Sobolev Spaces and Partial Differential Equations
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Maximum Principles – Part 2

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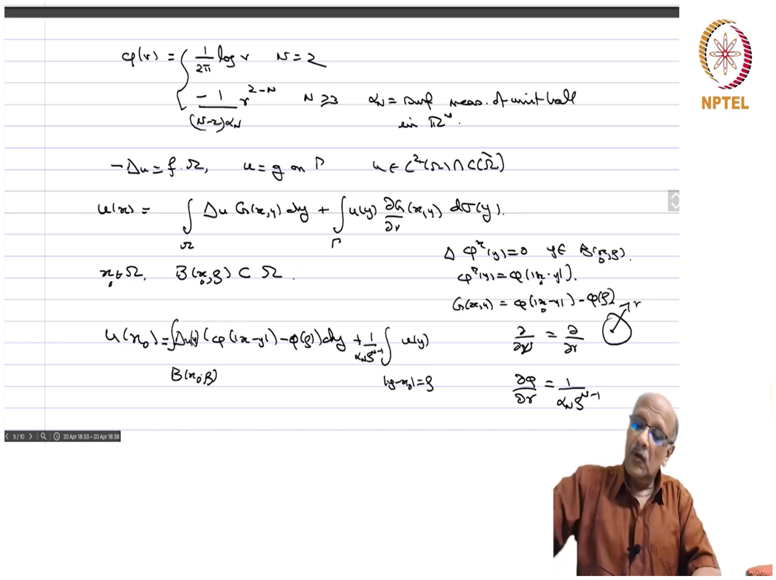

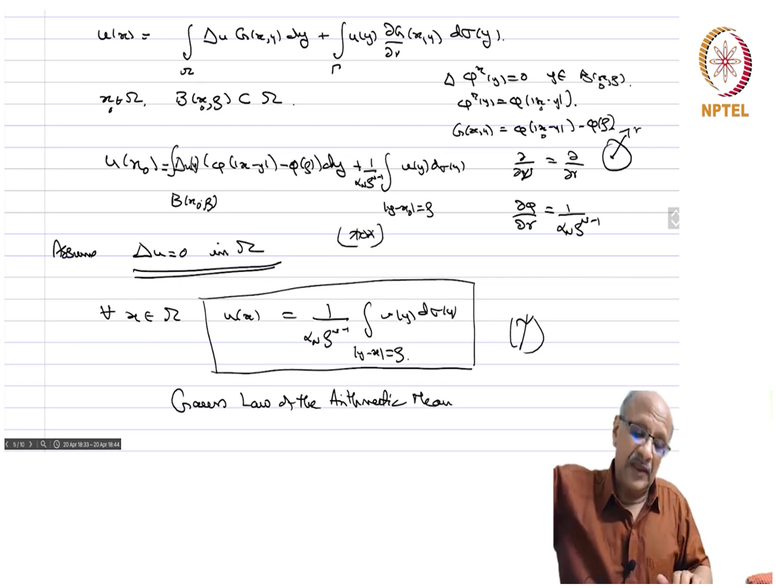

So now, we will recall what we did in the exercises so we take again

$$\phi(r) = \frac{1}{2\pi}\log(r), \; N = 2$$

and $\qquad \phi(r) = -\frac{1}{(2-N)\alpha_N} r^{2-N}$, $if\ N \geq 3$. Here α_N equals surface measure of the unit ball in $\mathbb{R}^N$.

And recall this is the fundamental solution of the Laplacian, so if you have that

$$-\Delta u = f$$

So, then, you have that $u = g$ on the boundary Γ, then we prove that $u \in C^2(\Omega) \cap C(\overline{\Omega})$. So then,

$$u(x) = \int_{\Omega} \Delta u\, G(x, y)\, dy + \int_{\Gamma} u(y) \frac{\partial G}{\partial \eta}(x, y) d\sigma(y).$$

So, we already saw the resist representation formula. So, then we now look at $x_0 \in \Omega$ and we take $\rho > 0$ such that $B(x_0, \rho) \subset \Omega$. And we apply this formula to $B(x_0, \rho)$, now what is the Green's function in $B(x_0, \rho)$.

