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(5) Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^p(\Omega)$, $1 < p \leq \infty$. Assume that there exists $c > 0$ s.t. $\forall \phi \in D(\Omega)$, $\forall 1 \leq i \leq N$,

$$|\int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx| \leq c|\phi|_{0,p',\Omega}, \quad p' - conjugate\ of\ p.$$

Show that $u \in W^{1,p}(\Omega)$.

Solution: So, now consider $L(\phi) = \int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx$. Then

$$|L(\phi)| \leq c|\phi|_{0,p',\Omega}.$$

So, $\phi \to L(\phi)$ is a continuous linear functional on $D(\Omega)$ with the norm $|.|_{0,p',\Omega}$. But $p \neq 1$, so $p' \neq \infty \Rightarrow D(\Omega)$ is dense in $L^{p'}(\Omega) \Rightarrow \phi \to L(\phi)$ extends uniquely to a continuous linear functional on $L^{p'}(\Omega)$.

By Riesz rep. thm $\exists v_i$ $1 \leq i \leq N$ $v_i \in L^p(\Omega)$

(dual of $L^{p'}(\Omega)$) $L(\phi) = \int_\Omega v_i \phi \, dx$. $\forall \phi \in \mathcal{D}(\Omega)$.

$\Rightarrow \dfrac{\partial u}{\partial x_i} = -v_i \in L^p(\Omega) \Rightarrow u \in W^{1,p}(\Omega)$.

(6)

(5) Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $u \in L^p(\Omega)$ $1 < p \leq \infty$.

Assume that $\exists \, c > 0$ s.t. $\forall \phi \in \mathcal{D}(\Omega)$, $\forall \, 1 \leq i \leq N$,

$$\left| \int_\Omega u \frac{\partial \phi}{\partial x_i} \, dx \right| \leq C |\phi|_{0, p', \Omega} \qquad p' = \text{conj. exp.}$$

$p = \infty \Rightarrow p' = 1$.

$1 < p < \infty, \; \frac{1}{p} + \frac{1}{p'} = 1$.

Show that $u \in W^{1,p}(\Omega)$.

Sol. Consider $L(\phi) = \int_\Omega u \frac{\partial \phi}{\partial x_i} \, dx$. $1 \leq i \leq N$

$|L(\phi)| \leq C |\phi|_{0, p', \Omega}$.

$\phi \mapsto L(\phi)$ cont. lin. fnl on $\mathcal{D}(\Omega)$ with the norm $|\cdot|_{0, p', \Omega}$.

But $(p \neq 1)$ $p' \neq \infty \Rightarrow$ $\mathcal{D}(\Omega)$ is dense in $L^{p'}(\Omega)$

$\Rightarrow$ $L(\phi)$ extends uniquely to a cont. lin. fnl on $L^{p'}(\Omega)$.

So, by Riesz representation theorem there exists v_i, $1 \leq i \leq N$, $v_i \in L^p(\Omega) - dual\ of$

$L^{p'}(\Omega)., \ s.t. \quad L(\phi) = \int_\Omega v_i \phi dx, \ \forall \phi \in D(\Omega)$.

$$\Rightarrow \frac{\partial u}{\partial x_i} = - v_i \in L^p(\Omega) \Rightarrow u \in W^{1,p}(\Omega).$$

(6) Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^p(\Omega)$, $1 < p \leq \infty$. Assume that $\exists c_1 > 0 \ s.t. \ \forall \Omega' \subset\subset \Omega \ and \ \forall h \in \mathbb{R}^N \ s.t. \ |h| \leq d(\Omega', \mathbb{R}^N \backslash \Omega)$, we have

$$|\tau_h u - u\backslash_{0,p,\Omega'} \leq c|h| \qquad -------- (*).$$

Then show that there exists $C > 0$, $s.t. \ \forall 1 \leq i \leq N$, $\forall \phi \in D(\Omega)$

$$|\int_\Omega u \frac{\partial \phi}{\partial x_i} dx| \leq C|\phi|_{0,p',\Omega} \ .$$

In particular, by Ex 5 above, if $1 < p \leq \infty$, we have
$u \in W^{1,p}(\Omega)$ if (*) is true.

