Real Analysis II

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Lecture – 2.1

Normed Vector Spaces

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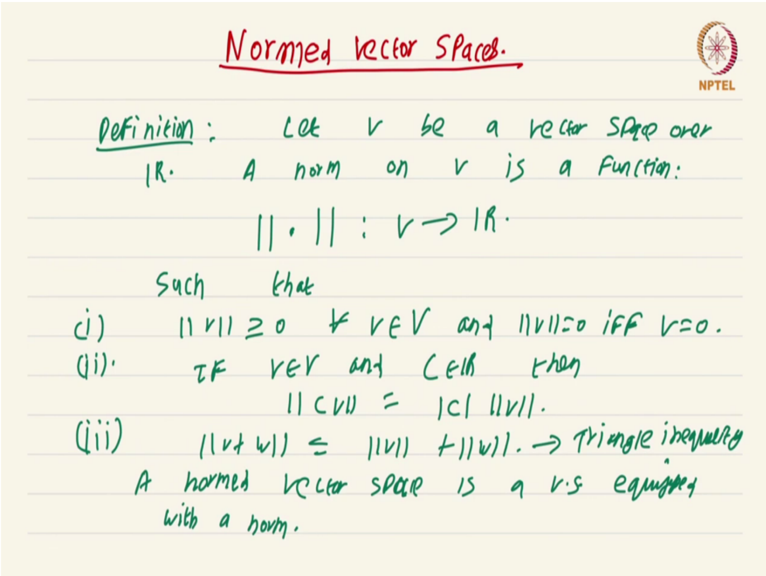

We will now study a class of metric spaces that are the most important for this course and in much of mathematics. These are the so-called Normed Vector Spaces, a vector space where you can measure the distance to the origin. So, let me just state the definition right away. We will deal only with real vector spaces, but much of what I say can be easily extended to the case where the vector space is a complex vector space.

Definition: Let V be a vector space over $\mathbb{R}$. A norm on V is a function and motivated by our use of parallel lines to denote the absolute value in the real numbers; we will use these double parallel lines $\|\cdot\| : V$ to $\mathbb{R}$ to denote the norm function. The bold dot is just an indicator that that is where the variable goes. So, we have the function norm $\|\cdot\|$ from V to $\mathbb{R}$ such that

(i) Positivity: $\| v \| \geq 0, \forall v \in V$ and $\| v \| = 0$ if and only if $v = 0$.

This is somewhat intuitive if you measure the distance to the origin. Of course, you will always get a positive quantity, and the only quantity that is 0 distance away from the origin is the origin itself.

(ii) If $v \in V$ and c is a scalar then $\| cv \| = |c| \| v \|$.

Again a perfectly reasonable requirement. If you scale the vector by a constant c, then the length of the vector or the distance to the origin of the scaled vector will also get scaled, but not by c, but by the absolute value of c because distance does not have a positive or a negative sign.

(iii) Triangle inequality: $\| v + w \| \leq \| v \| + \| w \|$.

This captures the intuitive fact that we have in the vector space $\mathbb{R}^2$ and the usual distance that you have in $\mathbb{R}^2$. The sum of the lengths of two sides of a triangle always exceeds the third side.

Definition: A normed vector space is a vector space equipped with a norm.

Let me just remark that many people call vector spaces as linear spaces, and normed vector spaces will be called normed-linear spaces. Usually, abbreviations are used NVS or NLS, or some such variant is used for the sake of brevity. So, this last one is the triangle inequality we spoke about, and it will be the one that we will use in the future repeatedly.

