Real Analysis II

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Lecture - 30.1
The Problem of Measure

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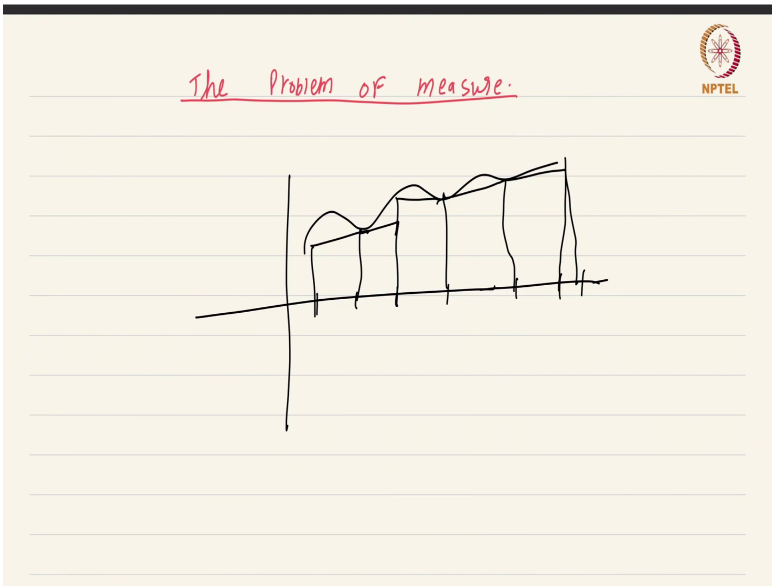

In the next set of videos we will study measurable functions measures etcetera defined using the Lebesgue integral we have, but before we do that let me describe the general problem of measure and the way in which Lebesgue viewed the Lebesgue integral. We have taken the approach via sequences of functions we first defined the Lebesgue integral and not the measure.

Originally, when Lebesgue came up with this construction of the Lebesgue integral he went the other way around he first defined what is known as the Lebesgue measure and defined the Lebesgue integral in terms of the Lebesgue measure.

This approach is the standard classical one going via measures then to integral. The advantage is it works in a lot more general situations for instance much of modern probability will simply be impossible without this idea of going through measures.

So, let us briefly talk about The Problem of Measure, to do that let me give you an analogy originally conveyed by Lebesgue in a letter of his to a colleague. Suppose you owe somebody money let us say x amount of money and you want to return it, let us assume that you have more than x amount of money in your pocket. So, if you meet the person your lender and you want to give this person back the money one way of doing that would be just put your hand inside your pocket and keep putting money on the table till the amount x is reached and give that much amount to your lender.

That is essentially the Riemann approach to integration, what you are essentially doing is you have a function whose graph might look like this to measure the area you just keep breaking down the intervals and then you just take the areas like so, and sum up the areas.

And as you make this mesh the distance between 2 subsequent points in the partition smaller and smaller essentially your approximation of the area becomes better and better. So, this is a somewhat of a loose analogy, but its going to serve as well in a moment.

Now, Lebesgue says his approach to the integral is slightly different what he is going to do is to return the money x what he is going to do is he is going to first take all the money from his pocket, then all the 1 rupee coins he is going to put it separately and stack them together, all the 5 rupee coins he is going to stack them together, the 10 rupee notes, the 20 rupee notes, the 50 rupee notes, the 100 rupee notes, he is going to stack all the currency based on the denomination and then he is going to bunch together all of these together. So, that you get x and give it to the lender.

So, instead of breaking up the domain essentially Lebesgue is going to break up the codomain depending on the value of the function he is going to break up the range of the function instead of breaking up the domain. And this is illustrated in the picture obtained from Wikipedia which I am going to paste now.

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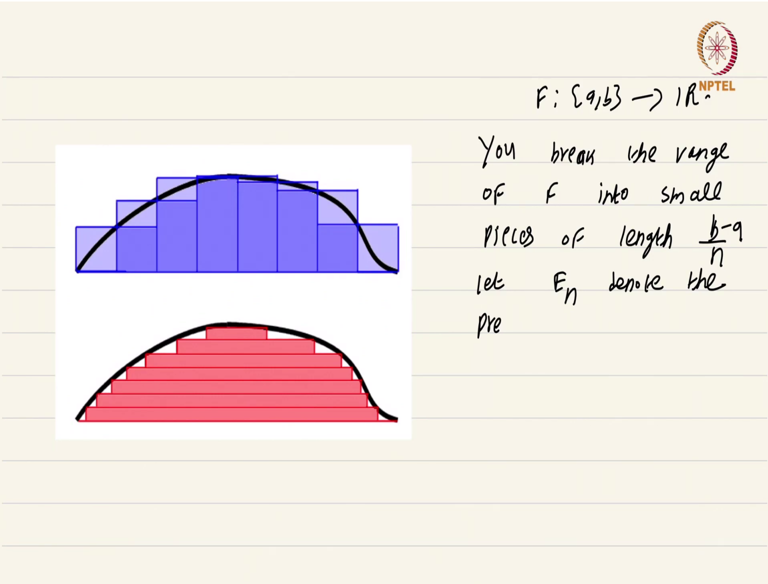

The blue color here is the Riemann's approach to the integral what I already drew and this is sort of the Lebesgue approach to the integral, what you do is you break up the range and you take the corresponding length of the interval where the function happens to have that range and then multiply that together.

