Our Mathematical Senses

The Geometry Vision

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Lecture-61

Video 12B: the matrix group of transformations of RP2

So, I want to now turn our attention to transformations of RP2. And in other words, maps from RP2 to RP2. And given an extended plane, pi, in P3, we've been studying projectivities from pi to pi. We've been studying transformations of an extended plane via projectivities. So my question is, can we understand these projectivities as maps from RP2 to RP2? We're kind of relating, we've upgraded our understanding of the extended plane to RP2 in some sense. RP2 is kind of a new avatar of the extended plane.

So earlier we had projectivities which mapped the extended plane to itself. Can we somehow, do we have some maps from RP2 to itself that can serve an analogous purpose? So what are some natural maps from RP2 to RP2? Can we think of any maps from RP2 to RP2? How do we even go about defining or generating a map from RP2 to RP2? Well again, RP2 is the set of lines through the origin in R3. So at the very least, we need a map from R3 to R3 which takes lines through the origin to lines through the origin. It shouldn't take a line through the origin to a circle or a parabola or some other weird set of points.

It should preserve linearity and collinearity of points. So luckily, we already have, if you've taken some linear algebra, we have a candidate which is the general linear group GL3R. Now the general linear group is just the set of 3x3 matrices with real entries that are invertible. So let's let A be some element of the general linear group. Let's let it be a 3x3 matrix with non-zero determinant and real number entries.

That's what the R stands for. So here's an element A of the general linear group. Each of these is a real number. There's nine entries because it's a 3x3 matrix and the determinant is non-zero. It's an invertible matrix.

That's all we know about it. That makes it an element of the general linear group. And

A, this matrix A, acts on R3 by left multiplication. We can take a vector x, y, z in R3, multiply it by this matrix on the left, and we get this other vector, this new vector in R3. The x-coordinate is this, the y-coordinate is this, the z-coordinate is this.

This is our product vector we get. This is our new vector we get, multiplying this vector by this matrix. So left multiplication by A sends an element x in R3. I'm using this notation. A bold x with a little arrow on top refers to the vector in R3, which is actually this triple here, x, y, z.

So multiplication by A sends this vector x to another vector, another element, A dot x, this guy, which is also in R3. So A, this matrix A, takes every vector in R3 to some other vector in R3, or maybe it fixes it, but it's a transformation of R3. It maps R3 to R3 and scrambles up all the vectors in some way. So just writing it another way, x goes to Ax. This vector goes to this vector.

That's what the action of A does. But notice something. A, if we take the vector lambda x, what does A send lambda x to? Well, lambda x is this. Lambda x, lambda y is L, lambda z. So if we're multiplying it on the left by A, you can just work out what that product is.

It's very similar to what we had earlier, but we have these lambdas here now. But then we can pull those lambdas out. We can just pull them out, because there's a lambda in every single entry. So we can pull it out, and this is lambda times A dot x. So since A dot lambda of x is equal to lambda times A dot x, this action takes lines through the origin to other lines through the origin.

Maybe it's a little bit easier to see this pictorially. Here is one line through the origin. This is the vector x, and this is the line consisting of all scalar multiples of x, all lambda x, where lambda ranges over non-zero real numbers. I guess the reason I'm not including zero, we'll see in a second. Over here, we have another line through the origin.

It's just the set of scalar multiples of Ax. Ax is this vector, and we can look at all the scalar multiples lambda Ax as lambda ranges over non-zero real numbers. And that gives us a different line through the origin. And left multiplication by A .

.. Well, since A times lambda x equals lambda Ax, any scalar multiple of x gets taken to some scalar multiple of the vector Ax. This line is taken to this line. We're saying it a slightly different way. Left multiplication by the matrix A maps the set of all scalar multiples of x onto the set of all scalar multiples of Ax. In other words, since we're taking lines through the origin to lines through the origin, we're inducing an action of the matrix A on RP2.

Putting it yet another way, we can write this line using homogeneous coordinates, and we can write this line using homogeneous coordinates. This is now a well-defined expression. The matrix A acts on homogeneous coordinates. We can take a homogeneous coordinate representation of this line, and it makes sense to say A dot this, because we can just take any representative, x, y, z, apply the matrix A to it, and then write the resulting thing in homogeneous coordinates to get the image. The action of GL3R on RP2 is well-defined, but it does have some redundancy to it.

It's not entirely faithful. That's a technical term, by the way, and we'll see what that means in a second. It means the following. Let's consider a line, V1, V2, V3, in RP2. I'm writing it in homogeneous coordinates.

We can write lambda times A of V1, V2, V3. Since A and lambda commute, this is just equal to A of lambda V1, lambda V2, lambda V3. This is just the same thing as this. These are both representatives of the same line. This homogeneous coordinate is the same as this homogeneous coordinate expression.

This is just A applied to this homogeneous coordinate. This is true for any non-zero lambda. In other words, all scalar multiples of a matrix A, all scalar multiples lambda A of a matrix A have the same effect on RP2. When we apply them to RP2, let them act on RP2, they're all going to have the same effect as one another. That's what I mean by redundancy.

Take any matrix here, all of its scalar multiples will have the same effect. It's not that any two elements of GL3 are will have different actions on RP2. We can rectify this. We can actually make it a cleaner action. If we equate all non-zero scalar multiples of A.

In other words, we can define an equivalence relation A tilde A prime to mean that A is equal to lambda A prime for some non-zero lambda. We're literally equating all non-zero scalar multiples of A. This is A, this is lambda of A. We'll call these two matrices equivalent. If we need two matrices that differ by a scalar multiple, we'll call them equivalent.

If we do that, we take GL3R mod out by this equivalence, we get the projective linear group PGL3R. The question is, what is this weird group we've constructed and can we visualize the action of PGL3R on RP2?