Our Mathematical Senses

The Geometry Vision

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Lecture-39

Video 8C: 1D projectivities a harder puzzle

Let's now try a slightly harder puzzle. So once again, I want to try constructing a projectivity from the line L to the line little l, capital L to little l. And this time, I have three points, capital A, B, and C, and little a, b, and c. So let's construct a projectivity sending a to a, b to b, and c to c. Remember last time we just connected a and a and b and b and used this point O? Well, unfortunately, that strategy won't work, even if the lines are coplanar. We're going to have to do something more clever because OC may not intersect C.

It's very unlikely that it's going to intersect C. So we're going to have to try something different this time. So we'll definitely need at least two perspectives to get this job done. And here's a good, if you'd like to try it yourself, this is a fun exercise to try.

So pause the video and you can try and construct a sequence of perspectives, a projectivity taking a, b, and c to little a, little b, and little c. Okay, so let's try this. Let's try and do this. So we're going to need an intermediate line. And which one should we use? Well, let's try what we did last time.

Let's just try that second strategy from last time when the lines were not coplanar. Let's connect a and c over here. That gives us an intermediate line M defined to be AC. And just like before, we have that M and capital L are coplanar. But an M is coplanar with little l.

So that might be useful as well. Okay, so we have our points a, b, and c. We want to move them to little a, b, and c. Well, let's start by connecting up c and c, little c and capital C. If we project, and let's choose any point o1 on the line cc.

And consider the perspective f o1 from L to M. Where is it going to send a, b, and c?

Well, we'll send a to itself because it's on the intersection. We'll send c to little c, which is exactly where we want to go. And we'll send b to some random new point on the line M. Let's just call it b prime.

So we made some progress. C has gone where we want it to go. But we still have to deal with a and b. But now we're in a nice situation because these two lines are coplanar. And we've basically done this before.

We just need to get b prime to b and a to a. How do we do that? Well, we need a perspective from M to L. But since these two lines are coplanar, a, little a is guaranteed to intersect b prime b somewhere. So let's call that intersection o2. And we can consider the perspective f o2 from M to L.

Where is it going to send a and b prime? Well, and c. Well, it's going to send a to a, b prime to b, and c is going to be fixed because that's in the intersection between M and L. So we're done. The projectivity f o1 followed by f o2 from L to little l sends a to a, b to b, and c to c. I didn't write that here.

But I should have written and c to c. So that does it. Now f o2 1 followed by f o2, that's just one sequence of perspectives from capital L to little l, which takes a, b, and c to little a, little b, and little c. We could have done something completely different. Like here's a very different construction that accomplishes the same thing.

But the construction that we did has a very nice property, which is that if the lines L and little l don't actually need to be coplanar. We never assumed that these two lines are coplanar. They could be just any two skew lines in space, and everything still works out. And that's because we chose M to connect these two lines, and therefore at a and c. But L and M were coplanar, and M and little l were coplanar.

So we used the fact that these two lines are coplanar, and that these two lines are coplanar. But the original lines L and L don't need to be coplanar. So to state that a little more formally, given two lines capital L and little l in P3, and given points a, b, and c on big L, and little a, b, and c on little l, there exists a projectivity from L to L taking those points a, b, and c to these other points little a, little b, and little c.