To be precise what you do is the following; you break up the range of F into small pieces of length 1 by n ok. Let E n denote the actually let me say length b minus a by n, let us take this function F from a, b to R ok and not b minus a to n.

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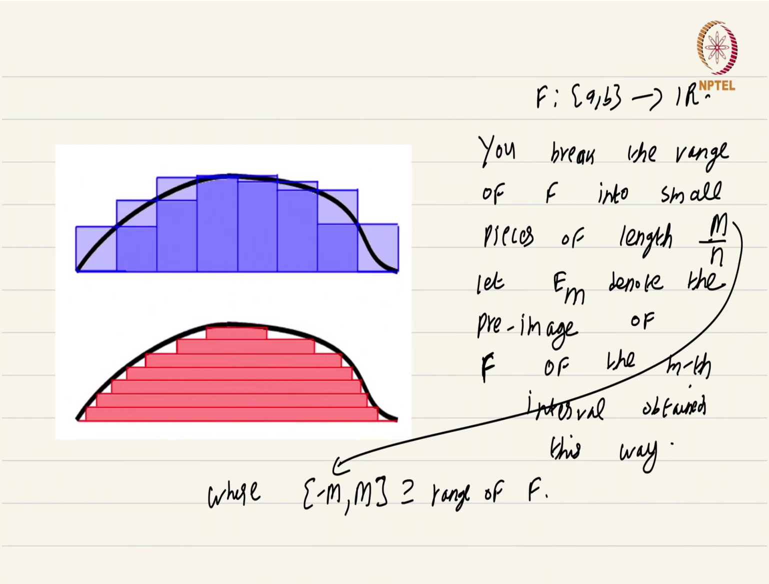

Let us say; let us say of length M by n, where this M where this minus M, M contains the range of F let us say for simplicity sake. This is just a simple situation do note that the Lebesgue integral will work even when the range is not contained in any such bounded set, but for simplicity sake let us say the range of F is contained in minus M, M you break up into pieces of length M by n and let E n denote the pre image let E m denote the pre image of F of the mth interval obtained this way obtained this way.

That is you look at the range split it into n equal pieces look at the mth piece and look at those points in the domain a, b where F takes the value lying inside the mth piece call that set E m ok.

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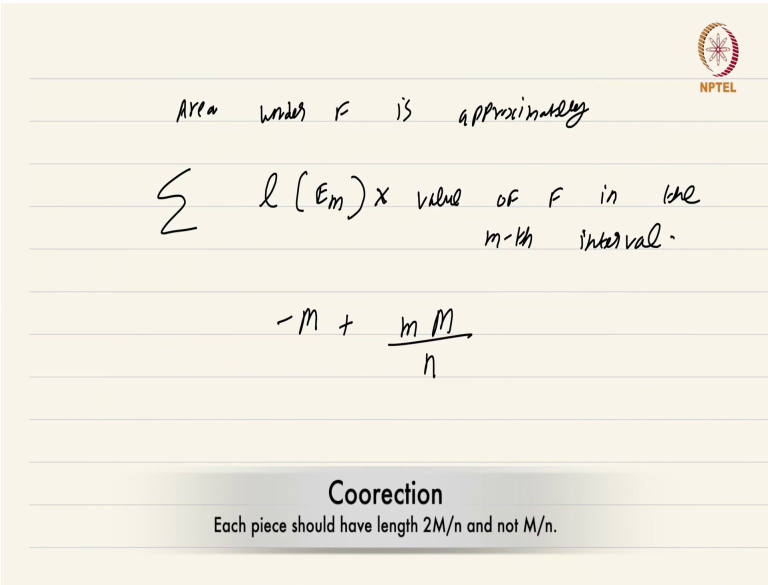

Then you can see that the area under F; area under F is approximately the length of this set E m length of the set E m times the value of F in the mth interval; in the mth interval. What is the value going to be? It is going to be minus M plus little m capital M by n because we are breaking up the interval into pieces of length M by n ok. So, essentially a process like this will also will also give you the area under the curve of course, I must take summation.

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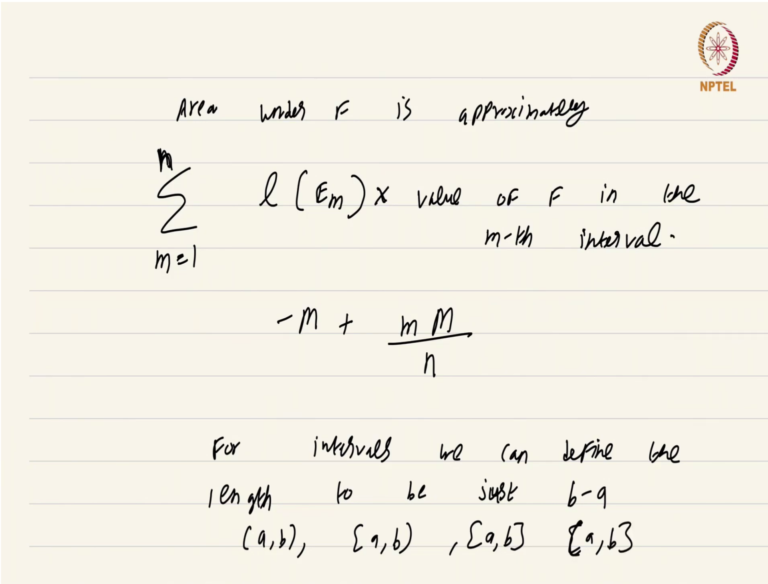

