**Real Analysis II**
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**Lecture - 26.1**
**Upper Functions and their Integral**

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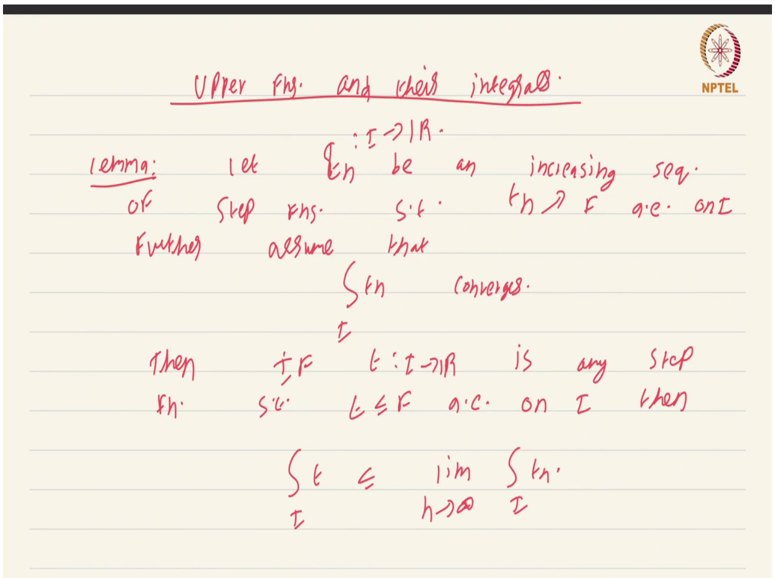


In this video we shall define a class of functions that arise as then increasing limit of step functions and we will also define how to define the integral of such functions. To begin with we need a simple lemma. This talks about the behavior of the limit of an increasing sequence of functions. Let t n be an increasing sequence of step functions increasing sequence of step functions such that S n sorry t n increases to a function F almost everywhere on the interval I.

So, these t ns are from I to R, ok. Further assume that integral of t n over I converges. Then if R if t from I to R is any step function such that t is less than or equal to F almost everywhere on I, then integral of t is less than limit n going to infinity of integral of t n. So, this is a somewhat technical sounding lemma, but this will be the basis for the definition of the integral of the upper functions.

What it is essentially trying to say is the following. If you have a sequence of a step functions that increase to a given function, then no step function that is less than or equal to the limit function can have a greater integral. That is essentially what this is saying as you can guess we are going to define the integral of the function f using this formula and this fact that integral of t over I is less than or equal to limit n going to infinity integral of I t n will sort of help us to prove the well definedness of the integral of the function F.

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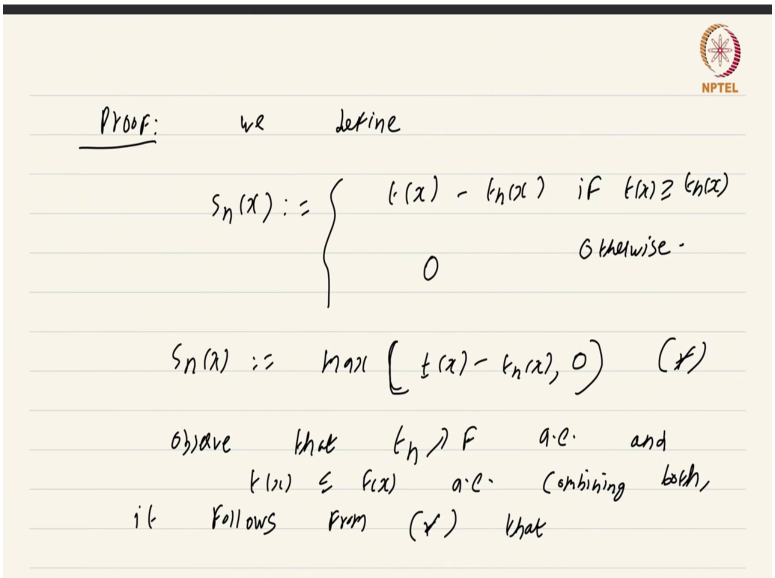


So, let us see the proof the lemma looks technical the proof is rather trivial. It just follows immediately from the previous theorem that says that if you have a sequence of step functions that decrease almost everywhere to a function a to 0 then the integrals also converge to 0. Now, what we are going to do is, we are we are not in the situation of the previous theorem. The previous theorem had a decreasing sequence of step functions.

