Our Mathematical Senses

The Geometry Vision

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Lecture - 15

Chapter Two: A Geometry of Coincidence

Welcome to week two of the Geometry of Vision, where we'll try to actually get our hands on these points at infinity. What are they, and where are they? Are they mathematical abstractions like the imaginary numbers, or do they actually exist in some sense? Last week, we saw that when we're drawing a picture or taking a photograph, vanishing points tend to emerge on our picture plane, seemingly out of nowhere. This is because a vanishing point is not the image of any existing point on the ground plane, but rather the image of a point at infinity, which doesn't actually exist on the ground plane at all. And yet, despite apparently not existing, the images of points at infinity, like that vanishing point over there, very much do exist and are very useful. They allow us to draw in perspective, and they even allowed us to solve the drawing challenges last week. So what if we start taking these points at infinity seriously and imagine that they actually exist? What kind of geometry would emerge? This week, we'll define points at infinity precisely, and in the process, expand our notion of the Euclidean plane.

We'll then look at some basic features of the geometry of this extended plane, known as projective geometry. We'll see that projective geometry is far more symmetric than Euclidean geometry, and full of surprising properties. To begin with, why don't we go for a train ride and try to take a closer look at that point at infinity. Over there.

We've seen how on any two-dimensional photograph, lengths, angles, and areas are completely dependent on our viewing perspective. And yet, nevertheless, our brains gather a great deal of geometric information from such images, like the fact that all three of these images represent square tiled floors. One response is to throw our hands up and say, okay, let's create a geometry in which lengths and angles do not matter. But what does geometry even mean in that case? Are there any geometric properties that continue to hold as we shift from one perspective to another? Well, as we shift perspective, points seem to remain points, and lines seem to remain lines. And even intersections between lines remain intersections.

So could we build a geometry entirely out of points, lines, and intersections? It would be a very basic geometry, but one that would be essential for knowing what changes and what stays the same when we shift perspective. There's a slightly broader term that'll come in very handy. Incidence. Incidence is just a fancy way of saying two objects meet someone. For example, two lines are incident if they meet at a point, a point of intersection, or at every point, which happens if the two lines are in fact the same line.

A point and a line are incident if the point lies somewhere along the line. And finally, two points can only be incidents if they're actually the same point. If we're going to build a geometry that makes sense under changes in perspective, then incidence relations between points and lines are pretty much all we have to work with. So the real question is, can we reach any interesting results with such few ingredients? But wait a second. Is it really true that points remain points under changes in perspective? After all, we've seen how some points, the points at infinity, seem to appear and disappear depending on our perspective.

So our geometry will also have to account for these points at infinity. Take that vanishing point over there. It seems just as out of reach now as it did back in the studio. And after all, it represents a point at infinity which can never be reached. Or can it? Maybe a point at infinity is just like any other point when viewed from the correct perspective.

In fact, something quite beautiful happens when we add points at infinity to our set of ordinary points that make up the Euclidean plane. Consider two distinct lines in this new extended plane. If they are not parallel, they'll intersect at a single ordinary point. But if they are parallel, they'll now intersect at a point at infinity. Moreover, there's now a full line at infinity, the horizon line, which intersects every ordinary line at a single point at infinity.

In this way, any two distinct lines now determine a unique point. These could be two ordinary non-parallel lines, or two ordinary parallel lines, or even two non-parallel lines, with the line at infinity. Moreover, any two distinct points continue to determine a unique line. Somehow by including points at infinity, we've discovered a surprising duality between points and lines. In fact, we're not at all crazy to imagine a geometry that includes points at infinity, but in which there are no lengths or angles, only points, lines, and intersections.

It goes by the name of projective geometry, and mathematicians have been studying it for thousands of years, discovering unexpected meetings of points and lines, which are also known as coincidences. Two of the most famous coincidences of projective geometry are Desarck's theorem, recorded in France around 1645, and the much older Pappus's theorem, which was recorded in Alexandria, Egypt around 340 AD. But who better to introduce these results than Gerard Desarck, the mathematician who first formalized projective geometry as a legitimate branch of mathematics? Bonjour! Welcome to the marvelous world of projective geometry! You must be wondering what interesting substance this geometry can possibly have, consisting as it does of only points, lines, and intersections. Let's go all the way back to Africa, the year 380. The mathematician Pappus of Alexandria recorded a remarkable coincidence in his critic's synagogue.