So, if you have a ball, how do we define $\Delta \phi^x (y) = 0\ for\ y \in B(x_0, \rho)$ and $\phi^{x_0}(y) = \phi(|x_0 - y|)$. Now phi f mod x minus y if you are in B x, rho this is nothing but phi of rho, so this is just phi of rho and the Green's function is equal to $\phi(|x_0 - y|) - \phi(\rho)$. So, if $\Omega = B(x_0, \rho)$.

So, if x naught is in omega, then you take b x naught rho is this function here and therefore you will get that

$$u(x_0) = \int_{B(x_0,\rho)} \Delta u(x_0) (\phi(|x_0 - y|) - \phi(\rho)) dy + \frac{1}{\alpha_N \rho^{N-1}} \int_{|y - x_0| = \rho} u(y)\, d\sigma(y)$$

u of x naught equal to integral b of x naught, rho phi of mod x minus y minus phi of rho dy plus integral u of y. And then what is d by d xn? So, we have to d by d nu is nothing but d by dr because you have a ball and then r is the normal direction so d by d nu is nothing but d by dr.

And so if you put it there so then you get mod y minus x naught equal to rho and if you d by dr if you compute it is just 1 by alpha n rho to the n minus 1 d phi by dr equals 1 by alpha n rho to the n minus 1 that is straight away immediate from this for the formula for n bigger than equal to 3. For n equals 2 log r will give you 1 by r that is 1 by 2 pi r 2 pi r is 2 pi is alpha 2 because the perimeter of the circle and rho to the 2 minus 1 is nothing but the row itself fine. So, you get this and now we also, delta u you get this formula.

So now, assume that $\Delta u = 0 \; in \; \Omega$

then the first term will disappear and therefore for every $x \in \Omega$, we have

$$u(x) = \frac{1}{\alpha_N \rho^{N-1}} \int_{|y-x_0|=\rho} u(y)\, d\sigma(y) .$$

So, this is called the Gauss law of the arithmetic mean for a harmonic function. So, the harmonic function satisfies the following property; any value at any point in the domain is the average of the value on any ball surrounding it on the sphere surrounding it.

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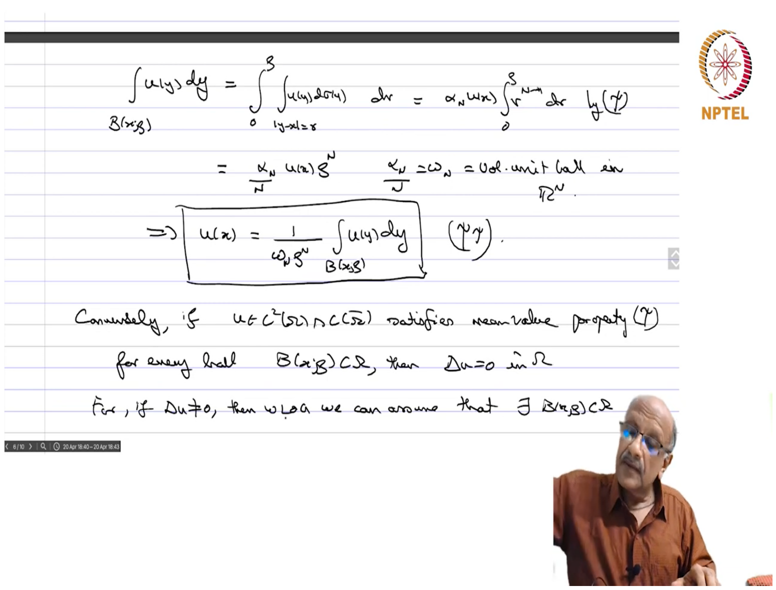

