Our Mathematical Senses

The Geometry Vision

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Lecture-59

Video 11D: bonus: join and meet as cross product

So, I want to look at a final application of this analytic framework, which will involve some linear algebra. And it'll also give us a sense of what all we can, the power of homogeneous coordinates. So remember that homogeneous coordinates allow us to represent a line through the origin by a single vector. We have a lot of choice in that vector, it can be any vector lying along that line. In other words, we can represent a projective point in RP2 by a vector. And if you recall, a projective line is simply a plane through the origin in R3.

So we'll say three projective points are going to be collinear if they all lie on a projective line. But how can we check this? Given three projective points, maybe we're given them by their vectors, by vector representatives, how can we check whether they're collinear, whether they all lie on a projective line? Well, here's an exercise. Show that three projective points represented by vectors v, w, and u are collinear if and only if the determinant of the 3 by 3 matrix u, v, w is 0. What do I mean by this notation? I'm saying let's let u, v, and w be column vectors, and let's form a 3 by 3 matrix using them.

So if u is u1, u2, u3, v is v1, v2, v3, and w is w1, w2, w3, then this is the matrix that I'm talking about here, this 3 by 3 matrix. And the exercises show that that is 0, the determinant of that is 0, if and only if these vectors are collinear. For example, in this picture here, if we say that this is u, this is v, and this is w, the way I've drawn it, these are not collinear. So if this statement is true, their determinant, the determinant of this matrix for these guys, should not be 0, because they're not collinear. They're not all lying in one plane through the origin.

So they're not all lying on one projective line. Remember, a projective line is a plane through the origin. A projective point is a line through the origin. So these three projective points in this picture are not collinear. They're not all lying on a projective line.

And therefore, the determinant should not be 0. And it is 0 if and only if they're collinear. So as a hint for that exercise, remember that the vectors should only span a plane through the origin if they're collinear. They shouldn't span all of R3. In this picture, they're actually spanning all of R3.

But really, to be collinear, they should only span a plane through the origin. So let's leave that for the time being and move on to a second application. So it's easy to represent projective points by vectors. But we can also represent a projective line by a vector. That's a little more surprising, maybe.

So a projective line is a plane through the origin. But we can represent it by the normal vector to that plane. Again, we have a lot of choice. We can take any normal vector to that plane. So we have a whole family of scalar multiples of a given vector that we can choose from.

But given a vector, any vector uniquely specifies a plane through the origin, which it's normal to. So a vector still uniquely specifies a plane through the origin, uniquely specifies a projective line. So for example, this vector that I've drawn here specifies this projective line, this blue plane here. Now, a second exercise will define the join of two points to be the unique line that they determine, the unique line they're coincident with. So can you show that the join of projective points is given by their cross product? Given two projective points, the projective line that they determine, that line, its vector representative is given by the cross product.

And as a hint for this one, the join of two projective points is the plane spanned by the corresponding lines through the origin. The join of this projective point and this projective point, sorry, is this plane through the origin. So you'll have to use the geometric interpretation of the cross product, but using this hint, that also shouldn't be too difficult. And a third application that I want to give as well, a third exercise, I want to define the meet of two lines to be the unique point that they are coincident with, the unique point that they intersect at. So given two projective lines, I claim that the meet of two projective lines is given by their cross product as well.

So this one is maybe a little harder to see. A projective line is a plane through the origin. So as a hint, the meet of two projective lines is just the intersection of the corresponding planes through the origin. Given two planes through the origin, they'll intersect in a line through the origin. And that is the point of intersection of those projective lines, the projective point of intersection of those two projective lines.

But the claim, the exercises show that that line through the origin that we get can also be thought of as the cross product of the normal vectors of the two planes through the origin that we intersected to get that line through the origin. So I'll leave it to you to think about that. But again, using the geometric interpretation of the cross product, that'll be key to solving this. So good luck.