Real Analysis II
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Lecture - 28.2
Monotone Convergence Theorem for Upper Functions

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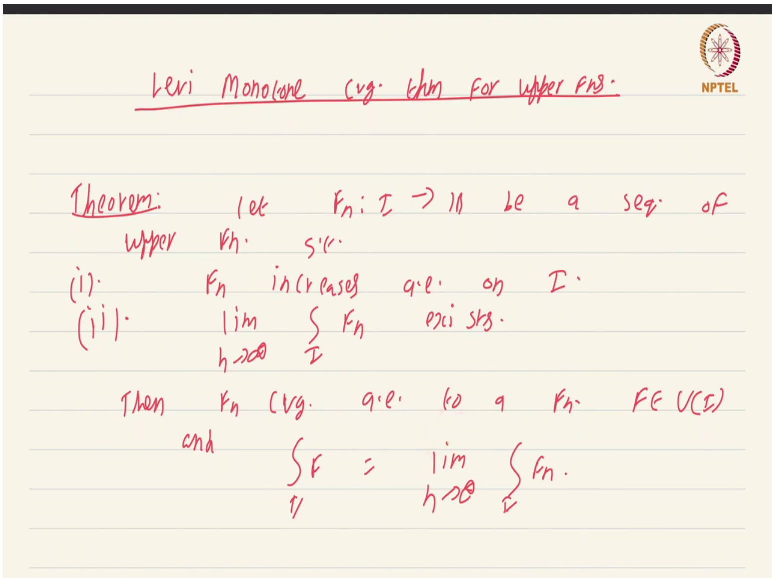

In this week we are going to see a proof of Levis Monotone Convergence Theorem for Upper Functions. We have already proved the theorem for step functions, in this video we will adapt the argument given there to the general case of upper functions. In the next video we will break up a Labesgue integrable function as a difference of two upper functions and get a Levi monotone convergence theorem for all Labesgue integrable functions.

So, here is the statement of the theorem its exactly similar to what we have seen for step functions. Let F n from I to R be a sequence of upper functions, such that number 1, F n increases almost everywhere on I.

Note there is a slight difference in this version compared to what we saw in the last video, there we assume that the step functions are increasing everywhere. But simply because step functions take only finitely many values, think about why assuming almost everywhere for step functions makes no sense.

Second; limit n going to infinity integral over I F n exists. So, the integrals are bounded above in a sense, then F n converges almost everywhere to a function F which is an upper function and the integral of F is nothing but the limit of n going to infinity integral of I F n. So, in some sense you can interchange the limit and the integral the limit and the integral can be interchanged.

One nice aspect about the monotone convergence theorem is that we are just assuming that the limit n going to infinity integral of I F n exists that automatically guarantees the existence of a limit function F whose integral is exactly the limit of the integrals which is something nice ok. So, as I said we are going to use the fact that we have already shown for step functions.

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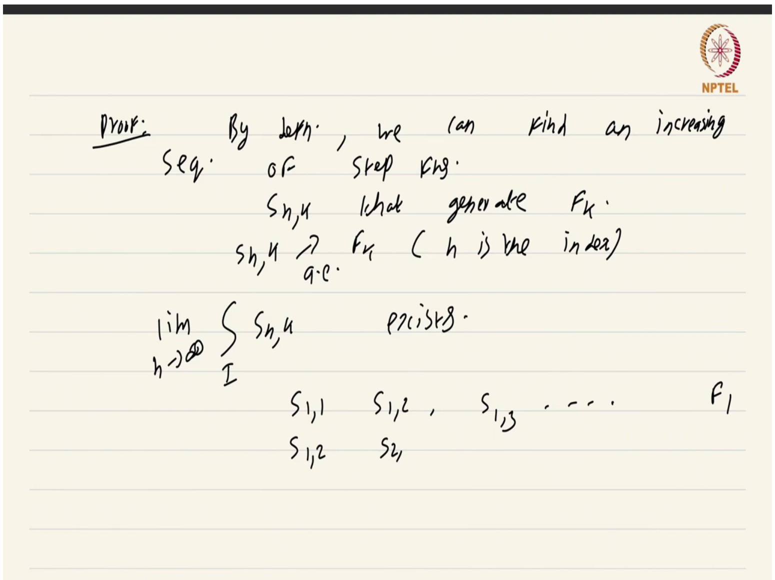

So, proof by definition we can find, we can find an increasing sequence an increasing sequence of step functions S n k that generate F k. Recall this means that S n k increases almost everywhere to F k here n is the index n is the index. So, I should write almost everywhere and this integral S n k over I this limit n going to infinity exists this is the meaning of S n ks generate F k.