So, we manufacture a decreasing sequence of step functions using this sequence of step functions t n. So, what we do is we define we define S n x by definition to be equal to t n x sorry, t of x minus t n x if t of x is greater than or equal to t n x; remember t is some step function that is less than or equal to f 0 otherwise. So, in more fancier notation S n x is just the maximum of t n x sorry, t of x t of x minus t n x and 0, ok.

Now, observe that t n increases to f almost everywhere and t of x is less than or equal to f of x almost everywhere. Combining both it follows from.

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$S_n \downarrow 0$ a.e.

From the previous thm.

$$\int_I S_n \to 0.$$

Because $S_n(x) \geq t(x) - t_n(x)$

$$\int_I S_n(x) \geq \int_I t(x) - \int_I t_n(x)$$

let $n \to \infty$

Let us call this star it follows from star that S n decreases almost everywhere to 0, ok. Since t n's are an increasing sequence of functions it is clear that S n's are decreasing and furthermore they must decrease to 0 because t n's converge almost everywhere to F. So, at some point of time t n has to become greater than t because t is less than or equal to F. So, this first term has to become less than or equal to 0 after a point in which case this maximum will become 0.

So, from the previous theorem from the previous theorem integral of S n over I converges to 0. Now, because S n of x is greater than or equal to t of x minus t n of x because it is the maximum of this quantity comma 0, using the various properties of step integral integrals of step functions we get integral over I S n of x is greater than or equal to integral over I t of x minus integral over of integral over I t n x, right.

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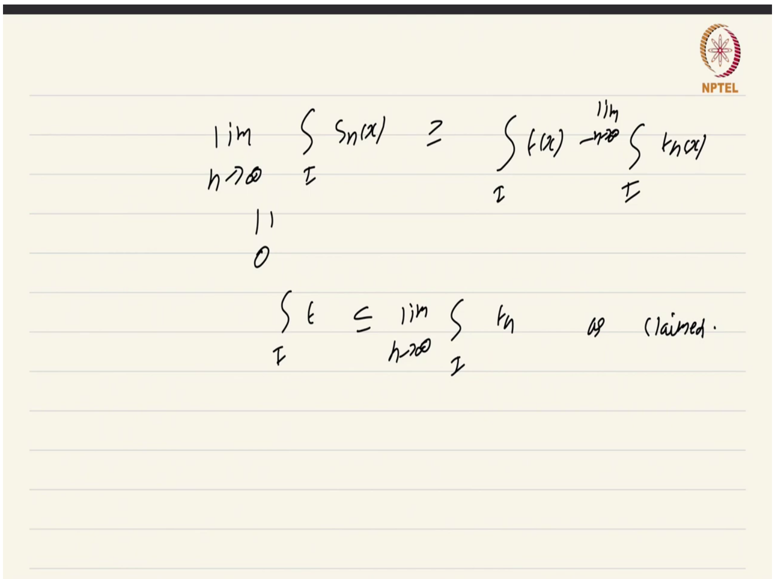

Now, let n go to infinity n go to infinity. So, we get limit integral over I S n of x n going to infinity which is just 0 and this is greater than or equal to integral over I t of x minus integral over I t n of x limit n going to infinity. In other words, we get integral over I of t is less than or equal to integral over I t n limit n going to infinity as claimed.

With this basic lemma in hand, we can now define upper functions which is going to be a class of functions and we can define the integral for upper functions.

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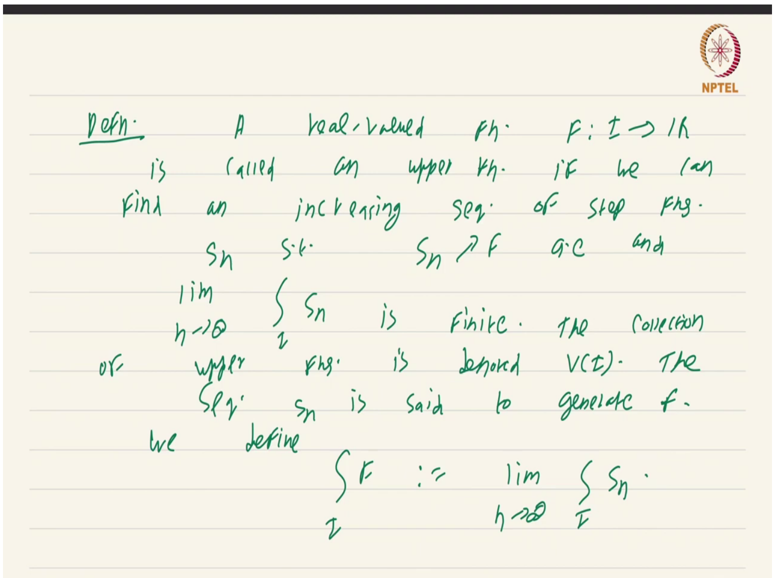