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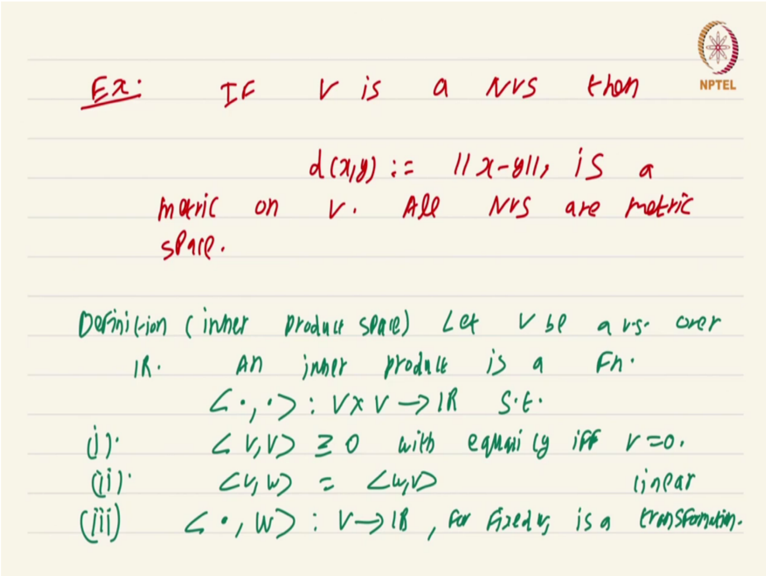

So, let me just immediately give you a very easy exercise.

Exercise: If V is a normed vector space, then

$$d(x,y) \coloneqq \|x - y\|$$

is a metric.

This means that all normed vector spaces naturally are metrics spaces. So, the idea behind choosing this as the definition of the metric is rather obvious. You know how to measure the distance to the origin; therefore, you can measure the distance between two vectors. You just take the difference between the two vectors and measure the distance from the origin of that vector.

Before giving examples, let me give a more refined notion of a normed vector space that you are undoubtedly familiar with from linear algebra called an inner product space. Now, I want to give this definition because the most interesting vector space in this course is, of course, $\mathbb{R}^n$ and that $\mathbb{R}^n$ The underlying norm comes from the inner product defined by the usual dot product or scalar product you have studied in multivariable calculus.

So, this is the definition of an inner product space.

Definition: Let V be a vector space over $\mathbb{R}$. An inner product is a function $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{R}$ such that

(i) Non-negativity: $\langle v, v\rangle \geq 0$ with equality if and only if $v = 0$.

Let me just remark that this first property is often called positive semi-definiteness.

(ii) Symmetry: $\langle v, w\rangle = \langle w, v\rangle$.

All these properties that I am writing down hold for any pair of vectors coming from this vector space.

(iii) $\langle \cdot, w\rangle : V \to \mathbb{R}$, for a fixed w is a linear transformation.

So, if you fix the second slot in a particular vector w, then when you substitute v in the first slot, it becomes a linear mapping from V to $\mathbb{R}$.

Needless let's say, combining three and two, we see that there is nothing special about fixing the second slot. You might as well fix the first slot; you will still get a linear transformation. And inner product space is nothing but a vector space along with an inner product.

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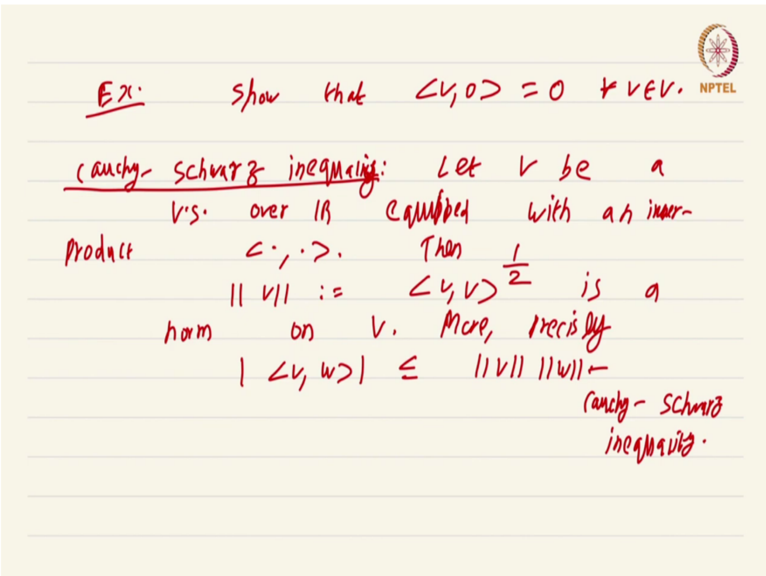

