Our Mathematical Senses

The Geometry Vision

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Lecture-38

Video 8B: 1D projectivities a puzzle

So we're going to look now at projectivities between lines. And I'm going to call those 1D projectivities for short because lines are one-dimensional. So although the lines might be sitting in P2 or sitting in P3, the lines themselves are one-dimensional. So I'll call projectivities between them 1D projectivities. Even if they're existing in P3. So I want to start this investigation with a simple puzzle.

Namely, given a line L and a line, a line capital L and a line little l, and points capital A and capital B on big L, and points little a and little b on little l, can we construct a projectivity from big L to little l, which sends a to a, and which sends b to b? So pause the video for a second and see if you can do that. For the time being, you can imagine that both the lines L are just in the plane, the extended plane P2. Can you construct a projectivity sending a to a and b to b? Try and do it, and now let's see how it's done. Well, we want to send a to a and we want to send b to b.

So the simplest way, let's just draw those lines out and see where they meet. They meet at this point which we'll call o. And if we imagine a perspective through o from L to L, where is it going to send a and where is it going to send b? Well, it'll send capital A to little a and it'll send capital B to little b. So we can do this with a single perspective, which we'll call fo from L to L, centered at the point aa intersect bb. But we're assuming here that capital L and little l are coplanar.

If capital L and little l are any two lines in P3 and not necessarily coplanar, like let's imagine that they're skew, then aa and bb may not meet. They might just be skew lines. And then this construction simply won't work. So let's try that situation. Let's assume that these are any two lines in P3 now and not necessarily coplanar.

And in this case, a single perspective is just not enough. We're going to have to make a proper projectivity, a composition of perspectives. So this example gets a little more interesting. So again, I would say I just gave you a hint, but pause the video and see if you can do it yourself. Once you've given it a shot, we can continue.

So let's see how this is done. And we can actually have a little trick which will help us, which is to draw an intermediary line m containing capital A and little b. Once we have this intermediate line m, the reason this is helpful is that capital L and m are now coplanar because they share the point a. Any two lines that share a point are guaranteed to be coplanar. So now we have these two coplanar lines meeting at a.

And let's not try and do it all in one go. We want to map a and b to little a and little b. But let's use this intermediate line and do a sequence of two perspectives. So let's start with a perspective centered at O1, where O1 is just any point along the line bb. We want to map b to b, so let's draw this line bb.

And O1 is any point along that line. Where will the perspective f01 from L to m send capital A and capital B? Well, it's going to fix capital A because capital A is in the intersection of m and L. But where is it going to send b? Well, it's going to send it over to little b. So we've done half of our work. We've gotten b to little b.

And a is still fixed. So what's remaining? Well, now we have to somehow get capital A to little a. We have to somehow get capital A to little a. How are we going to do that? Well, the simplest way is just draw the line between capital A and little a. And pick any point on that line and call it O2.

So once we've picked that O2, let's take a look at the perspective f02 from m to little l. Where is it going to send capital A and little b? Well, it's going to fix little b because that's in the intersection of m and L. But where is it going to send capital A? It's going to send it to little a. So that actually does it. f02 pulls capital A to little a.

And b is already at b. So the composition f01 followed by f02 will take capital A to little a and capital B to little b. Now just as a remark, it's easy to extend a projectivity of lines to a projectivity of planes. So we're eventually going to want to move up to 2D projectivity. So let's just quickly see that here we have a projectivity of lines.

But we can always choose planes pi1 containing l, pi2 containing m, and pi3 containing little l. And we can choose in such a way that these centers, O1 and O2, are not contained in any of those planes. Always choose any planes like that such that these centers are not in any of the planes just before or after them. And then we automatically get a projectivity of planes. f02 composed with f01 will give us a projectivity of planes pi1 to pi3, which sends capital A to little a and capital B to little b.

So that'll come in useful, come in handy later on.