Our Mathematical Senses

The Geometry Vision

Prof. Vijay Ravikumar

Department of Mathematics

Indian Institute of Technology- Madras

Lecture-29

Video 6E: Proving desargues's in three dimensions

Okay, so let's prove Desargues' theorem in 3D, in three dimensions, using some of the insights from that shadow drawing exercise. So let's first reexamine this whole axis of perspectivity. What does it actually represent? In the 2D version, it might feel a bit abstract, but if the two triangles lie on different planes in P3, then the axis has a much nicer interpretation. It's the edge where the plane containing one triangle meets the plane containing the other triangle. It might be helpful to shade in those planes, so the plane containing this triangle is going to meet the plane containing this triangle at this edge here. So that's the axis.

It has a very simple, clear geometric interpretation. One question you might ask, what if the planes don't meet in a line? What if the planes are parallel? We just said that the two triangles lie on different planes. What if those are parallel? Well, as we discussed earlier, or maybe you had a chance to think about it a little, if they're parallel, because we're in P3, those planes will still share a line in common at infinity. The line at infinity associated with this plane is the same as the line at infinity associated to this parallel translation of the plane.

So they'll still have an edge in common, and that edge is still going to be the axis of perspectivity. Keeping this interpretation in mind, proving most cases of Desarck's theorem becomes much easier. Namely, if two non-coplanar triangles, non-coplanar just means that they're in different planes, so if two triangles in different planes in the extended space P3, if they're in-person, if they're in perspective from a point, then they're also in perspective from a line. So I've modified my original statement of Desarck's theorem to move it into P3 and added this extra condition that the triangles are not coplanar, they're not in the same plane. So in this case, let's prove Desarck's theorem.

Why are these triangles in perspective from a line? Well, notice that the two yellow lines are coplanar, because this point capital A, capital B projects from this center of perspectivity to this line little a, little b. So therefore, they're coplanar. And therefore, they must intersect somewhere, again, due to the fact that everything here, the plane that they share is an extended plane, and therefore, any two lines intersect somewhere. So this line and this line have to intersect somewhere. So let's just mark that point here.

Similarly, well, yeah, and this is because the yellow plane, what I'm calling the yellow plane is the plane determined by capital A, capital B and little a, little b, that's an extended plane. And this point of intersection will have to lie on the edge where the two planes meet. This line a, b lies on this plane. This line little a, little b lies on this plane. So since little a, little b and capital A, capital B intersect somewhere, that point of intersection has to be on the intersection of this plane with this plane.

In other words, it has to be somewhere on this edge. So let's mark it here. Similarly, with the blue edges, the two blue edges are coplanar because this one is the projection of this one from the center of perspective. They'll have to meet somewhere and the place where they meet will have to be on the intersection of these two planes again. So it'll be somewhere that will mark here.

And for the red ones too, capital E, capital C, little b, little c will be coplanar, the two will have to meet somewhere and that point where they meet will be on this edge where the two planes meet. So that kind of does it. The edge where the two planes meet, we've just shown, serves as an axis of perspective for the two triangles. It contains the three points of intersection of the three pairs of corresponding lines. Putting it another way, a line drawn on a glass wall will meet its shadow when it reaches the bottom edge where the glass meets the floor.

If you draw this red line here all the way down to the floor, imagine its shadow extending, extending. This line will meet its shadow at this point, corner point here where the glass plane meets the floor plane. This yellow line will meet its shadow here. This blue line will also meet its shadow here where the glass plane meets the floor plane. So each edge of the triangle will meet its shadow somewhere along the glass plane, floor plane axis.