Summation as m runs from; m runs from 1 to n. So this process will sort of give you the area under the graph. Now the issue is what is the meaning of length of E m, E m is not in general going to be a nice set like an interval, it could be a very very complicated set. For intervals we can define; we can define the length to be just b minus a. So, the interval a b, half open a b, closed a b and open closed a b for all of these you can just define the length to be b minus a.

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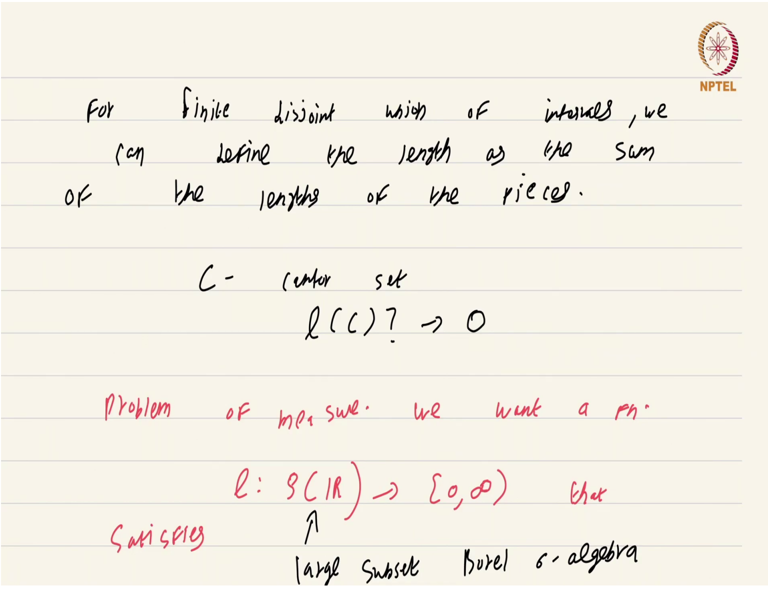

For finite disjoint union of; for finite disjoint union of intervals we can define; we can define the length as the sum of the lengths of the pieces. So, these pose no issues, but if you were to take a general set in R it is not clear what exactly the length is.

For instance if C is the cantor set, what is the length of the cantor set? Intuitively it should be 0; if you remember the net length of the intervals we remove from close 0 1 to obtain the cantor set will be of length 1. So, intuitively it should be 0, but how do we make this precise.

And there are all sorts of complicated sets on the real line can you assign a reasonable notion of length to all of them. Well, first we must understand what is the meaning of a reasonable notion of length and that is what I call the problem of measure problem of measure. What we want is, we want a function l from the power set of R to closed 0 open infinity a function that satisfies number 1 it should be countably additive.

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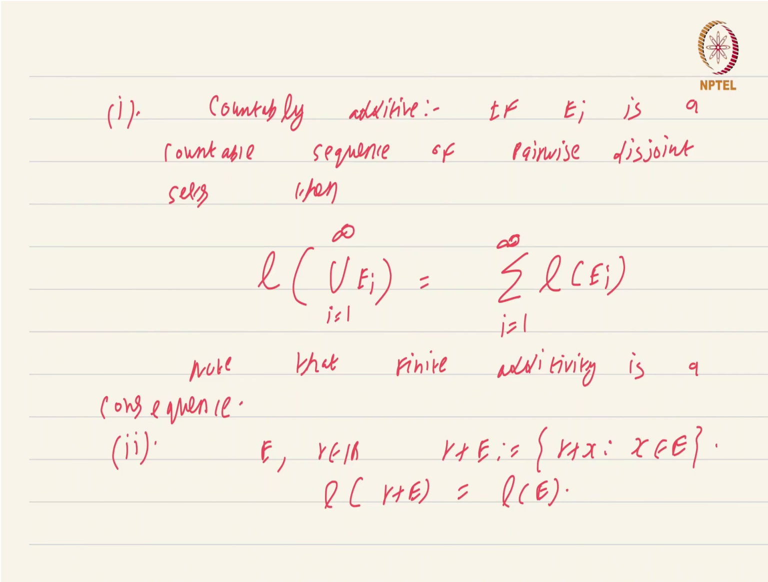

What do I mean by it should be countably additive? The l if E i is a countable sequence of pairwise disjoint sets, then the length of the union of the Ei's as you can guess must be length of E i sum ok. So, if you have a countable collection of sets then you must be able to countable collection of sets which are pairwise disjoint then the length of the union must be the sum of the lengths, this is what countable additivity is and this idea of countable additivity is both intuitively natural.

