2? μ_2 is the expected value of x^2.



The expected value of x^2, by definition, is the integral from $-\infty$ to $+\infty$ of $x^2 \cdot f(x)\,dx$. But this is non-zero only when x is positive, so we change the limits to 0 to $+\infty$. Therefore, we have the integral from 0 to $+\infty$ of $x^2 \cdot \lambda e^{(-\lambda x)}\,dx$. We will use the same transformation again. Let's take $z = \lambda x$, so that $dx = dz / \lambda$.

Now, $x = z / \lambda$, and the integral becomes the integral from 0 to $+\infty$ of $z^2 / \lambda^2 \cdot e^{(-z)}\,dz$. Then, $dx = dz / \lambda$, so we can cancel one λ, and the integral simplifies to $1 / \lambda^2 \cdot \int$ from 0 to $+\infty$ of $z^2 e^{(-z)}\,dz$.

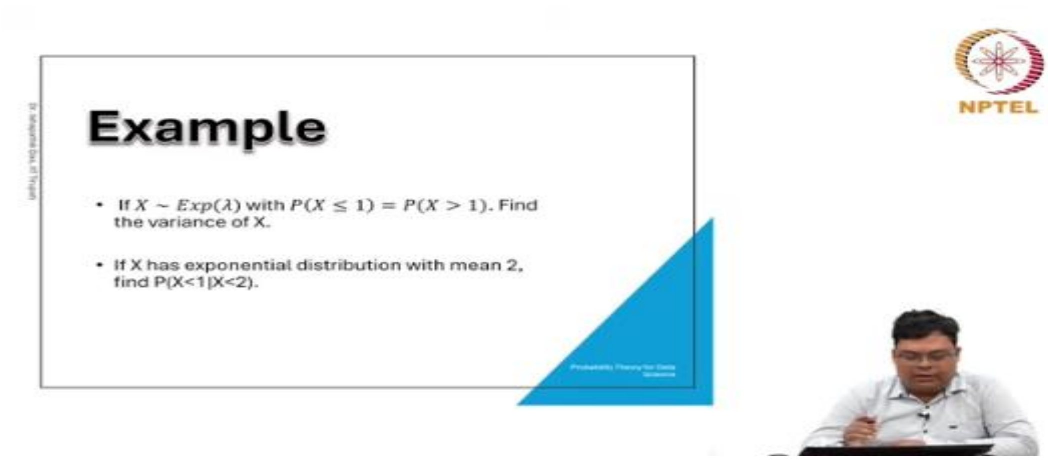
Now, remember that the gamma function formula is $\Gamma(\alpha) = \int$ from 0 to $+\infty$ of $x^{(\alpha - 1)} e^{(-x)}\,dx$. If we compare this with our integral, it looks very similar. This part is the same, but here we have z^2, which corresponds to $\Gamma(3)$ because we have a power of 2.

$\Gamma(3) = 2! = 2$. So, the integral becomes $1 / \lambda^2 \cdot 2$, which simplifies to $2 / \lambda^2$. Therefore, $\mu_2 = 2 / \lambda^2$.

Now, to find the variance, we subtract μ_1^2 from μ_2. We already know that $\mu_1 = 1 / \lambda$, so $\mu_1^2 = 1 / \lambda^2$.

Therefore, the variance is $2 / \lambda^2 - 1 / \lambda^2$, which simplifies to $1 / \lambda^2$. So, the variance of the exponential random variable is $1 / \lambda^2$, and the mean is $1 / \lambda$. So, we have found the probability density function and the cumulative distribution function, and we have also discussed their graphical representations.

Now, let us discuss an example of this exponential random variable. So, suppose X is an exponential random variable. Note that when we define an exponential random variable, we say that X has an exponential distribution with parameter λ, and the probability density function of X is given here.



Now, let us discuss one example. If X is an exponential random variable with parameter λ, and the probability of $X \leq 1$ is equal to the probability of $X > 1$, we need to find the variance of X. When X has an exponential distribution with parameter λ, the probability density function is

$f\_X(x) = \lambda e^{(-\lambda x)}$, for $x > 0$ and $x < \infty$, where $\lambda > 0$, and 0 otherwise.

Now, to find the probability that $X \leq 1$, we compute the integral from 0 to 1 of $f\_X(x)$ dx, which is

$\int$ from 0 to 1 of $\lambda e^{(-\lambda x)}$ dx.

This evaluates to $1 - e^{(-\lambda)}$.

The probability that $X > 1$ is the complement of the probability that $X \leq 1$, so it is $e^{-\lambda}$. Since the probabilities are equal, we have