So, let us arrange them all in an array we have S 1 1, S 1 2, S 1 3, dot dot dot and this increases almost everywhere to F1. Similarly, we have the second one S 1 2, S I made a slight mistake in the indexing the running index is n not k, so let me just fix that.

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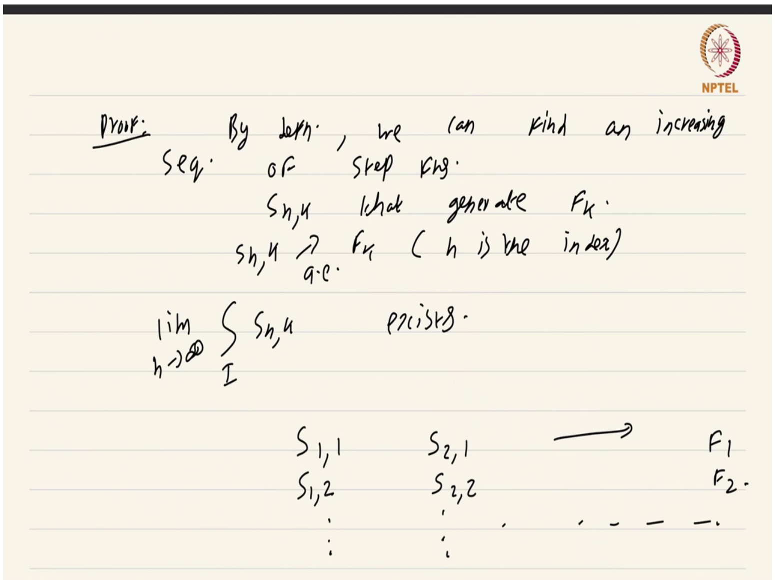

So, you have S 1 1 you have S 2 1 dot dot dot converging to F 1, then you have S 1 2 S 2 2 dot dot dot converging to F 2, you have this in both directions ok. And note these S n ks are increasing sequence of step functions they are an increasing sequence of step functions and they increase almost everywhere to F k ok.

Now, what we are going to do is we are going to produce new step functions combining together the data obtained by these step functions which converge to F k right.

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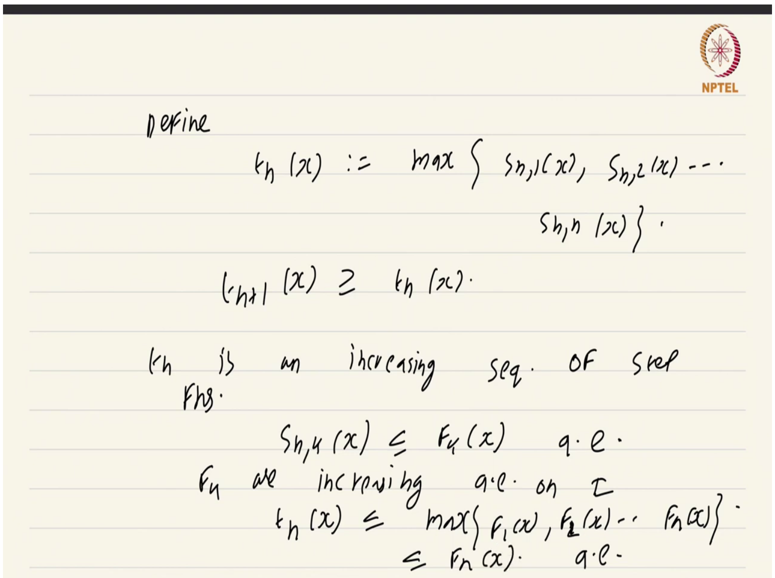

So, what we do is, we define t n of x by definition to be max of S n 1 of x, S n 2 of x comma dot dot dot S n n of x right. So, what are we doing? We are looking at the nth column we are looking at the nth column and we are taking the max all the way up till the nth entry ok. So, we start at for instance t 11 would be just S 1 1 sorry t 1 would be just S S 1 1 t 2 would be the maximum of S 2 1 and S 2 2 ok and t 3 will be the maximum of S 3 1, S 3 2, S 3 3 all the way up till the third entry ok.