It's almost as remarkable as my own. Perhaps, Pangolzian can tell you about his theorem. She was another brilliant mathematician who also lived in Alexandria during the 4th century. She's the earliest female mathematician that we have any written a record of. Did I hear correctly that you want to know about Pappus's theorem? An excellent idea, but there is one thing I must ask.

Why does everyone refer to it as Pappus's theorem these days? It is true that he wrote it down and he preserved it, but truth be told, nobody knows where it came from originally. In my opinion, it is one of the simplest, oldest, most fundamental theorems in existence and has probably been discovered dozens of times in human history. The theorem concerns a special feature of points and lines chosen at random. It doesn't seem like this feature should occur. And yet, it does, no matter how the points and lines are chosen.

Let me explain. Draw any two lines on the plane. Now take any three points on the first line labeled A, B and C. And any three points on the second line labeled a, b and c. Now, let X be the intersection of the line A, B with the line B, A. Let Y be the intersection of the line B, c with the line C, b.

Finally, let Z be the intersection of A, c with C, a. Notice anything about X, Y and Z? They are collinear. They lie on the same line. There is no reason to expect the point Z to lie on the line determined by X and Y.

And yet, it does. However you choose the two original lines and the six original points, the point Z will always be incident with the line determined by X and Y, which is why we call this a co-incidence. You can even label the points in strange orders or put them on different sides of the intersection of the lines. No matter how you set it up, X, Y and Z will be collinear. Take a moment to try this out yourself. All you need is a pencil, paper and straight edge.

Maybe you can try giving an explanation of why this theorem holds. Many proofs have been discovered over the millennia, each interesting in its own right. But each and every proof requires a special insight that is far from trivial. Perhaps you can discover one of your own. Either way, you will soon get to see one of my favourite proofs.

Excusez-moi. A delicious appetizer that was. But now, for the main course, it all happened one fortuitous night. I, Gerard Dessac, was making my way home after an evening of revelry, when I passed by a street lamp and noticed a triangular street sign standing nearby. As my perspective shifted, the lengths and angles of this triangle and its shadow kept changing. And I found myself contemplating how an artist might go about drawing this shadow.

And then it suddenly occurred to me that a far deeper relationship between the triangle and its shadow than I had previously considered. Hahaha! There was, of course, the obvious way the shadow triangle and the actual triangle were related. The three lights determined by the corresponding vertices of the actual triangle and the shadow triangle all meet as the light source. In other words, they are concurrent at the point from which the light is emanating. In this situation, I would say the two triangles are in perspective from a point.

But there was another relationship, dual to the first. Rather than look at the lines determined by corresponding points, I looked at the points determined by corresponding lines. For example, the two red signs determine a point R, the two blue signs determine a point B, and the two yellow signs determine a point Y. And for some reason, these three points R, B, and Y are co-linear. You could say the triangles were in perspective from a line.

I went home, got my coin and my papillé, and began drawing furiously. Every pair of triangles that were in perspective from a point turned out to be in perspective from a line as well. I drew many examples and observed this property in every one of them. Hah! You must try it yourself to fully appreciate this mystery. Pois, why was this the case? Moreover, the converse was also true.

Whenever two triangles are in perspective from a line, they must be also in perspective from a point. As for the proof, well, here's a drawing challenge. If you can solve it, there will be a point. Take the following image. Now, using just a pencil and a straight edge, can you draw the shadow of the triangle exactly as it should actually appear? If you can manage this, you can suddenly prove at least one direction of my triangle.

Good luck! Au revoir! Not only do the theorems of Desargues and Papists continue to hold under shifts in perspective, they also work perfectly well in the form of a circle. The circle is a circle, and it is a circle. The circle is a circle, and it is a circle. They also work perfectly well in the form of a circle. In fact, they actually require the existence of points at infinity in order to fully make sense.

In this way, they're truly theorems of projective geometry. Our main objective this week is to carefully understand the basics of projective geometry. So let's get started.