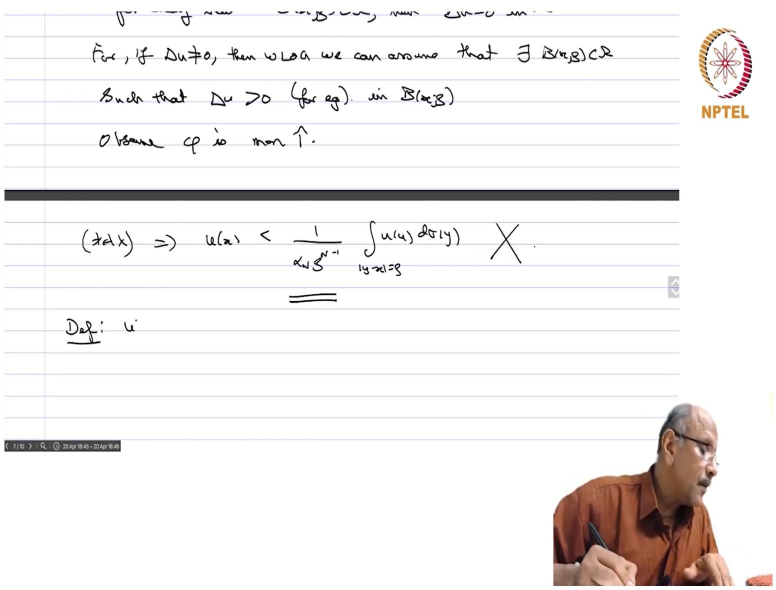

Now you also from this you can get another mean value theorem so these 2 are called mean value theorems, namely if you take this is true for any radius so as long as B of x r is contained in the domain omega this is true. And therefore now if you take

$$\int_{B(x,\rho)} u(y)dy = \int_0^{\rho} \int_{|y-x|=\rho} u(y)\, d\sigma(y)\, dr = \alpha_N u(x) \int_0^{\rho} r^{N-1} dr = \frac{\alpha_N}{N} u(x) \rho^N$$

$$\Rightarrow u(x) = \frac{1}{\omega_N \rho^N} \int_{B(x,\rho)} u(y)dy, \text{ where } \omega_N \text{ is the volume of the unit ball in } \mathbb{R}^N.$$

So, the first one says double dagger. The first one says that u is the average if you take any sphere of radius rho around the point and this tells you this is the average of any ball of radius rho taken around the point. So, for a harmonic function this is a very nice property which you have here.

Conversely, if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies mean value property, namely dagger for every ball $B(x, \rho) \subset \Omega$, then $\Delta u = 0 \; in \, \Omega$. For if $\Delta u \neq 0 \; in \, \Omega$, then without loss of generality we can assume that there exists a $B(x, \rho) \subset \Omega$ such that $\Delta u > 0 \; in \, \Omega$ or negative (for instance) in $B(x, \rho)$.

Because delta is a continuous function the function is c2 and therefore second derivatives are continuous so if it is positive at one point since it is not 0 then it will be positive in a small neighborhood. If it is negative you can take it to be negative also, it does not matter whether you are taking positive or negative. Now you then follow from this equation call this triple star if you like, observe that phi is monotonic increasing phi is a monotonic function. This function is monotonic increasing as you can readily verify and therefore if you look at the triple star here if the delta u in that small ball is strictly positive.

Now phi a phi x minus y minus phi rho will be negative so this whole integral is going to be negative and therefore you have triple star implies that

$$u(x) < \frac{1}{\omega_N \rho^N} \int_{B(x,\rho)} u(y)dy$$

which is a contradiction because we know that the mean value property holds in every ball. And therefore you have that this function is in fact harmonic.

So, we have a definition at this point:

Definition: you say that $u \in C(\overline{\Omega})$ is said to be sub harmonic if for every $B(x, \rho) \subset \Omega$, we have