Now, because we are taking the maximum always to the right that is t n plus 1 will be obtained by taking a maximum of functions which are to the right of the functions that you are taking for t n. And since each row the functions are increasing its straightforward to see that t n plus 1 of x is greater than or equal to t n of x, that is t n is an increasing sequence.

Furthermore, since it is a maximum of finitely many step functions it is also a step function it is an increasing sequence of step functions excellent.

Now, by definition we have S n k of x is less than or equal to F k of x almost everywhere ok, because this is because the sequence S n ks generate the function F k. And since F ks are increasing almost everywhere on I well we automatically have that t n of x is going to be less than the maximum of F 1 of x comma F 2 of x dot dot dot F n of x right.

Now of course, this will be true almost everywhere this will be true almost everywhere and notice that because this functions these functions F 1 F 2 dot dot dot F ns are also increasing almost everywhere I can write this is less than or equal to F n of x almost everywhere.

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$t_n(x) \le F_n(x)$ a.e. on I.

$\int_I t_n \le \int_I F_n$.

$\lim_{n\to\infty} \int_I F_n$ exists.

t_n must converge to a fn. $g \in U(I)$ (MCT for step fns.).

$\int_I g = \lim_{n\to\infty} \int_I t_n$.

So, the net upshot is t n of x is less than or equal to F n of x almost everywhere on I ok. Now both sides are upper functions, so we can use the results involving upper functions and the behaviour of integrals under order to conclude that integral over I t n is less than or equal to integral over I F n.

But our hypothesis is that limit n going to infinity integral of I F n exists and since this sequence of integrals is an increasing sequence, this means this quantity integral I F n is bounded above. Which means that integral of I t n is bounded above and since these t ns are increasing functions the net conclusion is that t ns must converge to a function g in U of I and this follows from monotone convergence theorem for step functions ok.

So, we have just constructed these new step functions which are increasing and because these new step functions are increasing and they are bounded above we get that we get a limit function and not only that we get that integral of g integral of g over I is nothing but limit n going to infinity integral over I of t n ok.

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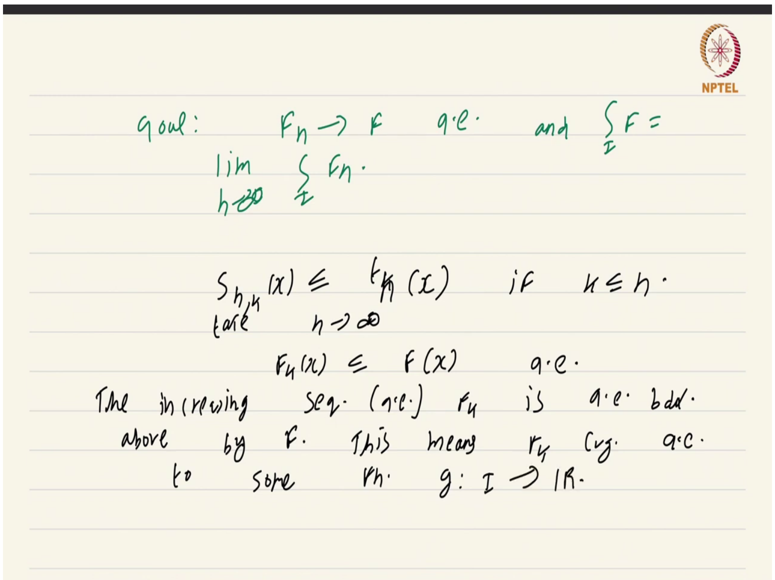

Now, what do we do now? We are going to show, goal is to show that F n converges to F almost everywhere and integral of I F is nothing but limit of n going to infinity integral of I F n let us keep this goal in mind ok. Now, as we have observed before S n k of x is less than or equal to t k oh sorry t n of x if k is less than or equal to n. This is because we are just taking the maximum of S n 1 S n 2 dot dot dot S n n, so this is just by definition ok.

Now, take n going to infinity take n going to infinity. So, what you will get is F k of x is less than or equal to limit n going to infinity t n of x which is just F of x. But this will be true only almost everywhere this will be true only almost everywhere. This means that the increasing sequence increasing sequence that is it is almost everywhere increasing of course, F k is almost everywhere bounded above by F.

This means F k converges almost everywhere to some function to some function g from I to R ok. But we already know that by our construction these functions t n x are less than or equal to F n x.