So, the definition is as follows definition is as follows. A real valued function a real valued function F from I to R, where I is some general interval is called an upper function an upper function if we can find we can find an increasing sequence of step functions S n such that S n increases to F almost everywhere and limit n going to infinity integral over I S n is finite, ok.

If you can find a sequence of increasing functions which increase almost everywhere to F then we say F is an upper function. The collection of upper functions is denoted U of I. The sequence S n is said to generate F that is for a sequence of step functions to generate F first of all it must be increasing, second of all it must increase almost everywhere to F, third point the integral should the limit of the integral should be finite.

We define we define integral of F over I by definition to be the limit as hinted before several times, ok. Now, notice that this sequence integral of I S n because S n's are increasing because of the property of integrals of step functions. This sequence integral I S n is also an increasing sequence. So, for this limit to exist it actually it is actually enough to assume that this sequence of integrals is actually bounded above. That is an another equivalent way of stating the same thing.

Now, the crucial part is there could be several sequences of step functions that generate F. What is the guarantee that all of them lead to the same value of integral of F over I? Essentially what is the guarantee that this collection of functions U of I the integral defined on this collection why is it well defined. Well, that is the content of the next theorem.

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Thm (well-definedness of the integral): let $F \in U(I)$ and let $s_n, t_n : I \to \mathbb{R}$ generate $F$. Then

$$\lim_{n\to\infty} \int_I s_n = \lim_{n\to\infty} \int_I t_n .$$

Proof: Observe that each $t_n \leq F$ a.e.

The previous lemma gives

$$\lim_{n\to\infty} \int_I t_n \leq \lim_{n\to\infty} \int_I s_n$$

So, this theorem says well definedness of the integral of the integral. So, let F be an upper function and let S n comma t n from I to R generate F then integral of S n limit n going to infinity is equal to limit n going to infinity integral of t n, ok. Now, observe the following I mean this proof is rather an easy consequence of the previous lemma. So, observe that each t n is less than or equal to F almost everywhere.

So, if you look at the previous lemma you look at the previous lemma it says that if you have a step function t which is less than or equal to F almost everywhere and the sequence of functions t n increase almost everywhere to F then integral of that t is less than limit n going to infinity of t n integral t n. So, what that shows the previous lemma immediately gives us the previous lemma gives integral of t n I is less than or equal to limit n going to infinity of integral of I S n, ok.

And, you can just pass to the limit on the left hand side also. So, we get that limit n going to infinity integral of I t n is less than or equal to limit n going to infinity integral of I s n.

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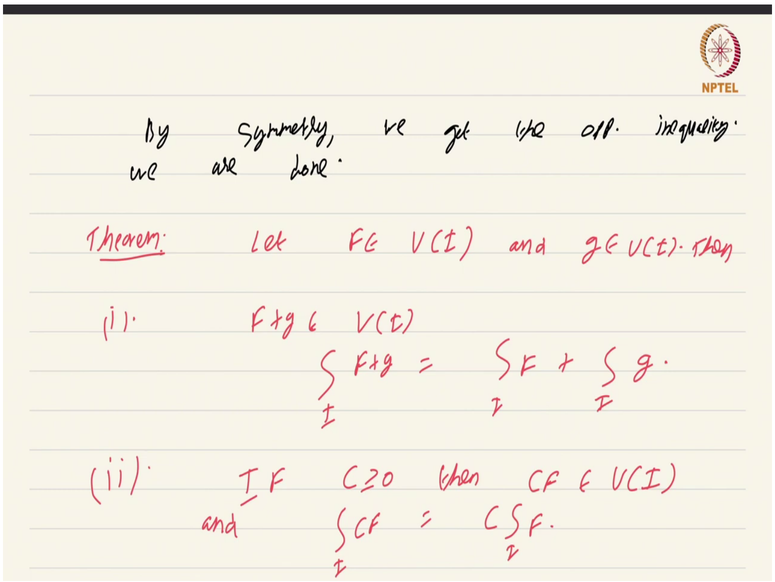


By symmetry we get the opposite inequality same argument we get the opposite inequality. We are done. So, the well definedness of the integral is very easy to show from the lemma and ultimately that lemma was a consequence of the behavior of decreasing sequences of step functions.