Now, I mentioned that the norm for $\mathbb{R}^n$, which is the most interesting vector space from our perspective, comes from the inner product the standard dot product on $\mathbb{R}^n$. But more generally, whenever you have an inner product, you automatically have a norm that arises from that inner product. Before that, I will just want to give a basic exercise.

Exercise: Show that $\langle v, 0\rangle = 0 \, \forall \, v \in V$.

The inequality that allows us to show that you can get a norm out of the inner product is called the Cauchy Schwarz inequality. It is a rather important inequality that crops up almost in all parts of analysis. It is one of those tools that you must be familiar with in and out, and you must know how to apply it and when to apply it at all times.

Cauchy Schwarz inequality: Let V be a vector space over $\mathbb{R}$ equipped with an inner product $\langle \cdot,\cdot \rangle$. Then

$$\|v\| = \sqrt{\langle v, v\rangle}$$

is a norm on V. (Of course, hereby square root, I mean the positive square root throughout. I always will take only the positive square root). More precisely,

$$|\langle v, w\rangle| \leq \|v\|\|w\|.$$

This is the actual Cauchy Schwarz inequality, and from this Cauchy Schwarz inequality, you will be able to get that it is a norm. Let us prove both of these facts together in one single stroke. So, let us first prove the Cauchy Schwarz inequality and see how the triangle inequality follows from the Cauchy Schwarz inequality. The other properties of a norm are rather obvious, and I will leave it to you.

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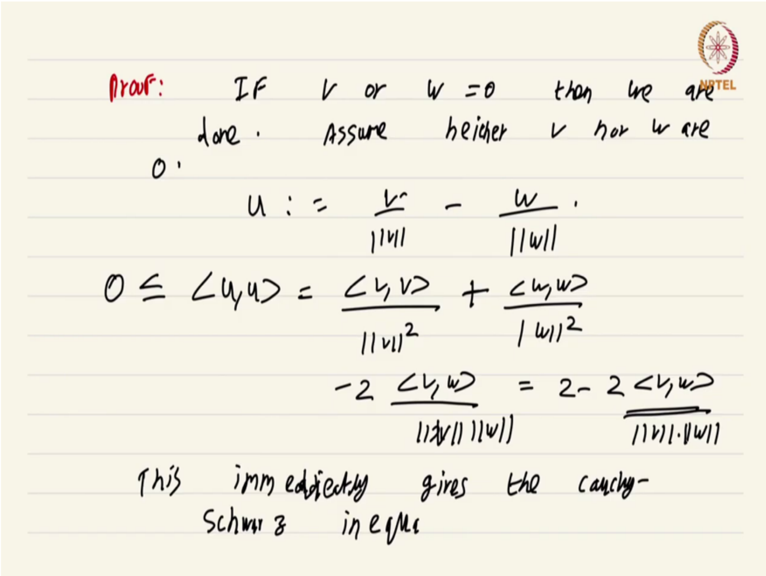

So, before we begin the proof, let me remark that since these inequalities are of paramount importance in all of the analysis, there are manifold proofs of this. And through my career, I have seen at least half a dozen proofs. All of the proofs will require you to memorize something or the other. Some of the proofs claim to be more conceptual than the others, but still, you have to remember something. So, I am going to present the proof that requires the least memory.

Proof: If v or $w = 0$, then we are done. By the previous exercise, it is rather trivial to see that $\langle v, w\rangle = 0 \leq \|v\|\|w\|$.