Note that note that finite additivity is a consequence, think about why is a consequence. Second notion second property that this notion of length must satisfy is if you define for a set E and a real number r. If you define r plus E to be by definition r plus x such that x is in E, in other words you are just translating this set E by the real number r then the length of r plus E must be the same as the length of E. So, this length notion of length must be invariant under translation this is yet another natural property you would expect of a putative length function.

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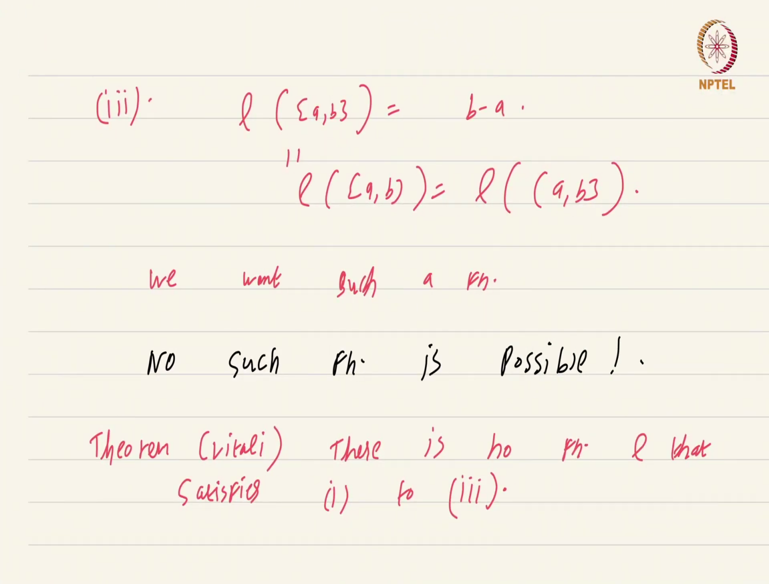

Then finally, it should agree with the notion of length for intervals. So, length of a b should be equal to b minus a, it should agree of course, it should also be equal for a b and open a close be all as well. So, the length of all these intervals must be b minus a.

So, in an ideal world we want such a function and once we get such a function we can proceed with this idea of Lebesgue to construct an integral by subdividing the range of a function as opposed to subdividing the domain of the function. And you would end up with an integral which will agree with the already defined Lebesgue integral we have from the previous videos.

Now, we are not going to pursue this line of approach I just wanted to highlight what original approach of Lebesgue was, so that you will be prepared when you see this when you take a course on measure theory. But one thing I want to do because it is such an interesting thing and sort of involves the axiom of choice which I had actually devoted a short video to way back in Real Analysis 1 in probably the very first week.

So, now, is the time that we at least see one of the applications of axiom of choice and also see why this axiom of choice is so unintuitive. We are going to show that no such function is possible, its not possible to construct a length function that satisfies all the properties we would want a nice length function to satisfy. So, that is Vitalis theorem by Vitali there is no function l that satisfies 1 to 3 ok. Let us see a proof.

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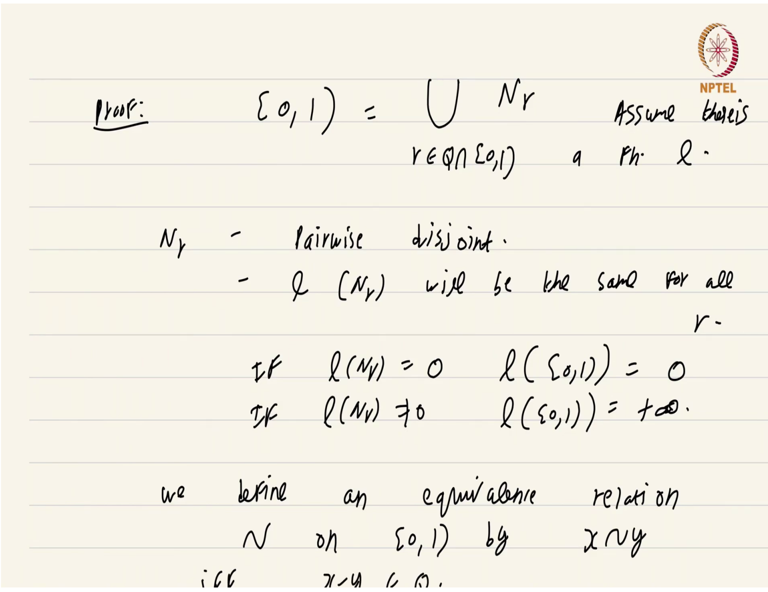

I shall be intentionally a bet sketchy about the proof simply because this belongs to our course on measure theory proper, but it is good to know it is good to have a feel for this. What we are going to do is we are going to write down close 0 open 1 as the union of a sequence of sets indexed by rational numbers that belong to 0, 1 which I am going to call N r ok.