So, ultimately everything hinges on the theorem that we did in the previous video, ok. Now, let us state some simple properties of this upper integral and we will in the next few videos we will extend this definition to a larger class and this next theorem sort of highlights the problem with this particular collection of functions. It is not really a vector space which is sort of not nice.

Let F be an upper function let F be an upper function and g also an upper function then number i, F plus g is an upper function and integral of F plus g over I is just integral of I F plus integral of I g. Number ii, here is the problem if C is greater than 0, then C F is an upper function and integral of C F over I is C times integral F. So, here we actually require C greater than or equal to 0.

There are upper functions whose negative is not an upper function and this is sort of problematic. This is sort of problematic and not nice. There are explicit examples you can check apostles exercises where there is an explicit example listed.

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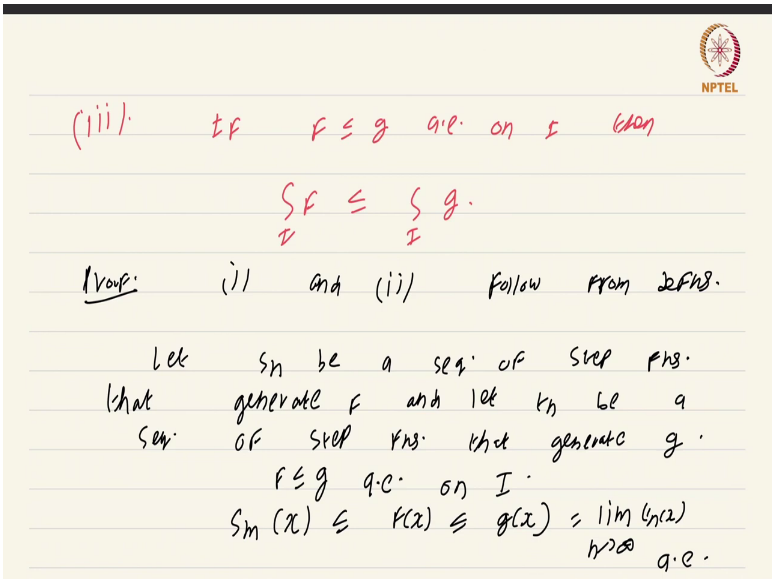

Property iii: if F is less than or equal to g almost everywhere on I, then integral over I F is integral less than or equal to integral over I g, ok. So, parts i and ii, i and ii follow from definitions there is not really nothing to prove follow from definitions, ok and I want you to prove ii in particular a i is pretty much trivial I want you to prove ii in some detail and see where the proof breaks down if C is less than 0 that is a very instructive exercise.

Let us move on to the proof of part iii. Let S n be a sequence of step functions of step functions that generate F and let t n be a sequence of step functions that generate g. Now, the hypothesis is that F is less than or equal to g almost everywhere on the interval I ok; that means, S m of x is less than or equal to F of x is less than or equal to g of x which is nothing but limit n going to infinity of t n of x and these equal inequalities and equalities are true almost everywhere ok.

Now, that means, from the previous theorem that we had the big theorem about what happens if a step function sequence of step functions converge to a function F.

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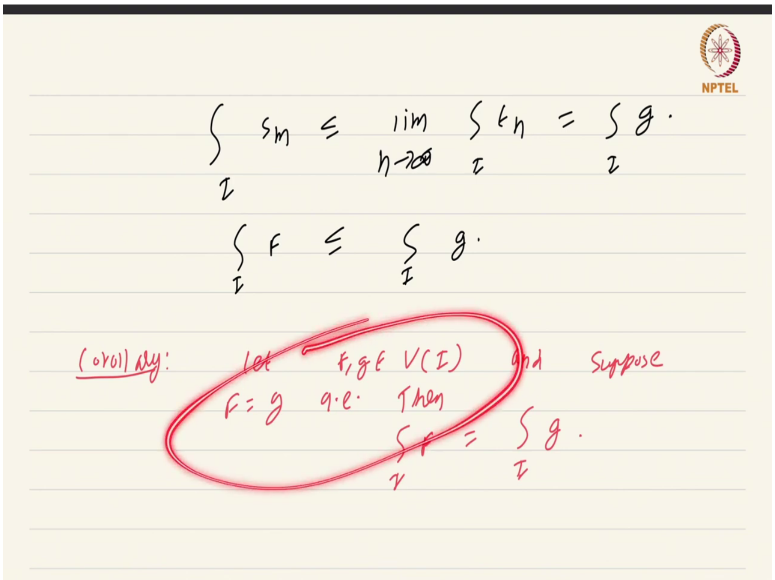


If you take any other step function that is lesser than or equal to that function, then we definitely have integral over I of S m is less than or equal to limit n going to infinity integral over I t n which is equal to integral over I g this just follows from the big theorem which we proved in the last video.

