Our Mathematical Senses

The Geometry Vision

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Lecture - 18

Vidoe 4C: Extending the euclidean plane

So now let's put these ideas together and work on extending the Euclidean plane. This is where things get interesting. So in order to understand what changes and what stays the same under a perspective shift, we want to build a geometry of the plane that's based entirely on incidence relations. But there's another important ingredient here, which is points at infinity. We've seen that vanishing points can appear out of nowhere almost when we shift perspective. That's because they're the images of points at infinity, which don't exist within the Euclidean plane, but are somehow attached to the Euclidean plane.

So our new geometry of the plane must somehow include points at infinity. So let's extend the Euclidean plane to include points at infinity. And in other words, let's create a larger linear space that includes the Euclidean plane as well as the points at infinity. And to do this, we'll just start adding new points.

How many new points do we need to add? Well, every family of parallel lines has a point at infinity associated to it. So we're going to have to add one point at infinity for every family of parallel lines. So how do we do this in a way that's well-defined, that's precise? And what do we even mean by a family of parallel lines? Well, given a line L in R2, let's let bracket L denote the set of all lines in R2 that are parallel to L. All of these guys are parallel to L, all of these guys, all of these guys. So all of the lines in R2 that are parallel to L, all together, let's denote the set of all of them by bracket L.

It's a very, very, very big set, infinitely large. This set is a family of parallel lines. That's what we mean when we talk about a family of parallel lines, the entire family. The relation of parallelism partitions the lines of R2 into distinct disjoint subsets. A line in R2 will belong to exactly one family.

It can't belong to this family and also that family. All of these parallel lines will form one family. On the other hand, all of the lines that are maybe in this, that are parallel to this guy, that'll be another infinitely large family, but it'll be disjoint from this one. And in this way, the set of all lines in R2 is partitioned into many, many, many, infinitely many distinct disjoint families. And this is because parallelism is an equivalence relation, if you know what that is.

But we don't need to know what that is for what we're doing. So bracket L is defined to be the set of all lines in R2, M within R2, such that M is parallel to L, set of all lines parallel to L. For each such family bracket L, let's define a point at infinity, PL, which I've drawn here. Here's my family. And I'm just adding in a point at infinity associated to that family.

And I'm going to declare that PL is incident to all lines M in this family and only to those lines M. So it's incident to this, to this, to this guy, to every guy in this family, every line in this family, PL is going to be incident to all of those. I'm declaring that incidence relation to be true. But it's not going to be incident to any line that's not in that family. Now, note that every single line in R2 is going to meet exactly one point at infinity, the one associated to its unique family.

It won't meet any other point at infinity. So the question is, is this a linear space? Does it satisfy these most fundamental axioms that we want to make sure are satisfied? Does the collection of all points at infinity along with Euclidean plane R2 form a linear space? Well, is L1 satisfied when... So L1 says that two distinct points have to determine a unique line.

Is that satisfied when one of these points is at infinity? For example, here's a point at infinity. Here's an ordinary point. So Q and PL, is there a unique line between them? Well, yes. It's just the line in this family that runs through Q. Remember, given a line at a point not on the line, there's a unique line through that point that's parallel to our original line.

So we can definitely draw a parallel through Q to this line L. And that is going to hit PL. So yes, and that's the only line through Q which is going to be in that family. Every other line through Q is going to be in a different family. It's not going to be parallel to L.

So yes, L1 is satisfied. What about for two distinct points at infinity? Let's say we take PL and PK for some other line K which is not parallel to L. In this case, is there a unique line connecting PK and PL? Well, we have a problem here because no ordinary line can connect distinct points at infinity. Every ordinary line hits exactly one point at infinity.

So we're going to have to add new lines to our system.

We don't want to add too many new ones, so let's see how efficiently we can do this in order to satisfy this axiom that any two points determine a line. Well, it turns out we just need one line. And let's just do this in the kind of simplest possible way we can think of. Let's make all the points at infinity collinear. Let's define a line at infinity, L infinity, which is just the collection of all the different points at infinity.

It's going to contain all the points at infinity. And let's now define this extended Euclidean plane. So this line at infinity, it's going to be the unique line that contains both of these guys. It's going to connect them, and L1 will be satisfied. Now that we have that, let's define the extended Euclidean plane, which we'll denote P2.

And it's just defined to be R2 union L infinity with all of these incidence relations we've added, that every line will be incident to the point at infinity that's associated to its family. And there's a line at infinity connecting all of the points at infinity. So this is a linear space.