Our Mathematical Senses

The Geometry Vision

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Lecture- 19

Video 4D: The shape of this exended plane

So now that we've built this extended Euclidean plane, let's try and take a closer look and understand the shape of it. How does it fit together? What does it look like? And let's just refresh our memory. So P2 is the extended Euclidean plane. And it's just, as a set of points, it's just all the ordinary points in R2, along with all these new points at infinity that we've added, one for each family of parallel lines. But how do we imagine this strange collection of points, and how do we work with it? So I'm going to just represent visually R2 by this grid here. Actually, this represents R2.

One approach is to just say, okay, let's not worry about how it looks or how we can imagine it as a whole, because anyway, it might be very counterintuitive and we might get tripped up over our own preconceived notions. So let's just work with the incidence relations. Let's just work with those incidence relations and definitions directly and not worry too much. And there's some merit to that approach.

But on the other hand, you also feel like, okay, really, can't we just try seeing it or imagining it? What can we say about it? So I want to talk a little bit about how we can view the extended plane. And one thing that immediately jumps into the picture is that we've actually been seeing points at infinity all along so far. Whenever we view the plane, whenever we view this plane R2 in perspective, we've already been looking at points at infinity. So clearly we can visualize it to some extent and we do that using perspective. So let's view the points at infinity of several families of parallel lines and see what they look like.

So here's two families of parallel lines in R2, the blue vertical lines and the red horizontal lines for this picture. How do those look in this perspective view? Well, they

look like this. And we see very clearly that the blue lines all meet at a point at infinity here. The red lines all meet at a point at infinity over there. And here's another family, the green diagonal lines.

 And can we see those in perspective? Well, yeah, of course we can. And if we draw them in this picture, they all meet at a point at infinity over here. So we can see yet another point of our extended plane very clearly. And there's another family of diagonals given by the other diagonals, another family of parallel lines. What do those look like in perspective? Well, they're these purple lines here and they give us another point at infinity.

 So that's four points. And as a little exercise, see if you can identify any more families of parallel lines in this picture using the corner points of the tiles. There are of course many, many more. Can you identify any more yourself? So okay, this is one step towards viewing the extended plane, but here's a good place to mention a certain warning because our intuition can be misleading at times. So let's notice that in order to build our extended plane, we actually added exactly one point at infinity for each family of parallel lines.

 So the point at infinity in this direction, if we look at these blue vertical lines, imagine that whole infinite family, the point at infinity up in this direction, which is right here, is the same as the point at infinity in this downward direction, which you see down there. So we get to the same point at infinity going up as we do going down, which is a little counterintuitive. And the reason for that, why can't we add two points at infinity, one for each direction, you might think? Why can't we add a PM up here and a QM down there? And if we were to try it, let's see what would happen. Unfortunately it doesn't look too good though. If we were to try it, remember we're trying to build a linear space.

 And in a linear space, two points determine a line, a unique line. Well, how many lines are there between PM and QM? Well, between these two points, there's going to be an entire family of parallel lines running between them. All of the lines in this vertical family would hit PM up there and QM down there in this imaginary situation. And that's no good. Rather than one unique line between them, there's infinitely many.

 Therefore, the point at infinity over there must be the same as the point at infinity over here in our extended plane. PM up there and PM down here. Whichever direction you go, you'll get to the same point at infinity, PM. And in particular, this tells us something kind of interesting or disturbing or however you want to see it about this line at infinity, which is that it's not a wall. There's no boundary or end to the extended plane.

And you can go up through a point at infinity, pass through it, and keep coming back and come back to where you started. There's no end to things. You can just keep going and going and get back to where you started. So a consequence of this is that lines in P2 behave a bit like circles. Like I said, you go far enough along and you get back to where you started.

So another warning is that, okay, fine, our lines are kind of like circles. But it's actually not that easy to visualize how all of these different circles patch together to form the entirety of P2. Perspective views give us little local glimpses of portions of P2, which are very useful to see. But the global shape of P2 is legitimately strange and hard to wrap your mind around. So let's do a little experiment demonstration to see that.

So for example, what happens if we subtract an entire line, including its point at infinity, from P2? In particular, what is the remainder of P2 minus the line L look like? What's left over when we subtract a line L? So here's a schematic diagram of, well, here's P2. Here's a line L in P2. This is kind of just a diagram of P2. And my line L meets this point at infinity PL in this direction, and the same point at infinity PL in this direction. So what happens when we subtract L? Well, remember, L is actually a closed loop in some sense.

You keep going and going, and you come back to where you started. Both ends meet in the point at infinity PL. So you might expect that L separates P2 into two distinct regions, two chunks. And therefore, that P2 minus L should consist of two disconnected pieces. That seems like a reasonable guess.

You might think there should be a region A and a region B. When we subtract L, we split up P2 into these two regions. So let's examine this. In fact, it's not the case. And to see that, let's draw another line M.

M intersects L at some point here. And let's imagine that we've subtracted L from P2. So imagine now we travel along M. Let's say we start at this point here, and we travel, travel, travel, travel, travel, travel, travel, get to the point at infinity PM. Now we're down here, and we keep traveling, traveling, traveling, and eventually we get to this point here.

So maybe I'll draw it in a color. So you start at this point, travel, travel, travel, travel, travel, travel, get to this   point.   What have we done? We've gone from region B to region A via this path along the line M. So since M provides a path from region A to region B via that point at infinity, P2 minus L is in fact connected. You can get between these via this path.

So you subtract this circular line, but you don't actually break up P2, which is a little weird, maybe a little hard to imagine. It's actually just one region. P2 minus L is just a single region. Now as an exercise, we've seen that a single line does not divide the extended plane into separate regions. But what about two lines? If we subtract both L and M from P2, how many regions are we left with? So as a hint, try starting somewhere in your region and traveling along a line.

Maybe draw some more lines through this center point here, travel along those lines, and see if you can get from one region to another. Can you get from any region to any other? Maybe they're all one region, or maybe there are some distinct regions. Maybe it's not four regions, like it looks like here, but how many regions are there? So that's an exercise for you to try. Now I want to make a final observation, which is that we'll often draw configurations of points and lines in P2 via their restrictions to R2, where the points at infinity won't be visible to us. We'll just have to imagine them off in the distance.

We'll basically see P2 from this bird's eye view of R2, not from a perspective view. So here are three lines in P2, where we're imagining P2 as R2 plus some points at infinity. We're not doing any weird perspective view business. So how many points do these three lines determine? How many points of intersection are there that are determined by these three lines? Well here's one, and here's another. So clearly there's at least two points of intersection, but these two lines are parallel, so if you imagine them going and going and going and eventually meeting at infinity, they're going to share a point at infinity, and that's going to give you a third point of intersection.

So these three lines actually determine three points. One, two, three. And this shows that even very basic configurations of points and lines, like three lines which intersect in three points, which you might call a triangle, can look surprising when they involve points at infinity. And indeed, to a projective geometer, this configuration here, which is shown here, is just simply a triangle with one vertex that happens to lie at infinity. So for a projective geometer, this is just another triangle.

And maybe one thing that you'll do in the homework is try and visualize other strange ways of drawing triangles in P2 with other selections of points at infinity.