Our Mathematical Senses

The Geometry Vision

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Lecture - 24

Video 5D: Coincidence #3 Desargues's theorem

So the third coincidence that I want to look at is Desargues' Theorem, which you saw in the intro video but which I'll quickly review here. So two triangles, capital A, capital B, capital C, and little a, b, c in the extended plane P2, are said to be in perspective from a point if there's three lines, capital A, little a, capital B, little b, and capital C, little c, which are concurrent, if those three lines are concurrent at a point. So if this line, this line, and this line are concurrent at a single point, which we'll call O, then we say that these two triangles are in perspective from a point. In this case, we call O the center of perspective. So that's our first definition. Now there's another definition that we have to make before we state the coincidence, the theorem.

So going back to these same two triangles, we'll say that they're in perspective from a line if this time, instead of drawing the lines between corresponding points, which we did last time, let's draw the points between corresponding lines. In other words, let's extend a, c, and a, c. Oops, sorry. Let's extend a, b, and a, b to get this point here.

Let's extend b, c, and b, c to get a point here. And let's extend a, c, and a, c to get a point here. This gives us three points, and our definition that I'm making is that these triangles are said to be in perspective from a line if these three points a, b intersect a, b, b, c intersect b, c, and a, c intersect a, c. If those three points are collinear, then we say that the triangles are in perspective from a line. And in this case, we call this line the axis of perspective.

So that's the second definition that we need in order to state Desargues' theorem. What does Desargues' theorem say? Well, two triangles in the extended plane are in perspective from a point if and only if they're in perspective from a line. In other words, these two definitions, these two statements about triangles are equivalent. So here's an image of two triangles perspective from a point. Desargues' theorem is saying that they're in fact also in perspective from a line.

If we draw these corresponding sides, extend them, and extend them, the points of intersection will be collinear. And similarly, if we start with triangles that are perspective from a line, we'll see that in fact these lines a, a, b, b, and c, c are concurrent. They'll intersect a single common point. So I just want to give a quick warning. So far, we've only discussed triangles in the extended Euclidean plane P2.

We haven't actually discussed this triangle's lying in a three-dimensional space. Everything is taking place in the extended plane P2. But this picture looks, it's easy to imagine that the image represents triangles sitting in space. And in fact, if you saw the intro video, which you should see, we introduced it as this idea of a center of perspective as a light source projecting one triangle onto its shadow. So we introduced the idea via triangles in space.

So there's a good reason for that, and we're going to get to that in the next lecture. But for now, the statement of Desargues' theorem, the inspiration came from space, from projections in space. But the statement of the theorem and the theorem itself is fully about triangles in a plane. And for the moment, that's what this image is. It's just two triangles that are coplanar, that lie in the same plane.

And they are not, this is not a perspective view or an image, a projection of triangles in space for the time being. But there is a good reason why it looks like that. And yes, we'll come to that and settle this a little more in the next section. So yeah, so we'll get to that next. Right now though, there is something else that we need to think about, just when we're concentrating on, when we look at the setting as being P2.

So in particular, why does it not hold in R2? Why do we need these points at infinity, and where do they enter the picture? Well, just like with the first and second coincidences, we do need them. So let's see a quick example, which makes it very clear why we need points at infinity, to state Desargues' theorem. So here is another configuration, where ABC and ABC clearly look like they're in perspective from this point. By Desargues' theorem, they should be in perspective from a line. So let's extend our corresponding lines.

AB intersect AB gives us this point. AC intersect AC gives us this point. BC intersect BC. Well, uh oh, we have a problem here. Where is the axis of perspective? Where is the intersection between BC and BC? Well, BC intersect BC.

Since they're parallel, it's going to lie at infinity. It will exist, but only because we're in the extended Euclidean plane. And as a result, what is the axis of perspective? Well, it'll go through these two lines. So if it exists, it better go through these two lines. But it's supposed to also intersect the point of intersection of BC and BC.

