

Sobolev Spaces and Partial Differential Equations
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Exercises - Part 5

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(5) Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $u \in L^p(\Omega)$, $1 < p \leq \infty$.
 Assume that $\exists c > 0$ s.t. $\forall \phi \in D(\Omega)$, $\forall 1 \leq i \leq N$,

$$\left| \int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx \right| \leq c |\phi|_{0,p',\Omega}$$
 $p' = \text{conj. exp.}$
 $p = \infty \Rightarrow p' = 1$
 $1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1$
 Show that $u \in W^{1,p}(\Omega)$.
Sol. Consider $L(\phi) = \int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx$.
 $|L(\phi)| \leq c |\phi|_{0,p',\Omega}$.
 $\phi \mapsto L(\phi)$ cont. lin. fun. on $D(\Omega)$ with the norm $|\cdot|_{0,p',\Omega}$.
 But $(p \neq 1) \Rightarrow p' \neq \infty \Rightarrow D(\Omega)$ is dense in $L^{p'}(\Omega)$.
 $\Rightarrow L(\phi)$ extends uniquely to a cont. lin. fun. on $L^{p'}(\Omega)$.



(5) Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^p(\Omega)$, $1 < p \leq \infty$. Assume that there exists $c > 0$ s.t. $\forall \phi \in D(\Omega)$, $\forall 1 \leq i \leq N$,

$$\left| \int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx \right| \leq c |\phi|_{0,p',\Omega}, \quad p' - \text{conjugate of } p.$$

Show that $u \in W^{1,p}(\Omega)$.

Solution: So, now consider $L(\phi) = \int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx$. Then

$$|L(\phi)| \leq c |\phi|_{0,p',\Omega}.$$

So, $\phi \rightarrow L(\phi)$ is a continuous linear functional on $D(\Omega)$ with the norm $|\cdot|_{0,p',\Omega}$. But $p \neq 1$, so $p' \neq \infty \Rightarrow D(\Omega)$ is dense in $L^{p'}(\Omega) \Rightarrow \phi \rightarrow L(\phi)$ extends uniquely to a continuous linear functional on $L^{p'}(\Omega)$.

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By Riesz rep. thm $\exists u_i, 1 \leq i \leq N, u_i \in L^p(\Omega)$
 (dual of $L^p(\Omega)$) $L(\varphi) = \int_{\Omega} \varphi dx, \forall \varphi \in D(\Omega)$
 $\Rightarrow \frac{\partial u}{\partial x_i} = -v_i \in L^p(\Omega) \Rightarrow u \in W^{1,p}(\Omega)$

(6)



(5) Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $u \in L^p(\Omega), 1 < p < \infty$.
 Assume that $\exists C > 0$ st. $\forall \varphi \in D(\Omega), \forall 1 \leq i \leq N$,
 $\left| \int_{\Omega} u \frac{\partial \varphi}{\partial x_i} dx \right| \leq C \|\varphi\|_{q, p, \Omega}$ $p' = \text{conj. exp.}$
 $p = \infty \Rightarrow p' = 1$
 $1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1$
 Show that $u \in W^{1,p}(\Omega)$.
 Sol. Consider $L(\varphi) = \int_{\Omega} u \frac{\partial \varphi}{\partial x_i} dx, 1 \leq i \leq N$
 $\|L(\varphi)\| \leq C \|\varphi\|_{q, p, \Omega}$
 $\varphi \mapsto L(\varphi)$ cont. lin. fcn on $D(\Omega)$ with the norm $\|\cdot\|_{q, p, \Omega}$.
 But $(p \neq 1), p' \neq \infty \Rightarrow D(\Omega)$ is dense in $L^{p'}(\Omega)$.
 $\Rightarrow L(\varphi)$ extends uniquely to a cont. lin. fcn on $L^{p'}(\Omega)$.



So, by Riesz representation theorem there exists $v_i, 1 \leq i \leq N, v_i \in L^p(\Omega)$ – dual of

$L^{p'}(\Omega)$, s. t. $L(\phi) = \int_{\Omega} v_i \phi dx, \forall \phi \in D(\Omega)$.

$\Rightarrow \frac{\partial u}{\partial x_i} = -v_i \in L^p(\Omega) \Rightarrow u \in W^{1,p}(\Omega)$.

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$\Rightarrow \frac{\partial u}{\partial x_i} = -v_i \in L^p(\Omega) \Rightarrow u \in W^{1,p}(\Omega).$

⑥ Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 < p < \infty$ and $u \in L^p(\Omega)$.
 Assume $\exists c_1 > 0$ s.t. $\forall \Omega' \subset \subset \Omega$ and $\forall h \in \mathbb{R}^N$ s.t.
 $|h| \leq d(\Omega', \mathbb{R}^N \setminus \Omega)$ we have
 $|\tau_h u - u|_{0,p,\Omega'} \leq c|h| \quad (*)$
 Show that
 Then $\exists C > 0$ s.t. $\forall 1 \leq i \leq N, \forall \phi \in D(\Omega)$
 $|\int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx| \leq C|\phi|_{0,p',\Omega}.$



(6) Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^p(\Omega)$, $1 < p \leq \infty$. Assume that $\exists c_1 > 0$ s.t. $\forall \Omega' \subset \subset \Omega$ and $\forall h \in \mathbb{R}^N$ s.t. $|h| \leq d(\Omega', \mathbb{R}^N \setminus \Omega)$, we have

$$|\tau_h u - u|_{0,p,\Omega'} \leq c|h| \quad \text{-----} (*)$$

Then show that there exists $C > 0$, s.t. $\forall 1 \leq i \leq N, \forall \phi \in D(\Omega)$

$$|\int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx| \leq C|\phi|_{0,p',\Omega} .$$

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In particular, by ex 5 above, if $1 < p < \infty$, we have
 $u \in W^{1,p}(\Omega)$ if \mathcal{U} is true.

