

Our Mathematical Senses

The Geometry Vision

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Lecture-30

Video 6F: Lifting Desargues's theorem from the plane

Okay, but what if the two triangles are coplanar? How do we prove that Desargue's theorem still holds? So here's a diagram showing a representation of two triangles, capital A, capital B, capital C, and little a, little b, little c, which are coplanar. They lie in the same plane, but they're in perspective from this point o. So that's all in one plane. Can we somehow invoke the separate plane version of Desargue's theorem that we just proved in order to prove this coplanar version? Well, yes, we actually can. So let's imagine that this plane that they lie there on is the ground plane in P^3 . So this diagram I just drew, we're just putting it on the ground, on the floor, in P^3 .

And let's imagine, yeah, that they're both on the ground plane, and I'm going to give one way of lifting them up into separate planes. So we're going to construct associated triangles, we're going to construct new triangles that are almost like these, but are lifted into separate planes, so they're no longer coplanar. So to do that, let's just pick any point P that's off of the ground plane. And we can drop a line from P to the point capital C, and let's just lift capital C along that line to some point C prime, somewhere on that line.

Doesn't matter where. So I'm defining a point C prime somewhere along this line. I'm just pulling it up from, via this point P. Now I'd like to do the same thing with this little c here, this other triangle. I'd like to also lift it out of the plane, because, notice, we have a problem here.

We want our two triangles to be in perspective from the point O. Well this new C prime is not in perspective from, is not associated to any other point via O. So we have to lift little c up, but we have to do it a little cleverly, so that it's related to C through O. It has to be in perspective from the point O with respect to C prime. In other words, we have to choose C prime to be right here, so that O projects C prime to little c prime.

So we had a lot of freedom in choosing our capital C prime, we don't have any freedom in choosing our little c prime. But now that we've lifted these points, little c and capital C to little c prime and capital C prime, now we can lift our triangles. So let's consider these new triangles, capital A, capital B, C prime, and little a, little b, little c prime. These are two triangles that live in P3, and they're non-coplanar. They don't share a plane.

So we can apply the Desargues little a, capital B somewhere, little a, little c prime will meet capital A, capital C prime somewhere, and little b, C prime, and capital B, capital C prime will meet somewhere. And moreover, these three somewheres all lie on a single line, they're all collinear. And that line is the axis of perspective for these two triangles. So from our Desargues's theorem that we just proved, we have this. These two triangles are in perspective from this line.

Now we haven't finished our proof yet, we want to show that these original two triangles are in perspective from a line on the ground plane. This axis here is probably not on the ground plane. And it's also not the axis of perspective for our original triangles. It's only the axis of perspective for these lifted triangles. So what do we do next? How can we finish our proof? Well, one idea, maybe we can project this axis, this line, back to the ground plane via our projection point P, that same point P.

So let's try that. Notice that in our axis, this point in particular, capital A, capital B, intersects little a, little b at this point here, the two yellow lines intersect here. Well, this is going to be in the ground plane because both this yellow line and this yellow line lie on the ground plane. A and B and little a and little b were never lifted. So these lines are in the ground plane, their intersection is in the ground plane.

This axis goes through the ground plane over here. So the projection of the axis isn't going to touch this point, that's going to stay there. But the other two points are going to get projected somehow. So it's going to be a new axis, but it's going to be in the ground plane. I claim that this new axis is the axis of perspective for our original two triangles.

So as an exercise, can you verify that this projected line must be the axis of perspective for the original two triangles? When we project our new axis back down to the ground plane, can you prove that we get the axis of perspective for our original two triangles? So see if you can do that, and good luck.