$$u(x) \leq \frac{1}{\omega_N \rho^N} \int_{B(x,\rho)} u(y)dy$$

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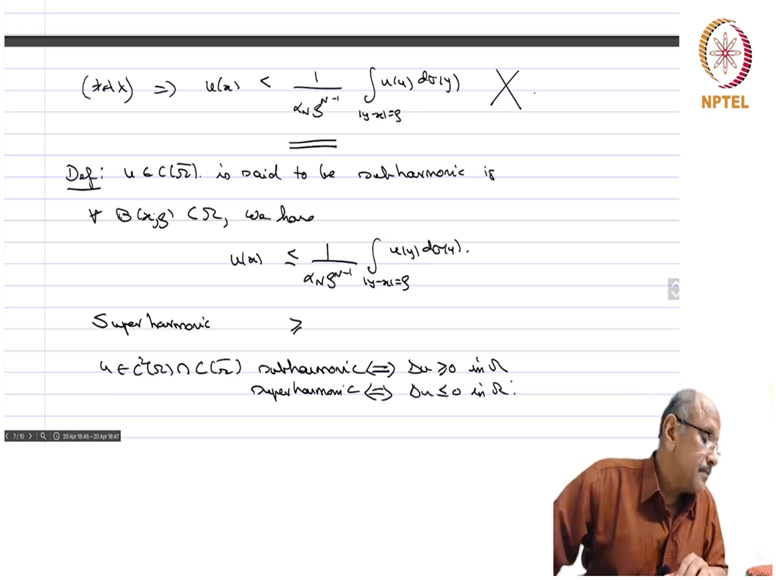

So, it will be super harmonic if you have greater than or equal, so this is the condition.

So, if you have $u \in C^2(\Omega) \cap C(\overline{\Omega})$, then it is sub harmonic if and only if $\Delta u \geq 0 \; in \; \Omega$ and super harmonic if and only if $\Delta u \leq 0 \; in \; \Omega$ it immediately follows from the definition .

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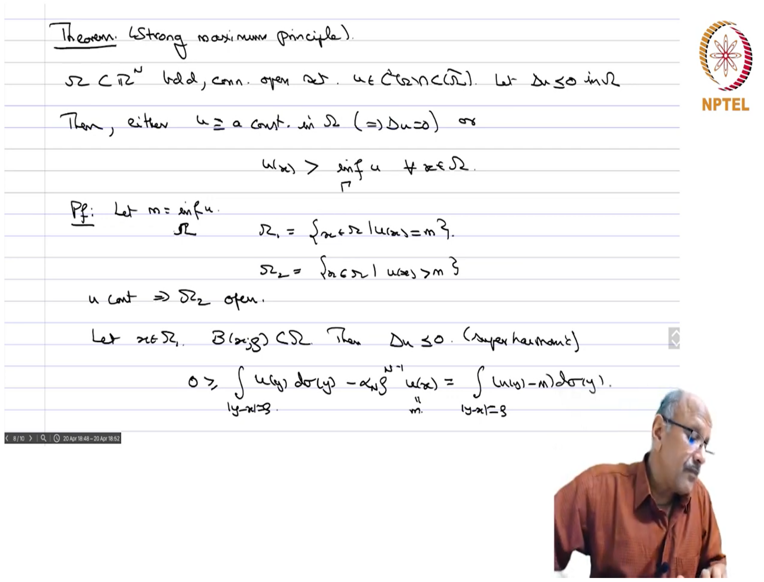

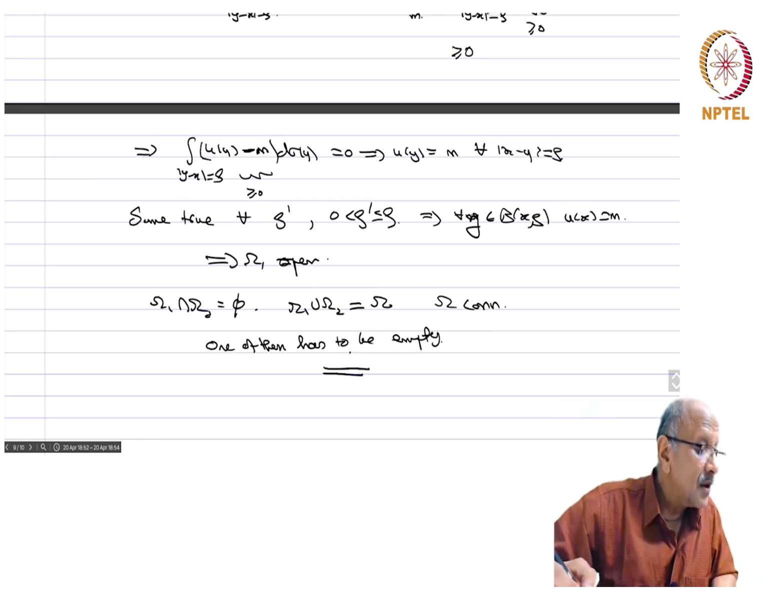

So now, we have an important theorem that is called the strong maximum principle.