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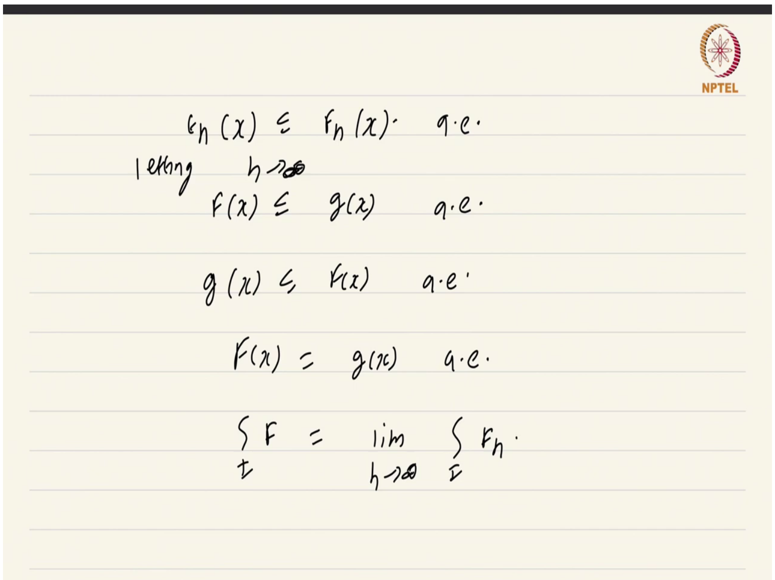

Recall, these t n xs were obtained by taking the maximum along the nth column all the way up till the n. And since the functions here to the right are all almost everywhere greater than or equal to we get this inequality that t n of x is less than or equal to F n of x almost everywhere ok.

Now letting n go to infinity letting n go to infinity we get that the limit of the left hand side is F of x by definition and the limit of the right hand side is g of x and this is true almost everywhere ok. But we also have we also have g of x is less than or equal to F of x almost everywhere.

Why do we have g of x is less than or equal to F of x almost everywhere? Well, we already know that F k of x is less than or equal to F of x and this is just the point wise limit of F k of x, therefore g of x will be less than or equal to F of x combining both we get F of x equal to g of x almost everywhere excellent.

Now, we have to show what remains to be shown is integral of I F is limit n going to infinity integral of I F n ok.

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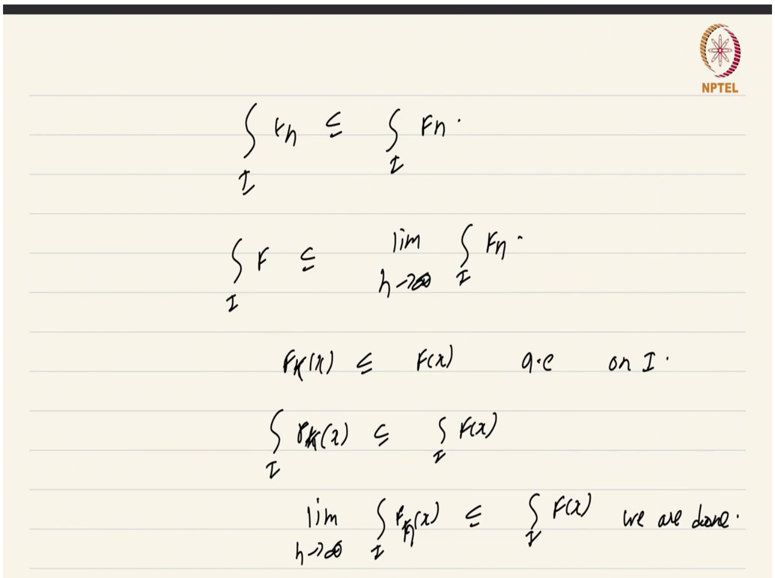

We have the following relationship that we have obtained earlier integral of I t n is less than or equal to integral of I F n and taking limits will tell you that integral of I F is less than or equal to limit n going to infinity integral of I F n. We also have the relationship that F k of x is also less than or equal to F x almost everywhere on I ok.

And again that means, that integral of I F k of x is less than or equal to integral of I F of x and taking limits on both sides tells you that limit n going to infinity of integral of I F k of x or rather limit F n of x is less than or equal to integral of I F of x and we are done we are done.

So, this concludes the proof of Levis monotone convergence theorem for upper functions, in the next video we shall use this result to show the similar result for Labesgue integrable functions. This is a course on Real Analysis and you have just watched the video on Levi monotone convergence theorem for upper functions.