$$1 - e^{-\lambda} = e^{-\lambda},$$

which allows us to solve for λ. This implies

$$e^{-\lambda} \cdot e^{-\lambda} = 1,$$

so

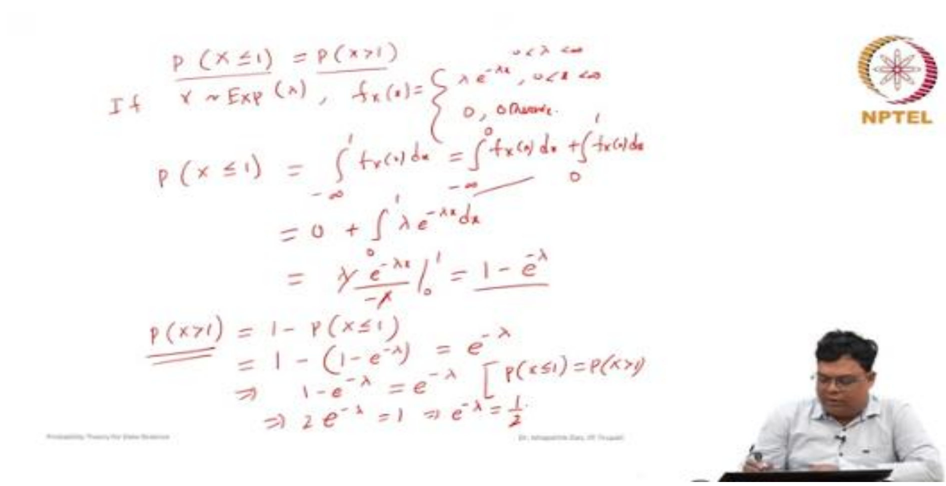
$$e^{-\lambda} = 1/2.$$

This implies that

$$-\lambda = \log(1/2) = \log(1) - \log(2) = -\log(2).$$

So,

$$\lambda = \log(2).$$

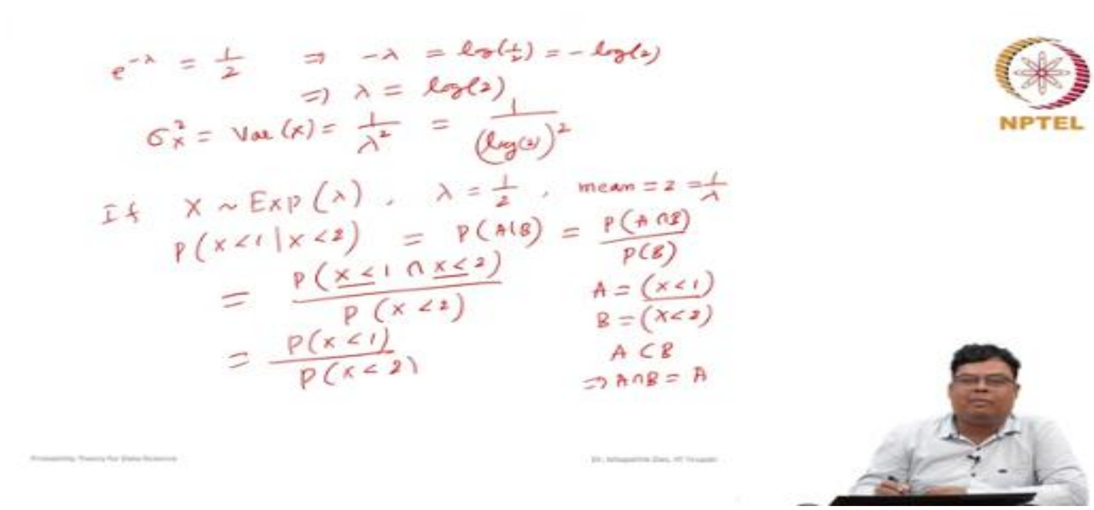


Now, given this value of λ, the question is to find the variance of X. The variance of X is $1 / \lambda^2$, so the variance of X is $1 / \lambda^2$, which is equal to $1 / (\log 2)^2$. This is the variance of the random variable X. This is one example using the exponential distribution.

Another question is, if X has an exponential distribution with a mean of 2, find the probability that $X < 1$ given that $X < 2$. This is a conditional probability question. We are told that X has an exponential distribution and that the mean is 2. We know that the expected value of X is 1 / λ, so with a mean of 2, we have 1 / λ = 2, which implies that λ = 1 / 2.

Now, to find the probability that X < 1 given that X < 2, we use the definition of conditional probability, which is P(X < 1 | X < 2) = P(X < 1 ∩ X < 2) / P(X < 2). This can be written as the intersection of the events, X < 1 and X < 2, divided by the probability that X < 2.

Let's denote the event X < 1 as A and the event X < 2 as B. Since A is a subset of B, whenever X < 1, it's also less than 2. So, the intersection of A and B is just A, and we can compute the conditional probability as P(A) / P(B). That is why A is a subset of B, so the intersection of A and B is just A. So, this will be nothing but P(X < 1) / P(X < 2).



Now, we need to find the probability of X < 1 and the probability of X < 2. Since X is an exponential random variable with λ = 1/2, we can find the cumulative distribution function (CDF). You can directly find the probability of X < 1, or you can use integration. The probability of X < 1 is the integral from 0 to 1 of the probability density function (PDF), which is (1/2) e^(-x/2). So, integrating this gives (1/2) times the negative exponential, resulting in 1 - e^(-1/2).

Now, for the probability of X < 2, we do the integration from 0 to 2 of (1/2) e^(-x/2). After performing the integration, we get 1 - e^(-1). So, the conditional probability of X < 1 given X < 2 will be the ratio of the probability of X < 1 to the probability of X < 2, which is (1 - e^(-1/2)) / (1 - e^(-1)). Then, you can substitute the value of e and simplify the fraction to get an approximate value. So, this is another problem we discussed.

$$X \sim Exp\left(\frac{1}{2}\right)$$

$$P(X<1) = \int_0^1 f_X(x)\,dx = \int_0^1 \frac{1}{2} e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\,\frac{e^{-x/2}}{-1/2}\Big|_0^1 = 1 - e^{-1/2}$$

$$P(X<2) = \int_0^2 f_X(x)\,dx = \int_0^2 \frac{1}{2} e^{-x/2}\,dx$$

$$= \frac{1}{2}\,\frac{e^{-x/2}}{-1/2}\Big|_0^2 = 1 - e^{-1}$$

$$P(X<1 \mid X<2) = \frac{P(X<1)}{P(X<2)} = \frac{1-e^{-1/2}}{1-e^{-1}}$$