Our Mathematical Senses

The Geometry Vision

Prof. Vijay Ravikumar

Department of Mathematics

Indian Institute of Technology- Madras

Lecture-37

Video 8A: Projectivities sequences of perspectivities

So having established perspectives, I want to now take a look at how much freedom they give us in manipulating images. So the concept of a perspective appears to nicely capture the idea of a perspective shifting map. The this perspective view maps to this perspective view under a perspective. And here's another way of seeing it, another viewpoint. But we have to be a little bit careful. Does it actually capture every perspective shifting map that we might want to capture? So here's a slightly trickier example.

Here's a plane. Here's one perspective view of these railway tracks. And here's another perspective view of the same railway tracks. So in other words, we have two different perspective images.

And can we create a perspective mapping the perspective view in pi prime to the perspective view in pi double prime? In other words, can we construct a perspective from the plane pi prime to the plane pi double prime, taking this perspective view to this one? And in particular, where would its center of perspective lie? We have to choose a specific center point in space that will relate points in this image here to points in this image here. For example, it'll have to relate this side rail image to this side rail image. But if we try and do that, we're quickly going to run into a difficulty. This line and this line are actually skew in space. They're skew lines.

And as a result, they're not going to be coplanar. No plane is going to contain them. And hence, they simply cannot be related by a perspective. Remember that any perspective at any point, if you look at where it's going to map this line here, well, it's going to form a plane. The line here and the point O together are going to form a plane in space.

They're going to span out a plane. And the image of that line is going to lie somewhere in that plane, wherever the plane hits this image plane pi prime, double prime. So these two lines would have to be coplanar if they're related by a perspective. So we have a problem. We simply can't relate pi prime to pi double prime by a perspective that takes this to this.

But they can be related by a sequence of perspectives. Namely, we can bring in this intermediary ground plane pi. So we can map the image here down to the ground plane in pi and then up to this second picture plane pi double prime to get from this image to this image. So this point here would map down to this point here, which would then map up to this point here. So via two perspectives, we can do it.

So we need a new definition, a projectivity, like projection. And a projectivity is a bijective map from a plane pi 1 in P3, like this one here, to a plane pi n in P3, like this one here. So a bijective map from pi 1 to pi n is known as a projectivity if it can be constructed as a composition of a finite number of individual perspectives. Which are centered at distinct points, possibly distinct points, o1, o2, up to o n minus 1. So in this image here, we have a perspective centered at o1, mapping from pi 1 to pi 2, followed by a perspective centered at o2, which takes the plane pi 2 to the plane pi 3.

And it takes this point to this point, followed by a third perspective centered at o3, which maps this point in pi 3 down to this point in pi 4. And in fact, maps the entire plane pi 3 to the plane pi 4. So this composition in this image, in this example, there's a composition of three perspectives, mapping the plane pi 1 to the plane pi 4, and taking this point here to this point here. So notice that each of these perspectives is a bijective map, since these are extended planes in pi 3, in the extended space pi 3. So each of these is individually a bijective map.

So we're looking at the composition of three bijective maps, which will clearly be bijective then. But of course, if we restrict our map to ordinary planes in the ordinary Euclidean space R3, then none of these are going to be bijective. They're each going to have lines where they're not defined. And the composition is going to fail to be defined on multiple lines within pi 1. But thankfully, the extended version is bijective.

So that's what we're going to focus on, the extended version of extended planes in pi 3. Now let's look at a different example, because this is a very special, important example that is going to keep coming up, which is projectivities from a plane pi 1 back to itself. So in some sense, it's circular. We're looking at a perspective from pi 1 to pi 2, a perspective from pi 2 to pi 3, but then a perspective back from pi 3 back to pi 1. So the composition of three perspectives in this case gives us a map from pi 1 to pi 1.

It's a transformation of pi 1. So the set of projectivities from pi 1 to itself is especially interesting. And the reason it's interesting is it not only captures, it'll turn out to not only capture all our perspective shifting maps, but we're also going to recover many, many familiar transformations of the plane, like isometries, dilations, and shear maps, basically all the maps you would have studied if you've taken linear algebra. So it's a bit challenging to visualize this at first with planes in three dimensional space, with lines in infinity, et cetera. So to help us out, let's actually go down a dimension.

And let's understand projectivities between lines. So forget planes completely for a moment. Let's just look at projectivities between lines.