Our Mathematical Senses

The Geometry Vision

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Lecture-45

Video 9D: 2D Fundamental theorem of projective geometry

So now let's look at the two-dimensional version of the fundamental theorem of projective geometry. So our initial goal was to understand projectivities between planes. In particular, how many degrees of freedom are there in the set of projectivities between planes, which is another way of saying what all changes as we shift perspective. So this is our attempt to address that initial question. So what is the fundamental theorem saying in 2D? Well, it's saying the following. Let's take two planes, pi and pi prime, sitting in P3.

And these are, of course, extended planes because they're sitting in P3. Let's let P, Q, R and S be four points in pi. And let's let P prime, Q prime, R prime and S prime be four points in pi prime. These can be any two ordered sets of four points with the restriction that no three are collinear.

In either set of four points, no three are collinear. In other words, they should really be able to form a quadrilateral of some sort of quadrilateral. If three are collinear, that's no good. Then we'll get something like this and they won't really form a quadrilateral. So that's the only restriction.

Otherwise we have full freedom in choosing these four points in each of these two planes. Now in this situation, the fundamental theorem states that there exists a unique projectivity from pi to pi prime, which takes P to P prime, Q to Q prime, R to R prime and S to S prime. There's a unique projectivity from pi to pi prime, which maps these four points onto those four points in order. The order matters here. If I switched, there'll be one projectivity taking these guys in this order.

But if I wanted R to map to S prime and S to map to R prime, I'd also be able to get a projectivity that did that. And that would also be unique. So to prove this, I'm going to follow a proof from Norman Wildberger's video lectures, actually. It uses the fundamental theorem of projective geometry in 1D. So the key to this proof is to first extend the line through PQ and the line through SR.

Those are going to intersect at a point A. Similarly, let's extend P prime Q prime and S prime R prime. Those are going to intersect at a point A prime. Now by the fundamental theorem, there exists a projectivity taking P, Q and A to P prime, Q prime and A prime. Remember, the fundamental theorem holds in P3.

Between any two lines in P3, we can take three points here and three points here, and we can find a projectivity taking these three to these three. So we're just literally applying the fundamental theorem, taking P to P prime, Q to Q prime and A to A prime. It's a projectivity from this line to this line. But we can extend this projectivity to a projectivity of planes, which we'll call gamma. We just need to choose planes that contain each of the intermediary lines, but which don't intersect any of the points, any of the centers of perspective.

So that's relatively easy to do. There's lots of degrees of freedom there in choosing those planes. So let's take a projectivity of planes gamma from pi to some plane pi star. And it needs to take P to P prime, Q to Q prime and A to A prime. And we just want to make sure that pi star is not equal to pi prime.

Again, we have a lot of freedom in choosing those planes. So we can choose the final plane pi star such that it's not equal to pi prime. So it'll look something like that. So we have a projectivity gamma now, which maps pi to pi star and takes these three points to these three points. So notice that in this, under this projectivity gamma, it's going to take S and R somewhere, somewhere in this plane pi star.

But this line, it's going to take this line SR A to some line going through A prime. And on that line, it's going to take S to some point S star and R to some point R star. So we get these new points S star and R star, which do form a line that hits A prime. And they live somewhere in pi star. So the points R star, which is just gamma of R and S star, which is just gamma of S.

Yeah, they lie on pi star. But the thing to notice here is that R star S star and S prime R prime, this is S prime R prime is a line in the plane pi prime, they're going to be coplanar because they share a point A prime. So they're going to be coplanar. And as a result, we can look at the intersection of S prime S star and R prime R star. So let's call that

intersection O.

And we can look at the perspective centered at O from the plane pi star to the plane pi prime. And that's guaranteed to take R star to R prime. It's going to take S star to S prime. But it's also going to fix everything on the intersection between these planes. It's going to fix P prime, it's going to fix Q prime.

So all in all, it's actually taking this quadrilateral down to this quadrilateral. And what happens when we compose it with our map f zero, sorry, with our map gamma? Remember, our map gamma took pi to pi star. So if we do gamma and follow it with this f O, we end up with a map from pi to pi prime. And it does exactly what we want. It's going to take P to P prime, Q to Q prime, S via S star to S prime, and R via R star to R prime.

So it's the desired productivity that we were seeking. So that takes care of the existence of a projectivity. And we still have to prove the uniqueness of it. But we'll do that in the next video.