And these N r's will be pairwise disjoint will be pairwise disjoint and each one of them will be mu of N r will be the same not mu l of N r that is if you assume that there is a function l. So, assume there is a function l there is a function l then it will turn out that l of N r will be the same for all r.

Now that is going to put us in a difficulty, because if l of N r is 0 then l of 0, 1 is also 0 by countable additivity and this will contradict three, if l of N r is not 0 then l of 0, 1 would be plus infinity which is also going to be nonsensical. So, we are essentially going to get a contradiction to one of the three properties assuming that the function l exists we will reach a contradiction ok.

So, how are we going to construct this sets l r what we do is the following. We define an equivalence relation; we define an equivalence relation on opens close 0 open 1 by x is related to y if and only if x minus y is a rational number ok, that this is an equivalence relation is easy to see.

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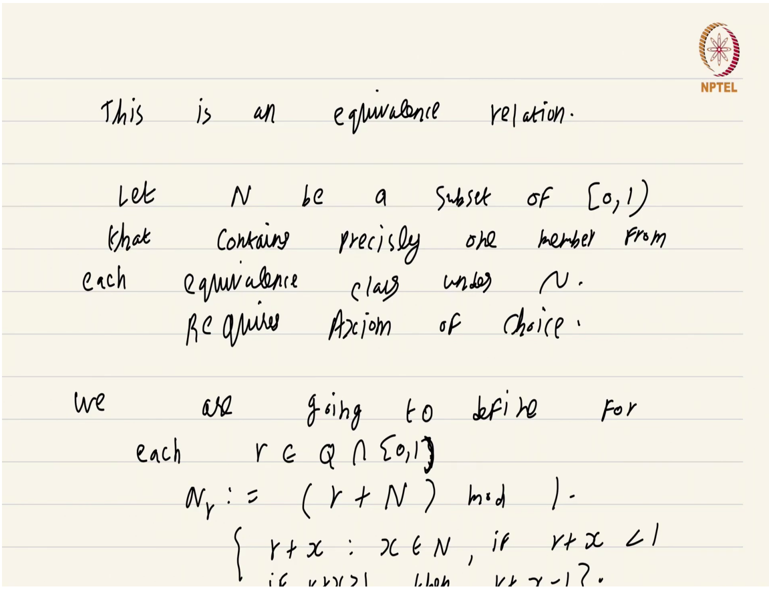

This is an equivalence relation it is just basic algebra ok. Now what you do is and at this step is the one where we require the axiom of choice what you do is, let N be a subset of close 0 open 1 that contains that contains precisely one member from each equivalence class equivalence class under this tilde under the sequence relation.

This equal insulation is going to partition the set close 0 open 1 into a number of pieces take exactly one element from each piece the fact that you can do this requires the axiom of choice no exception you really require the axiom of choice to do this ok. So, revisit that lecture if you have no if you have taken real analysis 1 if not just check the notes that I had posted in the very first week of this course.

You will notice that this is the axiom of choice is stated and think over why we require the axiom of choice and how exactly the axiom of choice is being used ok. Now what we are going to do is, we are going to define; we are going to define for each r in the rational numbers N r to be r plus N modulo 1 ok. What I mean by this is, I am going to so written out in more detail.

This is r plus x such that x comes from N if r plus x is less than 1, if r plus x is greater than or equal to 1 then r plus x minus 1 ok. So, a picture will illustrate what is going on.

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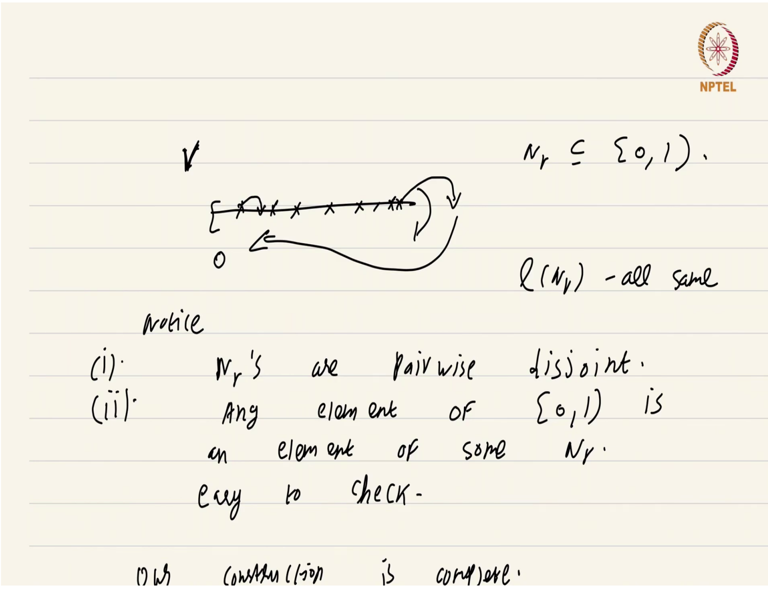

Suppose this is that close 0 open 1 we have this set N here somewhere ok, then we are adding this real not real number rational number r if this point moves here when you add r. Then you just add that to the set N r, on the other hand if you have a point almost towards closed to 1 and when you add r if it exceeds then just subtract 1. So, that you are back; you are back to the interval close 0 open 1.

