Our Mathematical Senses

The Geometry Vision

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Lecture-44

Video 9C: Proving pappus's theorem

Okay, so let's put together all these ideas and let's finally prove Pappus' theorem. So let's first recall Pappus' theorem, which states that given one set of collinear points a, b, and c, and another set of collinear points a, b, and c, the cross joins, this one between the a's, ab and ab, the second cross join ac and ac, and the third cross join bc and bc, their intersection, those three points of intersection, those three cross joins are going to be collinear, lying on this Pappus line. So how do we prove this? Well, we'll show that our strategy is going to be that we get the same intermediate line in any two crisscross projectivities from capital L to little l. So we're going to use that crisscross construction and we'll show that no matter which of the three points we center it at, we get the same intermediate line. So in some sense, that'll let us construct the Pappus line, that intermediate line will be the Pappus line. So let's start with a crisscross projectivity centered at a.

Let's let ma be the line connecting cross joins ab intersect little a capital B and ac intersect little a capital C. So let's let ma be that line that connects these two cross joins. And let's compose perspectives fa followed by f capital A, f little a followed by f capital A. And that's going to send a, b, and c down to little a, little b, and little c.

So we can see our points up here. F little a is going to pull those points down. Then f capital A from here centered there is going to push those points down all the way to little a, little b, and little c. So that's our crisscross construction centered at a. So let's compare that with the crisscross projectivity centered at c.

Here let's let mc be the line connecting the cross joins ac intersect ac and bc intersect bc. So mc is the line connecting those two cross joins. Now composing perspectives f little c followed by f capital C, what do we get? Well f little c is going to pull these points down to this line mc centered at little c. f capital C is now going to push these points down further to little a, little b, and little c. It's the same crisscross construction once again.

So in this case we had a line mc. Earlier we had a line ma, an intermediary line ma. We need to show that these are in fact the same line. So both of these lines contain this particular point ac intersect ac. Is there any other point these two lines have in common? If they're, so this is, oops sorry, this is mc, this is ma.

So mc was created from these two cross joins, ma was created from these two. So they share this one, but otherwise they don't seem to have any other points in common. So how do we show they're in fact the same line? Well there is something else to work with. There's this line here. They're intersections with the line l.

Both ma and mc are going to intersect l, and if we can show that that intersection is the same for both of them, that's going to help us out. Both of them intersect l, we need to show that's the same point that they intersect l at. So they appear to do it, how do we prove it? Let's let mc denote the point, capital mc, denote the point where the line mc meets l. And let ma denote the point where the line ma meets l. We need to show that ma is equal to mc.

We need to show that these two points ma and mc don't just appear equal, they're actually equal. So the question is where does mc map to under gamma c? That's the question we can ask. So mc is here. Remember gamma c is the crisscross projectivity centered at c. So if we apply that crisscross projectivity on this point mc, what do we get? Well remember gamma c is equal to f little c followed by f capital c.

So f little c is going to pull all the points from l down to mc. And mc is also going to get pulled, but mc is already on mc because it's the intersection, so nothing's going to happen, it's going to stay fixed. Now we'll look at f capital c, that's pushing points down from the line mc to the line little l. Where is that going to send capital mc to some new point? We haven't actually denoted this point yet. But we know that mc lies on the line capital l, and c also lies on the line capital l.

So the projection via c is going to send this to this point here where capital l meets little l. So let's call that point m, little m. Little m is defined to be the point where capital l meets little l, and gamma c sends capital mc to little m. Now let's look at what happens to ma. Remember ma is the intersection of the line ma with the line capital l.

Where does ma map to under gamma a? So gamma a is f little a followed by capital a. f little a is pulling points from capital l to ma. And what's it going to do to ma? Nothing because ma is already on ma, so it stays fixed. But now f capital a is pushing points down, projecting points down from via a, from the center a. And that's going to once again send capital ma.

Capital ma is on the line capital l. So where is it going to get sent? Well a is also on capital l. So it's going to send ma to the place where capital l meets little l. Namely it's going to send it to m again. So ma is also going to get mapped to m, but via this map gamma a.

So gamma of ma is equal to little m. So in other words, so but now what can we say? So gamma a and gamma c, the two crisscross projectivities, they agree on the three points a, b, and c. They both send them to these three points, little a, little b, and little c. And therefore by the fundamental theorem of projective geometry, they're identical maps. They agree on every single point.

They always do the same thing as each other. They're in total agreement. Now since gamma a of ma is m, little m, gamma a is sending ma to little m, and gamma c of mc is little m. Remember these maps are actually the same map. They're in total agreement.

Well they're also bijections, by the way. Not only are they in total agreement, but they're also bijective maps. So that means that ma is equal to mc. These points ma and mc are in fact the same point. Now these lines ma and mc now have two points in common.

This crossjoin ac intersect ac, both of those have the crossjoin in common, but they also now have this point ma slash mc, that's just the same point, in common. So they're actually the same line. If two lines have two points in common, they have to be the same line. Between those two points there's a unique line. So they're in fact the same line, they're the Pappus line.

And hence the three crossjoins are actually collinear. So that does it.