Now, taking limits on the left hand side taking m going to infinity we get integral over I F is less than or equal to integral over I g. So, this was a fairly straightforward proof. An easy corollary of this is something that you might have anticipated already. Let F comma g be upper functions on the interval I and suppose F is equal to g almost everywhere then integral of I F is integral of I g.

And, this follows immediately by observing that F equal to g almost everywhere is the same as saying F less than or equal to g almost everywhere and g less than or equal to F almost everywhere and applying the previous result. So, we have now shown one nice property of the Lebesgue integral. In there for the Riemann integral we know that a necessary and sufficient condition for the existence of the Riemann integral is the set of discontinuities being a set of measure 0.

But, you can take a function and modify it on a set of measure 0 to make it discontinuous everywhere. Please think about the remark I said carefully what will happen is you can start with the continuous function and just modify it on a really small set to not make it Riemann integrable. That is not something that you want from an elegant theory of integration.

Here in the case of Lebesgue integral which we are going to eventually define at least for upper functions if you have two upper functions that agree almost everywhere then their integrals are actually equal which is what we sort of expect, because these two functions are the same except on a negligible set, ok. So, the last few properties that I want to prove for the integral on upper functions is about their property when you break up the interval.

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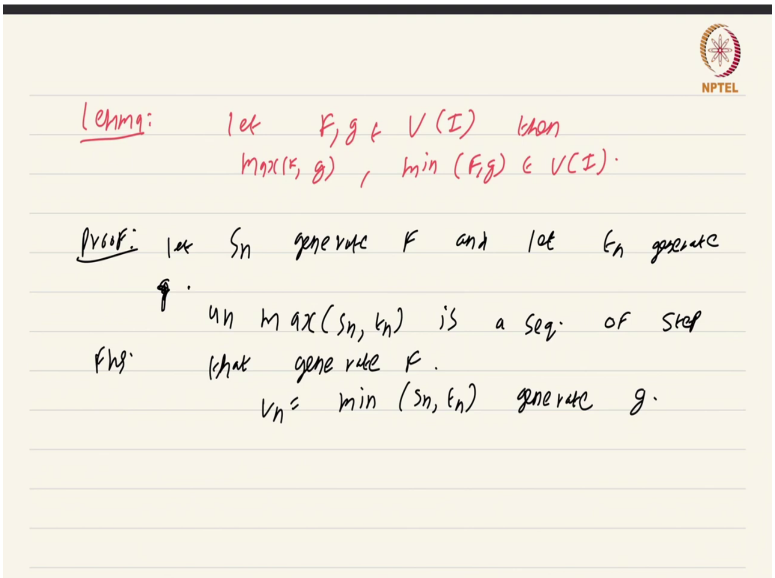


For that the following technical looking thing is useful. So, let me call it a lemma which is going to be used in the next theorem about breaking up the intervals, the lemma is as follows. Let F comma g be upper functions then both max F g as well as minimum F g are also upper functions.

And, the again the proof of this is rather straightforward pretty much everything I have done except for the proof of that theorem about monotone sequences where a little bit of analysis was actually involved pretty much everything is just a straightforward corollary of that theorem and the various definitions that we have.

So, what we do is let S n generate F and let t n generate g, this is sort of the standard first line boiler plate for every single lemma. Now, we want a sequence that generates max F g and min F g and you can easily check that max of S n, t n is a sequence of step functions sequence of step functions that generate F. Let us just call this sequence u n and similarly, v n equal to minimum of S n t n generate g.