Note that since we are taking a rational number so, this r is in Q intersect 0, 1 sorry about that 0, 1. Since this rational number comes from the interval 0, 1 this r plus N this no element of that can exceed 2. So, this will land up; this will land up always inside this set ok, excellent.

Now notice the following here is the part where I said I am going to be sketchy these are rather trivial, but you can check it notice that N r's are pairwise disjoint are pairwise disjoint and number 2 any element of close 0 open 1 is an element of some N r ok. So, both properties are really easy to check easy to check, the fact that property 2 is true is because we have taken one element from each equivalence class.

So, any element of close 0, 1 has to differ from one element of this set N by a real by a rational number and using that rational number you can find out which N r that real number is going to belong to. The fact that N r's are pairwise disjoint follows from the fact that we took only one element from each equivalence class. So, it cannot happen that the sums will be equal for two different numbers that that cannot happen. So, check these in detail they are not so hard they are quite straightforward.

So, once this is done our construction is complete, we have found these sets, we have found these sets N r who whose union is going to be the whole of closed 0 open 1. And of course, l of N r's are all same, because they are just all translates of the single set capital N.

So, this will sort of give you a contradiction. So, we have shown that it is not possible to simultaneously satisfy all the conditions. So, in a course of measure theory what you do is instead of considering a function l from the whole power set of R you consider on a large subset large subset ok. Essentially on what is known as the Borel sigma algebra it will contain all open sets, close sets, intersection of open sets, intersection of union of open sets so on and so forth.

You will consider a collection of sets that is quite large and is convenient for all purposes of analysis and on that you will be able to define a length function which is called the Lebesgue measure ok. We will do this in this course as well, but using the Lebesgue integral that we already have and as I mentioned once you have the Lebesgue measure you can define the Lebesgue integral in the way Lebesgue as classically proceeded.

So, this was just a brief motivational video for what is to come and for the future courses in measure theory.

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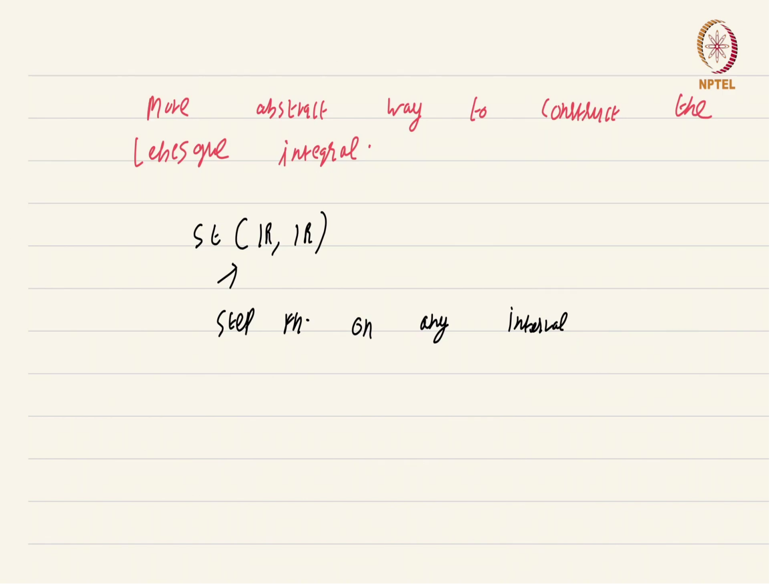

Let me also tie up one loose end that is more abstract way; more abstract way to construct the Lebesgue integral. We had sort of taken a hands on approach to define the Lebesgue integral, we can also take a more abstract approach. Let me just briefly illustrate what I mean by this, what you do is you look at the collection step R, R. What this means is step function on any interval a, b or rather better way to phrase it would be.

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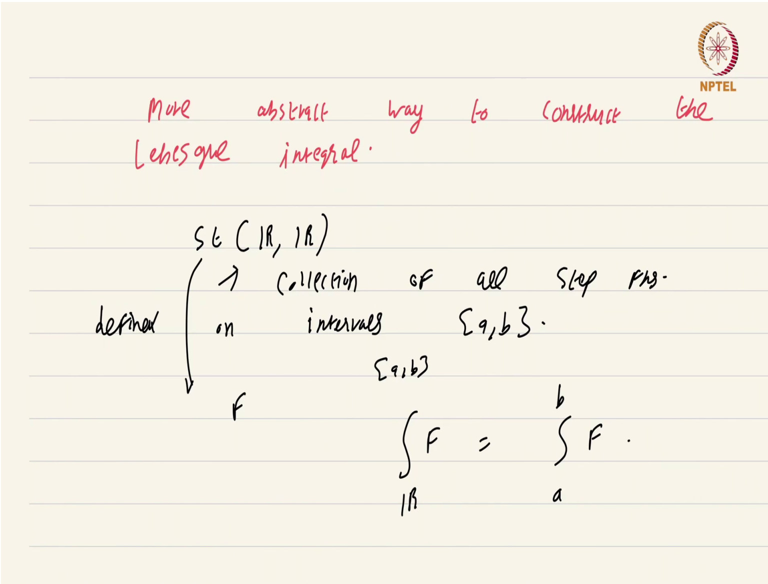

This is collection of all step functions; collection of all step functions defined on intervals a, b that is any step any function that is a step function on some interval closed a, b will be there in this collection. So, all possible step functions you have now put together in this collection called S t R R.