Theorem: So, $\Omega \subset \mathbb{R}^N$ bounded connected open set, $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Let $\Delta u \leq 0 \; in \; \Omega$. so it is a super harmonic function, then either $u \equiv$ a constant in Ω (implies $\Delta u = 0$) or

$$u(x) > \inf_{\Gamma} u \text{ , for all } x \in \Omega .$$

proof: so let $\inf_{\Gamma} u$. So, you write

$$\Omega_1 = \{x \in \Omega: u(x) = m\} \; ; \; \Omega_2 = \{x \in \Omega: u(x) > m\}.$$

So, u is continuous implies Ω_2 is open. So let $x \in \Omega_1$ and you, then have a $B(x, \rho) \subset \Omega$. Then $\Delta u \leq 0 \; in \; \Omega$ (so super harmonic) and therefore you have that

$$0 \geq \int_{|y-x|=\rho} u(y)d\sigma(y) - \alpha_N \rho^{N-1} u(x) = \int_{|y-x|=\rho} (u(y) - m)d\sigma(y).$$

$$\Rightarrow \int_{|y-x|=\rho} (u(y) - m)d\sigma(y) = 0 \Rightarrow u(y) = m \; for \; |y - x| = \rho.$$

So, the same is true for all ρ', $0 < \rho' \leq \rho$. And that means for all $y \in B(x, \rho)$, we have $u(y) = m$ and this implies that Ω_1 is open. So,

$$\Omega_1 \cap \Omega_2 = \phi, \; \Omega_1 \cup \Omega_2 = \Omega, \; \Omega - \text{connected}.$$

So, one of them has to be empty and that completes the proof because if Ω_1 is empty, then $\Omega = \Omega_2$.

So, that proves the theorem.

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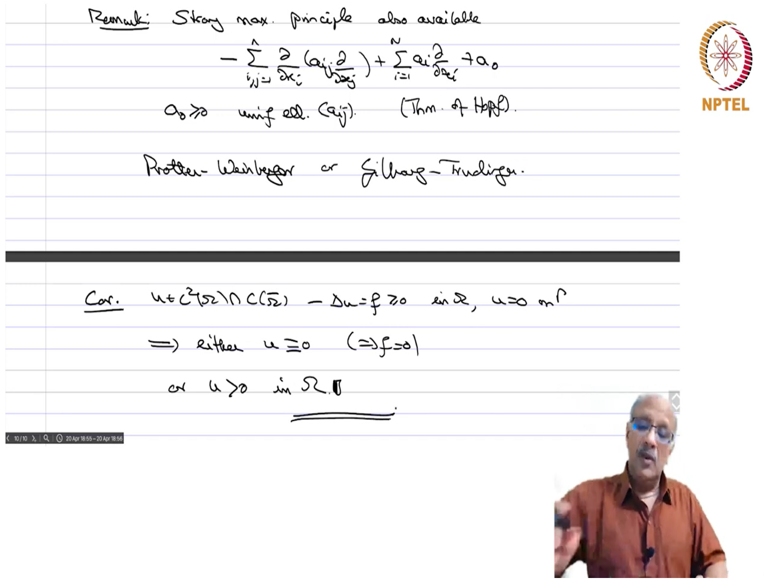

Remark: again strong maximum principle also available for

$$- \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial}{\partial x_j}) + \sum_{i=1}^{N} a_i \frac{\partial}{\partial x_i} + a_0 \ , \ a_0 \geq 0$$

and uniformly elliptic (a_{ij}) . So, if this is true, then this is a theorem of half theorem of half and you can see proof of either Protter Weinberger or Gilberger and Trudinger.

Corollary: so $u \in C^2(\Omega) \cap C(\overline{\Omega}),\ - \Delta u = f \geq 0\ in\ \Omega,\ u = 0\ on\ \Gamma,$ implies

either $u \equiv 0 (\Rightarrow f = 0)$ or $u > 0\ in\ \Omega$.

Because the infimum can be attained only on the boundary. And therefore, this is a very nice and useful result to know so this is just a corollary of the strong maximum principle so with this we will wind up the maximum principles.