Will it do that? Well, actually from the picture, yes. This line is parallel to these lines. So it will intersect these lines at their common point at infinity. In other words, it will intersect, all of these lines will intersect over at the point at infinity PBC, where BC is the line connecting B and C. So Desarque's theorem does hold, it's just that this axis of perspective will go to, one of the intersection points will lie at infinity.

Okay, so Desarque's theorem is forced to live in P2 in order to make sense. But now that we're here in P2, in the extended Euclidean plane, well then in the statement of Desarque's theorem, any point can actually lie at infinity. So that includes the center of perspective. So what would that look like? Can we draw two triangles, capital ABC and little abc, that are in perspective from a point at infinity? So the lines capital A little a, capital B little c, B, and capital C little c, they're all going to be concurrent at a point at infinity, meaning they're going to look parallel to each other. So our triangles are actually going to be related in this way.

The lines connecting corresponding vertices are going to be parallel, and this will be our Desarque's configuration. These parallel lines will all be concurrent at this point at infinity. So we can get configurations that look quite different once we start involving points at infinity. And in the restriction to R2, that we'll really see these triangles as looking very similar and being related by this kind of parallel projection. Another example, another way we can get new interesting looking configurations is to have other points lie at infinity.

Any point can lie at infinity, so that includes vertices of any of the triangles. So remember, to a projective geometer, this is a legitimate triangle in P2. It has vertices A, B, and C. It so happens that C lies at infinity, so from the top-down view we get a configuration, an image that looks like this.

But that's okay. This is a perfectly good triangle for a projective geometer in the extended Euclidean plane. So here's a point O, and as a small exercise, can you try drawing a triangle, little a, little b, little c, such that the two triangles, capital A, capital B, capital C, and little a, little b, little c, are in perspective from the point O. So can you draw a second triangle, little a, little b, little c, such that O is the center of the perspective relating those two triangles, such that they're both in perspective from the point O. So I'll let you think about that.

It's an exercise. But to help you think about that, I want to give an important caveat concerning triangles, because things might be getting a little confusing with these weird looking triangles. So in Euclidean geometry, a set of three non-collinear points carves out one closed region in space. If we have three points, a, b, and c, in the ordinary Euclidean plane, in R2, they're going to determine three lines like this, and those lines are going to carve out one closed region in space. The other regions are all kind of open.

They're not closed off. You can just keep going and going and going. There's one closed off region with a boundary. In projective geometry, things are a little weirder. Given a set of three non-collinear points, well, are they still going to carve out a single region in the plane? They'll certainly carve out this region here. But remember, if we take a line, say we take this line here, let's look at a line c.

That's supposed to be a line. It's going to go off to a point at infinity. Let's call this line l. Here's a point at infinity pl. We have the same point at infinity pl over here. So if we start at a point here and go on a journey along this line l, well, we can eventually get to another point here.

So this region and this region are connected like we saw in a previous video. So this is all a second region. And moreover, it's also a region that's kind of bounded. In particular, you can see the boundary by walking up along this line, you come back here and you get to here.

So this line is one boundary. On the other hand, if you go on a walk along this line, you'll come to here, keep walking, walking, walking, and get to this line, and get to this point. So this is another boundary. And finally, we have this boundary here. So all in all, this is like a well-bounded closed region. And similarly, we have a region here and we have another region here.

So we've carved out P2, our extended Euclidean plane, into four distinct regions. One, two, three, four. So there's four closed regions in this extended plane. So when we talk about the triangle ABC, we're going to let the word triangle refer specifically to a collection of three points, non-collinear points of course, and not to a particular region carved out in the plane. So triangle ABC refers to the above configuration, but not to a particular region like this one or this one or this one.

So it's a bit of a change because earlier in Euclidean geometry, we're used to triangle, we are a bit ambiguous. Sometimes by triangle, we mean these three points. Sometimes by triangle, we mean this region. From now on, triangle refers to a collection of three non-collinear points or equivalently non-collinear, non-concurrent lines that go along with those points, but not to a particular region. So this triangle ABC refers to the above configuration and again, not to any of the particular regions carved out by this configuration.

And maybe as a little exercise, can you see why this above configuration also divides P2 into exactly four regions?