Sol. Let \tilde{u} be the anti of u by zero outside Ω .

$\tilde{u} \in L^p(\mathbb{R}^n)$ let $\varphi \in \mathcal{D}(\Omega)$ let $\Omega' \subset \subset \Omega$ s.t. $\text{supp } \varphi \subset \Omega'$

$\{e_i\}_{i=1}^n$ std. basis vectors of \mathbb{R}^n

$R \in \mathbb{R}$, $|h| < \delta(\Omega', \mathbb{R}^n \setminus \Omega)$.

$$\begin{aligned} \int_{\Omega} u(x) \frac{\varphi(x+h e_i) - \varphi(x)}{h} dx &= \int_{\mathbb{R}^n} \tilde{u}(x) \frac{\varphi(x+h e_i) - \varphi(x)}{h} dx \\ &= \int_{\mathbb{R}^n} \frac{\tilde{u}(y-h e_i) - \tilde{u}(y)}{h} \varphi(y) dy \end{aligned}$$



$$\begin{aligned} \int_{\Omega} u(x) \frac{\varphi(x+h e_i) - \varphi(x)}{h} dx &= \int_{\mathbb{R}^n} \tilde{u}(x) \frac{\varphi(x+h e_i) - \varphi(x)}{h} dx \\ &= \int_{\mathbb{R}^n} \frac{\tilde{u}(y-h e_i) - \tilde{u}(y)}{h} \varphi(y) dy \\ &= \int_{\Omega'} \frac{\tilde{u}(y-h e_i) - \tilde{u}(y)}{h} \varphi(y) dy \end{aligned}$$



$|h| < \delta(\Omega', \mathbb{R}^n \setminus \Omega)$ $y \in \Omega'$, $y-h e_i \in \Omega$.

$$\Rightarrow \int_{\Omega} u(x) \frac{\varphi(x+h e_i) - \varphi(x)}{h} dx = \int_{\Omega'} \frac{\tilde{u}(y-h e_i) - \tilde{u}(y)}{h} \varphi(y) dy$$



$$|h| < d(\Omega', \mathbb{R}^N \setminus \Omega) \quad y \in \Omega', \quad y - he_i \in \Omega.$$

$$\Rightarrow \int_{\Omega} u(x) \frac{\phi(x+he_i) - \phi(x)}{h} dx = \int_{\Omega'} \frac{u(y-he_i) - u(y)}{h} \phi(y) dy$$

$$= \int_{\Omega'} (\tau_{-he_i} u - u)(y) \phi(y) dy$$

$$\leq C |u|_{0,p,\Omega'} \quad \text{Holder \& (*)}$$

Pass $h \rightarrow 0$ DCT

$$\left| \int_{\Omega} u \frac{\partial \phi}{\partial x_i} \right| \leq C |u|_{0,p,\Omega'} \leq C |u|_{0,p,\Omega}$$



So, in particular by exercise (5), if $1 < p \leq \infty$, we have

$$u \in W^{1,p}(\Omega) \text{ if } (*) \text{ holds.}$$

solution. Let \tilde{u} be the extension of u by 0 outside of Ω . Let $\phi \in D(\Omega)$ and $\Omega' \subset\subset \Omega$ s.t.

$\text{supp}(\phi) \subset \Omega'$. Let $\{e_i\}_{i=1}^N$ be the standard basis vectors of \mathbb{R}^N , $h \in \mathbb{R}$, $|h| < d(\Omega', \mathbb{R}^N \setminus \Omega)$.

$$\int_{\Omega} u(x) \frac{\phi(x+he_i) - \phi(x)}{h} dx = \int_{\mathbb{R}^N} \tilde{u}(x) \frac{\phi(x+he_i) - \phi(x)}{h} dx = \int_{\mathbb{R}^N} \phi(y) \frac{\tilde{u}(y-he_i) - \tilde{u}(y)}{h} dy$$

$$= \int_{\Omega'} \phi(y) \frac{\tilde{u}(y-he_i) - \tilde{u}(y)}{h} dy .$$

Since $|h| < d(\Omega', \mathbb{R}^N \setminus \Omega)$ and $y \in \Omega' \Rightarrow y - he_i \in \Omega$. So

$$\int_{\Omega} u(x) \frac{\phi(x+he_i) - \phi(x)}{h} dx = \int_{\Omega'} \phi(y) \frac{u(y-he_i) - u(y)}{h} dy = \int_{\Omega'} (\tau_{-he_i} u - u)(y) \phi(y) dy$$

$$\leq C |\phi|_{0,p,\Omega'} \quad (\text{Holder \& } (*)).$$

So, you pass to the limit as $h \rightarrow 0$ and dominated convergence theorem will tell you that

$$\left| \int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx \right| \leq C |u|_{0,p,\Omega'} \leq C |u|_{0,p,\Omega} .$$

this is exactly what we wanted to prove. And that completes this thing.