Our Mathematical Senses

The Geometry Vision

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Lecture-46

Video 9E: The four fixed points lemma

In order to prove the uniqueness in the fundamental theorem of projective geometry, we're going to need a lemma known as the four fixed points lemma. Now, the lemma states the following. Given a plane pi in P3 in the extended space P3, suppose we have a projectivity gamma from pi to itself, which fixes four points P, Q, R, and S, no three of which are collinear. Then gamma is actually the identity function on pi. In other words, another way of saying that, if gamma is fixed, so if no three of these points are collinear, it means that these four points can be thought of as the vertices of a quadrilateral. So if gamma fixes the vertices of a quadrilateral, then it fixes everything.

Every point is a fixed point for gamma if these four points are, and gamma is actually the identity function on pi. So that's what the statement says, that's what the theorem says. How do we prove it though? So we're going to follow a proof from Coxeter's textbook, Projective Geometry. And the proof uses the three fixed points theorem in a fundamental way.

So to prove it, let's first consider the lines P, Q, and S, R. What can we say about the, how does gamma treat those lines? Gamma is mapping pi to itself, it's taking various points of pi to various other points of pi, but it's fixing P, it's fixing Q, it's fixing R, and it's fixing S. That's all we know to begin with. So what is gamma going to do to the line P, Q? Well, gamma takes lines to lines. So it takes the line through P and Q, this line P, Q, to some other line in the plane pi.

But where is that line? Is there any, can we say anything more? Well yeah, gamma fixes P and it fixes Q. So gamma of P is P, gamma of Q is Q, so gamma of this line will contain P and Q. It'll be a line containing P and Q. There's not too many of those, there's only one. So gamma of this line is just this line.

Gamma fixes the line P, Q, but we have to be a bit careful. That terminology is a bit ambiguous because we don't know, we know it's taking this line to this line, but that doesn't mean it fixes every point in that line. It could scramble it up in various ways. We know it fixes this point and we know it fixes this point, but it may not fix all the other points. It does fix the line though.

And by a similar argument, we know that gamma fixes the line SR as a set. It sends this line SR to the same line. It also fixes S and it fixes R, but we don't know if it fixes this point or fixes this point. So all we know is that it fixes this line and it fixes this line as sets. But this line and this line intersect at a point which I'll call A.

But gamma is fixing this line and it's fixing this line. It's therefore going to have to take the intersection of these two lines, which is this point A. So A is lying in this line, so the image of A, gamma of A, will also lie in this line. A lies in this line, so the image of A, gamma of A, will also lie in this line. So gamma of A lies in this line and this line, meaning that gamma of A is equal to A.

Gamma also fixes A. Now we're in business because gamma fixes P, Q, and A. It fixes three lines all lying on this line. So it's going to fix this line point-wise. It's going to fix each and every point on this line by the three fixed points theorem.

And it's going to fix each and every point on this line by the three fixed points theorem. So suddenly it's fixing infinitely many points. And similarly, we can take this line and this line, call their intersection C, and by the exact same symmetric argument, gamma is going to fix every point on this line and every point on this line point-wise. So gamma of C is equal to C, and by the three fixed points theorem, the lines P, S, P, Q, Q R, and S R are all fixed point-wise. Each and every point on those lines is fixed.

So gamma is fixing a whole lot of points. And this line is also going to be fixed because this point of it is fixed. Its intersection with this line right here, that's fixed. This is fixed, and this is fixed. So it contains three points which are fixed by gamma.

So by the three fixed points theorem, the entire line is fixed by gamma. This line, we know it has three fixed points. So by the three fixed points theorem, gamma fixes the entire line. Same with all of these lines and every other line in the plane pi.

So that does it. And we can come, now let's return our attention to the fundamental theorem of projective geometry. By the four fixed points lemma, any two projectivities taking P, Q, R, and S to P prime, Q prime, R prime, and S prime must be equal. And how do we see that? Well, we know that if, let's say that gamma is one projectivity taking P, Q, and R and S to P prime, Q prime, R prime, S prime, and gamma prime is another projectivity doing that. Well, let's consider the composition gamma prime composed with, sorry, gamma inverse composed with gamma prime. So let's imagine performing gamma prime and then performing gamma inverse.

Well, that's going to take, it's going to take P to P prime, then back to P. It's going to take Q to Q prime, then back to Q. It's going to take R to R prime, back to R, S to S prime, back to S. So this is going to fix P, Q, R, and S. It's a map from pi to pi, which fixes P, Q, R, and S.

So by the four fixed points theorem, it is the identity, which means that gamma inverse undoes gamma prime. Everything that gamma prime did is now undone by gamma inverse, but gamma inverse also undoes gamma. So in fact, gamma inverse, gamma and gamma prime had to be the same map. So that completes the proof of the fundamental theorem of projective geometry in 2D.