Our Mathematical Senses

The Geometry Vision

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Lecture-56

Video 11A: the real projective plane

So today, I want to shift our approach to studying geometry. And let's just recall that from the time of Pappus, Pandrosian, and Hypatia, those ancient Alexandrian mathematicians who lived in the 300s CE, from their time period all the way up through the 19th century, projective geometry was mainly studied with straight-edge drawings, like the ones that we've done in this course, and axioms. And this approach to studying geometry was sometimes called synthetic. And the reason it was called synthetic is from this word synthesis. It's because it involved explicit hands-on constructions, building of something. So it's a slightly older use of the term synthetic than we might think of today.

Today when you hear the word synthetic, you might think of polyester clothing or electronic music, but this is a slightly older terminology, an older meaning of the word synthetic, which was really focusing on the fact we were building something from the ground up. But another approach to geometry came out of the work of Rene Descartes. And in his book La Geomatrie, he established Cartesian coordinates and analytic geometry. But projective geometry wasn't so easily framed in a coordinate system, because how do we assign coordinates when distances are not even fixed? At least that's how it seemed, until a remarkable new framework came into being in the end of the 19th century.

This really came out of the work of Moebius, but also Plucker, Grossmann, and many other mathematicians contributed to and led to this development. So as a first step into this analytic framework, we need to construct the real projective plane. So let's first briefly review our old framework. So up till now, our setting has been P2, the extended Euclidean plane. And P2, if you remember, was a linear space.

It's a set of points, a set of lines, and incidence relations between those points and lines, which satisfy two axioms. L1, which states that any two distinct points are incident with exactly one common line. And L2, which just states that every line is incident with at least two points. So the set of points in P2 is just the set of points in R2, union a bunch of points in infinity that we added, one for each family of parallel lines. But P2 can be a bit tricky to visualize.

So we've addressed this a little bit, but some basic questions that come up, which are not that easy to answer from this formulation, are just whether P2, does it look the same at every point? From this formulation, it looks like we have our ordinary points coming from R2, and we have these points in infinity that we've added on. So it looks like these and these could somehow be different from each other. They might have different structures, they might have different properties. It's not clear that they all have the same properties or whether P2 looks the same from any point. A second question is about the lines in P2.

We saw that they're kind of circular-ish. Like if you keep going in one direction, you come back to where you started. So to what extent are they circular? Do they have other geometric properties or circles? What more can we say? And finally, a big question that we've kind of left unanswered still, is just what is the overall shape of P2 globally? How can we think of P2 as a whole and visualize it as a whole? What is its overall shape? So these are some obvious questions about P2 that we can't answer very easily with our old framework. So let's introduce a new avatar, the real projective plane, RP2. It's the same thing as P2, but through a new lens in some sense.

RP2 is also a linear space, we'll see very soon, but it has much more structure. There's a much more natural way of thinking about its shape, i.e. its topology, and even seeing it as a manifold. So an important question to start with is just what is RP2 and where does it come from? So I want to go back to an even more basic question, which is what are our primary objects of study in the geometry of vision? What are we actually studying when we try and explore the geometry of vision? And one answer is we're looking at points in planes.

Here we have an image of the ground plane pi in the picture plane pi prime, and we're studying all these different points in pi which make up the railway tracks, and all these different points in pi prime which make up the image of those railway tracks under this perspective. Maybe in this perspective drawing, or in this photograph up here. So that's one possible answer to that question. But maybe another answer we could give is that maybe we're studying the sight lines through a fixed center point. This is the center point here, and we're looking at all of these different sight lines.

Maybe all along that's really what we've been studying. In any one plane, like pi, there are some ordinary points that we can see perfectly well, like the points that make these railroad tracks or this brown rail next to that. But there's also some points at infinity that we can't see. And yet all of these correspond to sight lines through a point O. So this point at infinity here corresponds to a sight line through O.

This point over here, this ordinary point, is a different sight line through O. So let's consider sight lines as the primary objects of study, and remove the apparent distinction between ordinary points and points at infinity. So to define RP2, let's just let RP2 denote the set of all lines through the origin in R3. So writing it out in a more mathematical notation, it's the set of lines L in R3 such that L contains the origin 0, 0, 0. So here's a bunch of lines through the origin, and each of these lines is an element of RP2.

So let's define a projective point to be a line through the origin in R3. In other words, an element of RP2. So a projective point is just an element of RP2. And let's define a projective line to be a plane through the origin in R3. So here's an example of a projective line.

It's a plane through the origin, but it also consists of a bunch of projective lines. Sorry, it consists of a bunch of projective points. Each of these lines through the origin that I'm drawing is a projective point, and there's a whole lines worth of them in this projective line here. So do the following statements hold? These are our kind of basic axioms, statements of incidence. And we can think of them also as axioms, actually, of a projective plane, of an extended plane.

So do these axioms hold, do these statements hold, for RP2, as we've defined it? So any two distinct projective points are incident with exactly one common projective line. Is that true? So here's a projective point here. Let's draw that a little more clearly. Here's another projective point. And are they incident with exactly one common projective line? Well, in this picture they are.

It's just this plane, this line that I've drawn, it's the plane through the origin that's marked in blue here. And is there exactly one plane through the origin containing these two lines through the origin? Well, yeah, it's just the span of those two lines through the origin. We can take any vector representatives, take their span, that's going to be a plane through the origin. And indeed, that will be the unique plane through the origin, which contains these two vectors.

So this holds. And what about the second statement? Any two distinct projective lines are incident with exactly one common projective point. So projective lines are planes through the origin. Any two planes through the origin, are they incident with exactly one common line through the origin, projective point? Well, yeah, any two planes through the origin will intersect in a single unique line through the origin. So this statement is also true. So I just want to remark that because these properties hold, you can say that RP2 is a model of a projective plane, just like P2.

These actually are considered or called axioms of a projective plane. Any space that you can build which satisfies them is a model of a projective plane. So RP2 is our new and improved model of a projective plane. And in RP2, we already see one advantage, which is that it's immediately apparent that any point, any projective point is geometrically similar to any other, indistinguishable from any other. They're all just lines through the origin in R3.

There's nothing to distinguish one line through the origin from another. So in this way, RP2 is said to be a homogeneous space.