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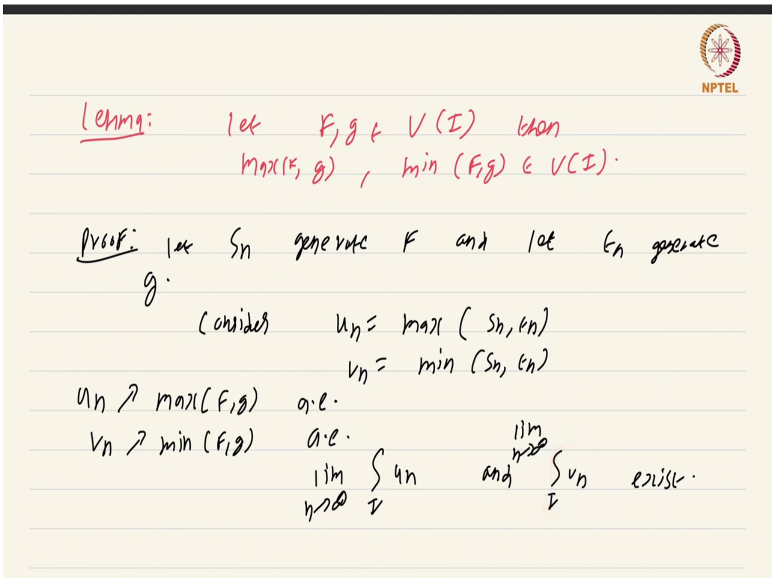


Now, actually I should not technically be using the fact that I mean I should not technically say generate F, so, to be 100 percent precise what we will do is we will consider u n equal to max S n comma t n and v n equal to min S n comma t n.

All we can assert is u n increases to max F g almost everywhere and v n increases to min F g almost everywhere. Remember, when you say that a sequence generates a given upper function you will have to first of all show that the integrals the limit of the integrals converge only then is actually the limit function even an upper function in the first place. So, we know that both u n and v n increase to max F g.

So, what we have to show is integral of I u n and integral of I v n exist and as I have remarked earlier actually the limits limit n going to infinity of both of this exist as I have remarked earlier because v n's are increasing this is actually going to be an increasing sequence the integral over I v n. So, it is suffices to prove that this sequence is actually bounded above.

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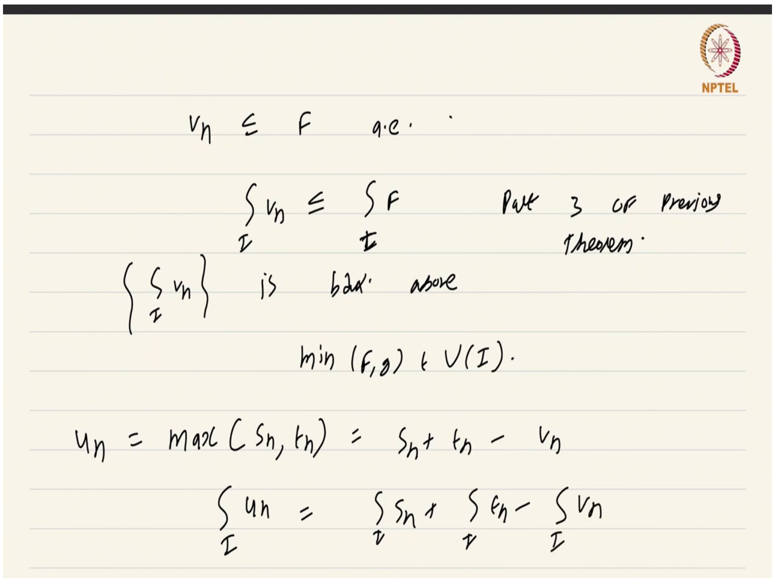


And, it is rather easy to see that this sequence is bounded above simply because v n is less than or equal to F almost everywhere, right and this just follows because v n is defined to be minimum of S n comma t n and S n is less than or equal to F almost everywhere.

So, because v n is less than or equal to F it follows that integral of I v n is less than or equal to integral of I F. This is by part 3 of the previous of previous theorem. This shows that integral of I v n this sequence this set is bounded above. So, that shows that minimum of F g is an upper function, ok.

Now, how do you show that max of F g is an upper function you cannot pull the same trick because you do not have u n is less than or equal to F for obvious reasons. So, what you do is you try to be slightly clever you observe that maximum of this S n comma t n is actually just S n plus t n minus v n ok. In other words, finding out the maximum of the two quantities is same as summing up the two quantities and subtracting the lesser of the two quantities intuitively obvious and that is what we have used here.

So, u n is S n plus t n minus v n therefore, by the algebraic properties of the integrals for step functions note these are all step functions I already remarked that when you have a function F it is not guaranteed that the negative of the function is an upper function you can write this simply because these are all step functions integral of I u n is integral of I S n plus integral of I t n minus integral of I v n ok.