Now assume neither v nor w is 0. The only part you have to memorize now is to define this auxiliary vector

$$u = \frac{v}{\|v\|} - \frac{w}{\|w\|}.$$

Take the inner product of u with itself. You already know that by the condition that this inner product is positive semi-definite, therefore

$$\langle u, u\rangle \geq 0.$$

And by the various properties of the inner product, you immediately get that

$$0 \leq \langle u, u \rangle = \frac{\langle v, v \rangle}{\|v\|^2} + \frac{\langle w, w \rangle}{\|w\|^2} - \frac{2\langle v, w \rangle}{\|v\|\|w\|}.$$

Since $\langle v, v \rangle = \|v\|^2$ and similarly $\langle w, w \rangle = \|w\|^2$. So, you just get.

$$0 \leq \langle u, u \rangle = \frac{\langle v, v \rangle}{\|v\|^2} + \frac{\langle w, w \rangle}{\|w\|^2} - \frac{2\langle v, w \rangle}{\|v\|\|w\|} = 2 - \frac{2\langle v, w \rangle}{\|v\|\|w\|}.$$

So, this immediately gives the Cauchy Schwarz inequality.

Now let us move on to the proof that the Cauchy Schwarz inequality proves that the definition that we gave that

$$\|v\| = \sqrt{\langle v, v \rangle}$$

is, in fact, a norm.

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$$\|cv\| = |c| \, \|v\|$$

$$\langle cv, cv \rangle^{\frac{1}{2}} = (c \ \langle v, cv \rangle)^{\frac{1}{2}}$$
$$= (c^2 \langle v, v \rangle)^{\frac{1}{2}}$$
$$= |c| \, \|v\|.$$

$$\|v+w\|^2 = \langle v+w, v+w \rangle$$
$$= \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle$$
$$\leq \|v\|^2 + \|w\|^2 + 2\|v\| \, \|w\|$$
$$= (\|v\| + \|w\|)^2$$

this shows $\|\cdot\|$ is indeed a norm.

The fact that $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$ is rather obvious. So, all we have to do is to show that triangle inequality is satisfied. Let me just prove the second property also for completeness sake. So, the second property says that we have to show:

$$\|cv\| = |c|\,\|v\|.$$

By the definition

$$\|cv\| = \sqrt{\langle cv, cv\rangle}.$$

We already saw that if you fix the second slot, this is a linear transformation in the first slot. So,

$$\|cv\| = \sqrt{c\langle v, cv\rangle}.$$

By symmetry, we have already remarked that if you fix the first slot, it will be a linear transformation in the second slot. So,

$$\|cv\| = \sqrt{c^2\langle v, v\rangle}.$$

And note that we are taking the positive square root every time. So,

$$\left\|cv\right\| = \sqrt{c^2\langle v, v\rangle} = |c|\left\|v\right\|.$$

So, the proof that whenever you take a vector and scale it, the distance to the origin scales in the way you expect is rather easy to prove. Now, finally, for the triangle inequality. This is going to be an immediate application of the Cauchy Schwarz. What you do is you consider

$$\|v+w\|^2 = \langle v+w, v+w\rangle.$$

Immediately expanding using the fact that the inner product is linear, you will get

$$\begin{aligned}\|v+w\|^2 &= \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle \leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \\ &= (\|v\|^2 + \|w\|^2)^2 .\end{aligned}$$

Here, I have applied the Cauchy Schwarz inequality. So, this shows that $\|\cdot\|$ is indeed a norm.

Now, let me end this video by remarking that inner products are very special. It is not always true that whenever you have a norm, it comes from an inner product. The importance of inner product spaces that I am sure you are familiar with from linear algebra is that on inner product spaces, you can measure the lengths of vectors, but you can also measure the angle between two vectors. So, these are spaces on which you have some type of geometry because you have both angles and distances.

The most general type of space in which you have both angles and distances is what is known as the Riemannian manifold. On this Riemannian manifold, all the usual notions from classical geometry like area and volume make sense. So, Riemannian manifolds are some special spaces in which you have inner products on each tangent space. So, I will not say more because this is I mean this is well beyond the scope of this course.

This is a course on real analysis, and you have just watched the module on normed vector spaces.