Our Mathematical Senses

The Geometry Vision

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Lecture-50

Video 10B: Evenly spaced points

So now that we've defined the cross ratio more precisely, let's take another look at evenly spaced points. So long ago we asked the question, how do we know that these photos represent the same evenly spaced square tiling? And let's just review the cross ratio of four evenly spaced collinear points. We did this in the intro video, but let's quickly see it again. So here's the cross ratio. Let's call these points A, B, C, and D. And let's just let the distance between them, the successive tiles, be one in every case.

So the cross ratio is AB over BC, all divided by AD over DC. Again if you want that way of remembering it, again this is just one way. It's AB over BC divided by AD over DC. So AB is one.

AB is one. BC is also one. AD is three. And DC is negative one. So we get that the cross ratio is one over one divided by three over negative one.

Well that's just negative one third. This is just one over one divided by three over negative one. That's negative one third. And we've seen how it's preserved under a perspective shift. We haven't proved it yet, and we're going to prove it, but right now we'll just take it as a fact.

And indeed, if we measure out these pixels, we can calculate that once again, if you actually plug these numbers in, AB is 51 pixels, BC is 81 pixels, the total AD is 279 pixels, and DC is negative 147 pixels. Put those in your calculator and you get negative 0.33. So negative one third. So this is very convenient.

We can take any set of evenly spaced points, calculate the cross ratio of any evenly spaced points, say this point, this point, this point, and this point, and we get negative 1 third. We can take this point, this point, this point, and this point, and we'll get negative one third. Now we can also test whether a set of points is not evenly spaced using the cross ratio. For example, we could take a set of four points, like these ones here. They're collinear.

And we could calculate the cross ratio. We could call them A, B, C, and D, and calculate the cross ratio and see what we get. And if we get negative 1 third, then it's quite possible, highly likely that they're evenly spaced. But if we get something that's not negative 1 third, then there's no chance that they're evenly spaced. There's going to be no perspective view of these four points that will make them evenly spaced.

So that's another property of the cross ratio that we're going to prove. We haven't quite proved it yet, but we will prove that very, very soon. In particular, that weird, bizarre minus sign is really necessary. Four points whose cross ratio is plus 1 third will not be evenly spaced under any perspective view. So again, we're going to prove that very soon.

So this will all come out of something called the injectivity of the cross ratio. So I also want to look again at the puzzle that we did in the intro video, the intro puzzle, where we saw a very different set of evenly spaced points. I mean, it's actually a very similar set of evenly spaced points, but we saw it from a different perspective. And we measured out the distances of the first few heights of the tiles. And the question was, what is the next one? So we saw that this bottom tile has a height of 30.

The next one has a height of 10. The next has a height of 5. The next had a height of 3. And the question was, what is the next one, and can we find any pattern? Well, the surprising thing is that these successive lengths don't form a geometric series. There's no common ratio between them.

A geometric series or geometric progression is one where the ratio between successive numbers is always the same. So for example, this is not our situation, but this would be a geometric progression. If the first was 20, the next was 10, the next was 5, the next was 2.5, the common ratio here is 1 half. And we keep multiplying by 1 half to get to the next length.

So unfortunately or unfortunately, that's not our situation. Rather, it's this stranger pattern where the ratio keeps changing. 30 to 10, that's a ratio of 1 third. 10 to 5, that's a ratio of 1 half. 5 to 3, well, that's a ratio that's even a little bigger.

The ratio keeps getting a little bit bigger. So there is no common ratio, and it's not a geometric progression. But this is an image of an evenly spaced tiling, and we can use the cross ratio. So zooming in a little bit, let's mark some points as A, B, C, and D. So we know that this is 5, and this is 3.

We want to know what x is, the length between C and D. So if we just calculate the cross ratio, we plug in 5 for A, B. We plug in 3 for B, C. AD is 5 plus 3 plus x, which is 8 plus x.

And DC is negative x. That has to be negative 1 third. So multiplying this out, we get that negative 5x is equal to negative 8 minus x. In other words, x is equal to 2. So 2 is the next tiling. So we can calculate that using the cross ratio.

But that doesn't necessarily reveal the pattern. I mean, it does, but it's a complicated pattern because each length relies not just on the previous length, but on the one before that as well. The length of x relied on this length and this length. So a geometric progression, each length will only rely on the previous one via that common ratio. But here, each one relies on the two previous ones.

So it's a little more complicated to figure out the pattern. And alternatively, we can also try and calculate x from first principles, which does maybe reveal something that feels more like a pattern that you can wrap your mind around. So for that, let's consider a side view of the picture plane and the same evenly spaced points. So here's the same picture plane.

Here's your eye. Here's the ground plane. And here are the evenly spaced points that you're observing. And for simplicity, let's just say that the distance between any two points is one. And the distance between your eye and the ground is one. And the distance between the picture plane and you is also one.

And I'll leave it to you to do the slightly more involved scenario where this and this may not be one. You might be further or closer to the picture plane and you might be higher or lower down. But this will give you a good sense of what the pattern is. So let's look at the total pattern of the, rather than look at just the pattern of the ratios, the pattern of the lengths of these individual tiles, let's instead look at the total heights as we keep adding successive tiles. So the first tile alone has a height of 1 half.

If this is one and this is one and this is one and this picture plane has height one, this is going to hit it at height 1 half. If we add on another tile, now if we look at the height of the first two tiles together, well, where does this point, where does this line meet the picture plane? It's going to meet at the point 2 thirds. Similarly, this line is going to meet it at 3 fourths. This one is going to meet it at 4 fifths. And this one that is just out of the frame is going to meet it at 5 sixths.

So you can see a much clearer pattern there. And if you take their differences, you can get a general formula for the length of the nth tile. So I'll leave it to you to do that and then see how that compares to the length that we're getting in our example.