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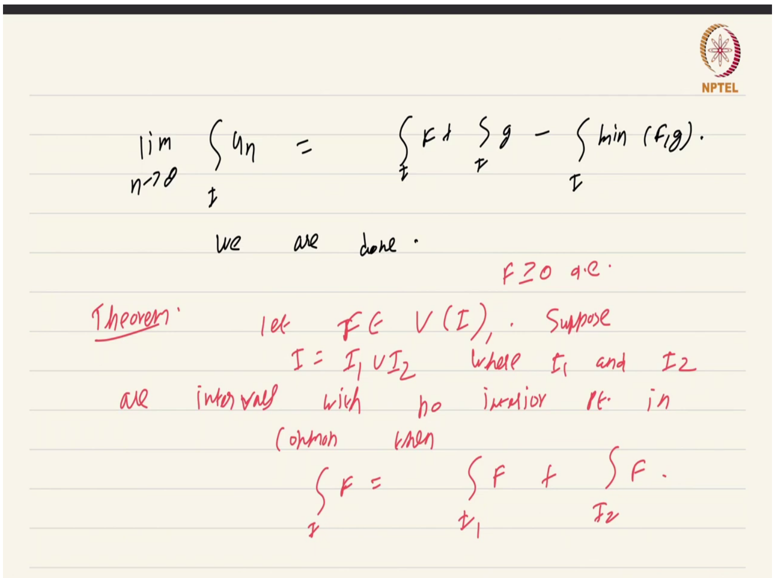


And, as limit n goes to infinity this is just integral of I u n limit n going to infinity of this is just integral of I F plus integral of I g plus integral of I is not plus minus, minus integral of I minimum of F g. And, that finishes the proof we have already shown that mean this integral exists and therefore, we are done. So, we are done. So, this somewhat technical looking lemma we are going to use to show that you can split the interval of definition of an upper integral upper function and evaluate the integral by breaking up the intervals.

So, let us state the theorem let F be an upper function on the interval I. suppose I am going to state it for just two intervals, you can use induction and prove the same thing for finitely many integrals. Suppose, I is I 1 union I 2 where I 1 and I 2 are intervals with no interior point in common are intervals with no interior point in common interior point in common. Then integral of I F is just integral of I sorry, integral of I one F plus integral of I 2 F. So, you can split the definition the domain of definition into two intervals and then you can evaluate by evaluating the integral on the two pieces separately.

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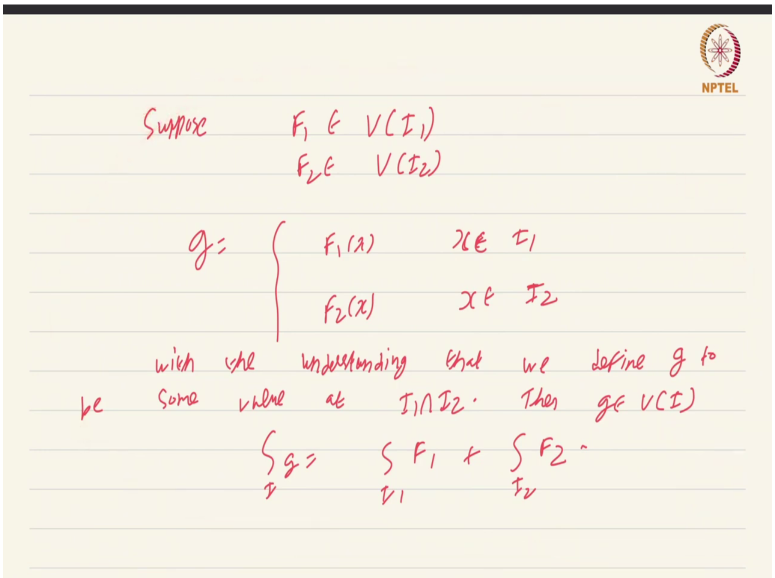


Similarly, suppose F 1 is an upper function on I 1 and F 2 is an upper function on I 2 and we define the new function g to be F 1 of x if x is in I 1 and F 2 of x if x is in I 2 technically this definition is not exactly precise because it can happen that there is an end point in common between I 1 and I 2 with the understanding with the understanding that understanding that we define F to be some value at the define F sorry, g to be some value at I 1 intersection I 2.