And if you take a function F here and you know that there is some interval a, b on which F is a step function you define this integral of F over R to be integral a to b F. And as we have already studied before this is actually well defined, this function F is a step function over several intervals it does not matter which you take the integrals will always be the same ok.

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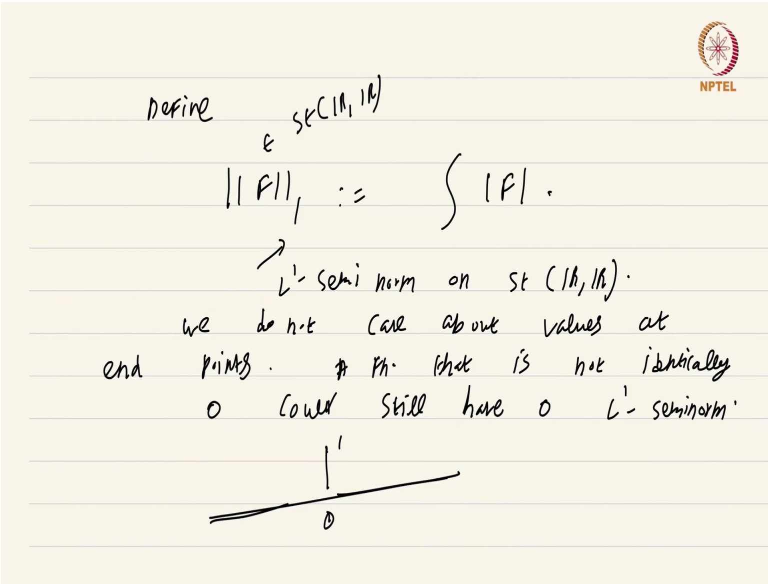

Moreover, define the l 1 on norm of an element where F is coming from the step function on R by definition to be integral mod F ok. Now this is called the L 1 semi norm on S t R R, now why is it a semi norm and not a norm well because the way we have defined step functions we do not care about values at the endpoints right.

We do not care about values at endpoints values at endpoints, I said that step function means you can find a partition such that on the interior of each interval determined by the partition the function is constant at the end points I do not care what the value is; that means, that to a function that is; function that is not identically 0 could still have could still have 0, L 1 semi norm right.

For instance, if I take a function which is just let us say at the point 0 it is 1 and everywhere else it is 0 its going to be a step function in our definition you can check that, but the integral is; obviously, going to be 0, but the function is not 0 that is why this is a L 1 semi norm ok.

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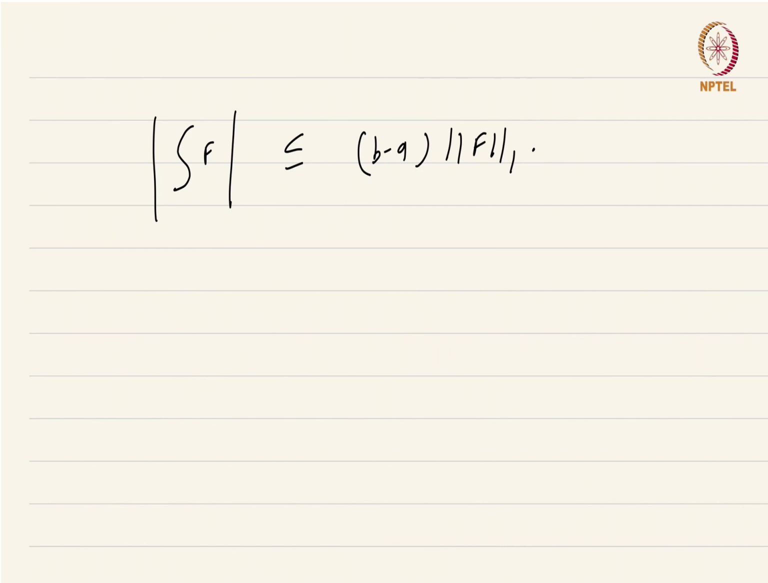

From what we have established we already know that this integral of F absolute value is going to be less than or equal to b minus a times the l 1 norm. Sorry there is no b minus a here the integral of F is just going to be less than or equal to the absolute value of the integral of F is less than the l 1 norm.

