Our Mathematical Senses

The Geometry Vision

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Lecture - 17

Video 4B: Linear spaces

So, there's another fundamental concept we're going to need known as a linear space. So as we're trying to build the geometry of incidents, the geometry of points and lines, and there are some very fundamental incidence relations that we want to make sure that these points and lines satisfy, certain basic axioms of incidents, you might say. So there's two axioms in particular, which I'm calling L1 and L2. L1 states that any two distinct points are incident with exactly one common line. We saw that this was true in R2, and generally we want this to be true of any geometry we build of points and lines. The second axiom is that every line is incident with at least two distinct points.

Every line contains at least two points. That's all it means. It seems almost trivial. Why would a line contain less than two points? Most lines we're used to dealing with contain infinitely many points.

But the thing is, we want to build a general enough framework that we could even consider finite sets of points. So geometries of finitely many points. So this axiom is actually important because it says there's enough points that every line is interesting. So mathematicians define something called a linear space, which is just a collection of points, lines, and incidence relations that satisfy these two axioms. So it's like the most basic entity you could create consisting of points and lines that has some interest because it satisfies these most basic axioms.

That's kind of the idea. So linear spaces can be finite or infinite. So let's just see some examples to get a better sense of what they are. So a question is, can we describe a linear space with exactly five points? Forget infinitely large spaces. Let's just go and try and make a really small space and see what it might look like satisfying these axioms.

So here's an example of a linear space with five points. One, two, three, four, five. And does it satisfy these axioms? Well, every two points seems to have a line between them. And if you're familiar with some graph theory or you've seen the complete graph on five vertices, you'll immediately know that, yes, there is an edge between every two vertices here. So this is certainly satisfied because every two points has a line connecting them, and exactly one line connecting them.

How many lines are there? Well, since there's five points and every pair of them has a line, there's five choose two, which is equal to 10. So there's 10 lines in this picture. And of course, every line has at least two distinct points. We don't have any lines that just have a single point. So this is a very simple example of a linear space.

Are there any other linear spaces consisting of exactly five points? Let's explore that just to get a better sense of how much freedom we have when we construct a linear space. So can we create a linear space on five points with less than 10 lines? So one strategy is to make three or more points collinear. And here's an example of that. If we make these three points collinear, we have one line connecting these three. We have to make sure every two distinct points has exactly one common line.

So these two need a line, these two need a line, these two need a line, these two need a line, these two need a line, oh, and these two need a line, and they need a line between them. So all in all, there's eight lines. And yes, every line here has at least two distinct points. This line even has three. Are there any more linear spaces on five points? What if we make more points collinear? So for example, we can make four points collinear.

This gives us these four share a line. There's one point off of that line. And it gives us four more points for a total of five, four more lines for a total of five lines. Or we can make all five of them collinear. Then there's just a single line.

That's a very simple linear space. Are there any others? Well, as an exercise, see if you can construct a linear space on five points with exactly six lines. Now, a more familiar linear space, which is going to be much more useful to us and much more pertinent to our exploration here, is especially because we want to know how the Euclidean space behaves under changes in perspective. So our whole reason for bringing in linear spaces is that we want to see things that we're used to working with, like R2, do indeed form a linear space. So let's check that R2 forms a linear space.

So any two distinct lines are incident with exactly one common line. Yes, that's true in R2. We know that. Any two points determine a line. And secondly, every line in R2 has to be incident with at least two distinct points.

Every line has to contain at least two points. In R2, does every line contain at least two points? Yes. Every line contains infinitely many points. So in particular, it'll contain more than one point, two or more points. So this L1 is satisfied, and L2 is also satisfied.