It really does not matter whatever value you put, you can choose your favorite real number and put F to be that in that place. One second, I think I missed one crucial hypothesis. I actually require F is an upper function and F is greater than or equal to 0 almost everywhere.

You will understand why I am going to put this hypothesis why I need to put this hypothesis here in the proof for this it does not matter.

For this second part it really does not matter, F 1 and F 2 could be any upper functions whatsoever. Then g is in u of I and integral of I g is just integral of I 1 F 1 plus integral of I 2 F 2, ok. So, this second part I am going to leave this second part I am going to leave it to you it is fairly easy. Let us deal with the first part.

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Proof: Let $S_n$ generate $F$.

$S_n^+ := \max(S_n, 0)$ $\quad F \geq 0$ a.e.

$S_n^+$ also generate $F$. $\quad S_n^+$ increase to $F$ a.e.

Now note that

$$\int_{I_1} S_n^+ \leq \int_I S_n^+ \leq \int_I F.$$

$S_n^+|_{I_1}$ generates $F|_{I_1}$. Similarly

And, see where that extra hypothesis that you need F greater than or equal to 0 almost everywhere where is that used. So, let us go on to the proof. So, let S n generate F. Now, I want to use this S n to generate sequence of positive step functions.

So, what I do is I look at max of S n comma 0 and I call that S n plus ok because F is greater than or equal to 0 almost everywhere, we have S n plus S n plus also generate F ok. Here is the crucial part where I required that F is greater than or equal to 0 almost everywhere. The fact that S n pluses are increasing is obvious because S n is increasing, but the fact that S n plus increases to F almost everywhere requires F to be greater than or equal to 0 ok.

Now, note that integral of I 1 of S n plus is obviously, less than or equal to integral of I S n plus which is in turn less than or equal to integral over I F ok which means S n plus generates F restricted to I 1. So, for this inequality that integral of I S n plus is less than or equal to integral of I S n plus I require the positivity of the functions.

What is essentially happening is I is a larger interval, but since the values of S n plus are non-negative this integral can only increase and not decrease. So, this is the crucial place that actually hinges upon the positivity or rather the non-negativity of the original function F. So, S n plus generates F restricted to rather S n plus restricted to I 1 generates F restricted to I 1.

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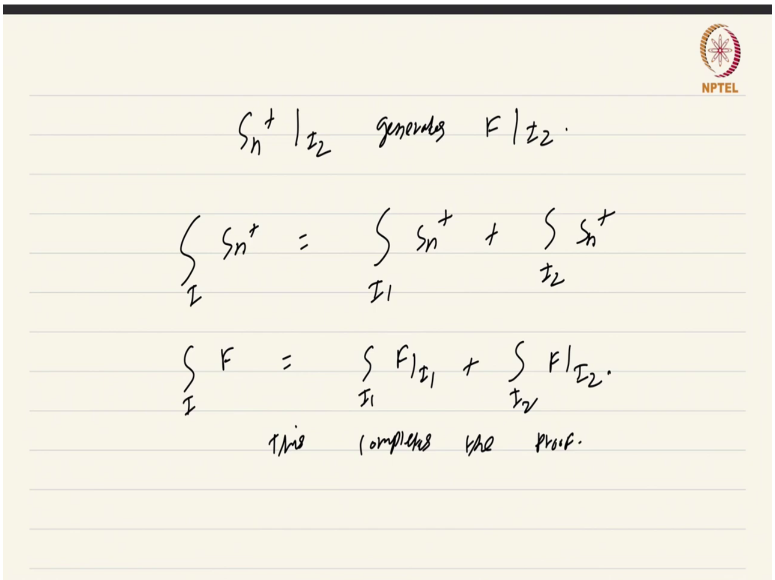


Similarly, S n plus restricted to I 2 generates F restricted to I 2. Now, it is obvious that for step functions we have this; when you break up the interval of integration for a step function you get this obvious equality. Integral of I S n plus is just integral of I 1 S n plus plus integral of I 2 S n plus and when you take limits you get integral of I F is equal to integral of I 1 F restricted to I 1 plus integral of I 2 F restricted to I 2, ok. So, this completes the proof.

So, more or less, all the basic properties of the Riemann integral are also enjoyed by this integral on upper functions. As a next step we are going to show in the next video that any Riemann integrable function is automatically an upper function and the Riemann integral coincides with the integral we have defined for upper functions. Once we are done with that we can move on to the definition of the Lebesgue integral.

This is a course on Real Analysis and you have just watched the video on Upper functions and their integrals.