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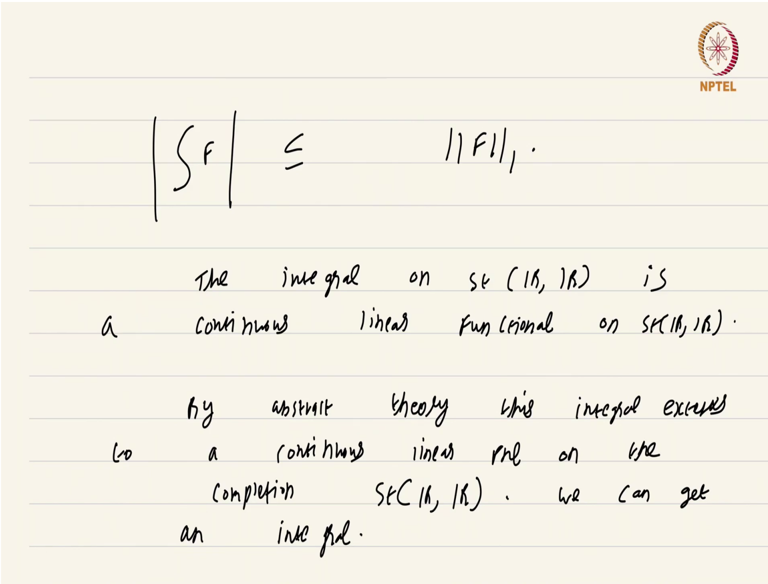

In other words the integral the integral on S t R, R is a continuous linear functional on S t R, R. Note there is a slight technicality because we do not really this S t R, R is not a normed vector space for the reason I mentioned we do not have a norm we have a seminorm, but that technicality can be fixed by defining an equivalence relation that says that two step functions are equal if the difference has 0 l 1 norm and so on, those are minor technical points which are going to distract from the main goal the main idea.

So, I am not going to emphasize these minor technical points. But just take it for granted that the integral is going to be a continuous linear functional on S t R, R the fact that linearity we have we have already established in all its detail. The fact that it is continuous is just this fact only minor technical point is this is not a normed vector space, but a semi normed vector space, but that is a minor point.

By abstract theory that we developed in the first part of the course by abstract theory this integral extends to a continuous linear functional on the completion of S t R, R. Again you can ask me, what is the completion this is not a normed vector space and so on just bear with me this is just going to highlight something these technicalities are rather boring to fix, but they can be easily done ok.

So, that way we can get an integral; we can get an integral, only issue is that the completion of S t R, R is some really complicated thing. The construction that we have done for the completion of a normed vector space is quite abstract; well, there is a theorem which is often known as the fundamental theorem.

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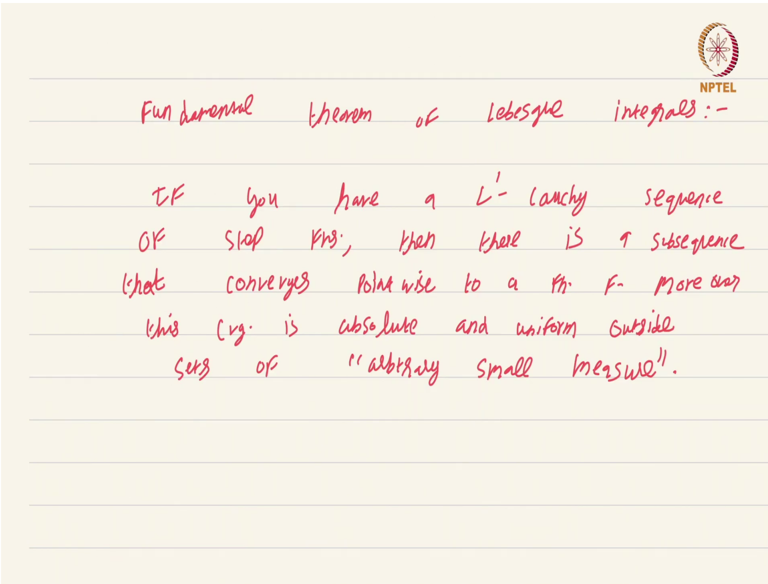

Fundamental theorem of Lebesgue theory, theorem of Lebesgue integrals that sort of says the following if you have if you have L 1 Cauchy sequence of step functions, then there is a subsequence that converges point wise to a function F. Moreover, this convergence is absolute and uniform outside sets of arbitrarily small measure very small measure I am not going to elaborate what this means.

So, essentially what happens is you have an abstract completion and you do not know what that space is its going to be either some equivalence classes or some other really complicated thing, but what happens is when the moment you have an L 1 Cauchy sequence of step functions you get a subsequence that converges point wise to a proper function and this subsequence not only converges point wise, but outside sets of arbitrarily small measure this C sub sequence converges uniformly and absolutely also ok.

So, this can be taken care of the fact that this you are no longer in an abstract space I mean you can make these abstract spaces very well down to earth function spaces without difficulty. Of course, I am not going to pursue this more abstract approach, but this is to tie up with the video on Riemann integrals where we completed the space of step functions when we began our study of the Lebesgue integral. The same thing can be done you can do adapt this abstract approach to the Lebesgue integral as well.

Anyway this was sort of just a motivational video for what is to come and tying up some loose ends we will now go on and proceed with some more study of the Lebesgue measure as well as studying measurable functions etcetera from the perspective that we have adopted, that the space of Lebesgue integrable functions was defined using step functions the way we did it straightforward and not so tedious at the same time completely concrete and down to earth.

This is a course on Real Analysis and you have just watched the video on The Problem of Measure.