Sol. Let $\tilde{u}$ be the extn. of u by zero outside Ω.

$\tilde{u} \in L^p(\mathbb{R}^N)$. Let $\varphi \in \mathcal{D}(\Omega)$. Let $\Omega' \subset\subset \Omega$ s.t. $supp\, \varphi \subset \Omega'$.

$\{e_i\}_{i=1}^N$ std. basis vectors of $\mathbb{R}^N$.

$h \in \mathbb{R}$, $|h| < \delta(\Omega', \mathbb{R}^N \backslash \Omega)$.

$$\int_\Omega u(x) \frac{\varphi(x+he_i) - \varphi(x)}{h} dx = \int_{\mathbb{R}^N} \tilde{u}(x) \frac{\varphi(x+he_i) - \varphi(x)}{h} dx$$

$$= \int_{\mathbb{R}^N} \frac{\tilde{u}(y - he_i) - \tilde{u}(y)}{h} \varphi(y) dy$$

$$\int_\Omega u(x) \frac{\varphi(x+he_i) - \varphi(x)}{h} dx = \int_{\mathbb{R}^N} \tilde{u}(x) \frac{\varphi(x+he_i) - \varphi(x)}{h} dx$$

$$= \int_{\mathbb{R}^N} \frac{\tilde{u}(y - he_i) - \tilde{u}(y)}{h} \varphi(y) dy$$

$$= \int_\Omega \frac{\tilde{u}(y - he_i) - \tilde{u}(y)}{h} \varphi(y) dy$$

$|h| < \delta(\Omega', \mathbb{R}^N \backslash \Omega)$ $y \in \Omega'$, $y - he_i \in \Omega$.

$\Rightarrow$

$$\int_\Omega u(x) \frac{\varphi(x+he_i) - \varphi(x)}{h} dx = \int_{\Omega'} \frac{u(y - he_i) - u(y)}{h} \varphi(y) dy$$

So, in particular by exercise (5), if $1 < p \leq \infty$, we have

$$u \in W^{1,p}(\Omega) \; if \; (*) \; holds.$$

solution. Let $\widetilde{u}$ be the extension of u by 0 outside of Ω. Let $\phi \in D(\Omega)$ and $\Omega' \subset\subset \Omega$ s.t.

$\operatorname{supp}(\phi) \subset \Omega'$. Let $\{e_i\}_{i=1}^{N}$ be the standard basis vectors of $\mathbb{R}^N$, $h \in \mathbb{R}$, $|h| < d(\Omega', \mathbb{R}^N \backslash \Omega)$.

$$\int_{\Omega} u(x)\frac{\phi(x+he_i)-\phi(x)}{h}dx = \int_{\mathbb{R}^N} \widetilde{u}(x)\frac{\phi(x+he_i)-\phi(x)}{h}dx = \int_{\mathbb{R}^N} \phi(y)\frac{\widetilde{u}(y-he_i)-\widetilde{u}(y)}{h}dy$$

$$= \int_{\Omega'} \phi(y)\frac{\widetilde{u}(y-he_i)-\widetilde{u}(y)}{h}dy \; .$$

Since $|h| < d(\Omega', \mathbb{R}^N \backslash \Omega)$ and $y \in \Omega' \Rightarrow y - he_i \in \Omega$. So

$$\int_{\Omega} u(x)\frac{\phi(x+he_i)-\phi(x)}{h}dx = \int_{\Omega'} \phi(y)\frac{u(y-he_i)-u(y)}{h}dy = \int_{\Omega'} (\tau_{he_i}u - u)(y)\phi(y)dy$$

$$\leq C|\phi|_{0,p,\Omega'} \quad \text{(Holder \& (*))}.$$

So, you pass to the limit as $h \to 0$ and dominated convergence theorem will tell you that

$$|\int_{\Omega} u\frac{\partial\phi}{\partial x_i}dx| \leq C|u|_{0,p',\Omega'} \leq C|u|_{0,p',\Omega} \; .$$

this is exactly what we wanted to prove. And that completes this thing.