Real Analysis II
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Lecture - 15.1
Taylor's Theorem

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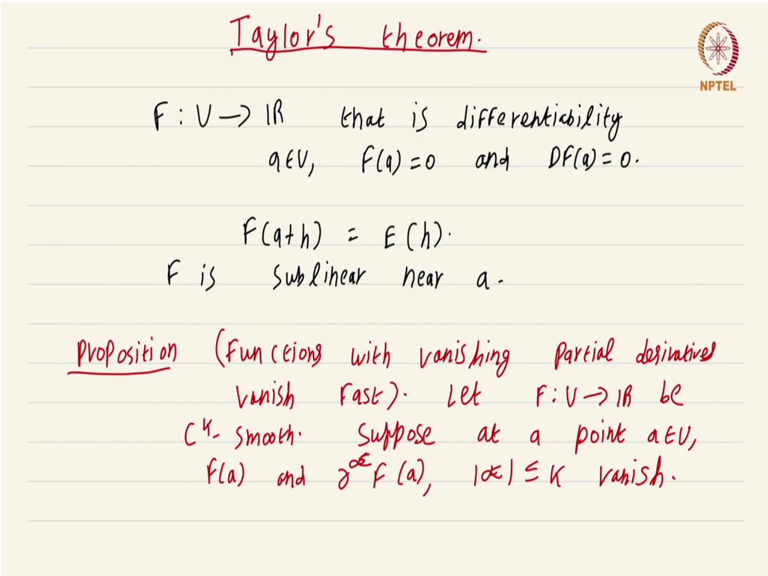

We will now present Taylor's Theorem in several variables. Our version will be a very simple and elegant version that exploits multi index notation to achieve its elegance.

So, the starting point is the function F from U to R that is differentiable. Now, the condition of differentiability says that F can be approximated in a nice manner by a linear function. Now suppose at some point a in U, we have both F of a equal to 0 and the derivative at the point a is also 0.

So, suppose it happens that at a particular point both the function and its derivative vanishes, then the definition of differentiability at this point merely says that F of a plus h is equal to E of h. In other words F is sublinear near a. Now the aim is to show that if not only the first derivative vanishes the higher derivatives also vanish then the function goes to 0 near the point a really fast that is captured in the next proposition.

Proposition: So, I am going to title this proposition with a fancy title functions with vanishing partial derivatives vanish fast. So, this is a colloquial way of saying what is to follow a more precise mathematical statement let F from U to R be C K Smooth. Suppose at a point a in U, F of a and d alpha F at the point a mod alpha less than or equal to K Vanish; vanish. So, not only does the function vanish, but all the partial derivatives still order K vanish.

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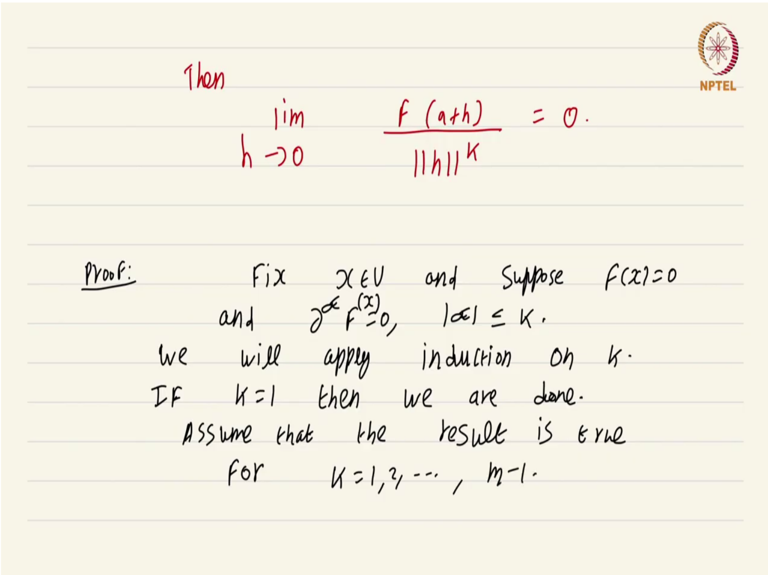

Then then limit h going to 0 of F of a plus h divided by norm h power K equal to 0 ok. So, the conclusion is that F vanishes at the point a really fast; that is captured by saying that even if you divide by norm h power K the function still goes to 0 ok. So, let us prove this the proof is not hard the proof just involves induction.

So, if you do not mind I am going to fix x in U and suppose F of x equal to 0 and del alpha F equal to 0 mod alpha less than or equal to K ok. So, I am changing the point because I am going to be applying induction. So, del alpha F at x ok.

Now so we will apply; we will apply induction on K. If K is equal to 1 then we are done because we just analyze that case right here F of a plus h is E of h and the characteristic property of the error function is limit h going to 0 E of h by norm h equal to 0 ok.

Assume that the result is true result is true for K equal to 1, 2 dot dot dot m minus 1 ok.

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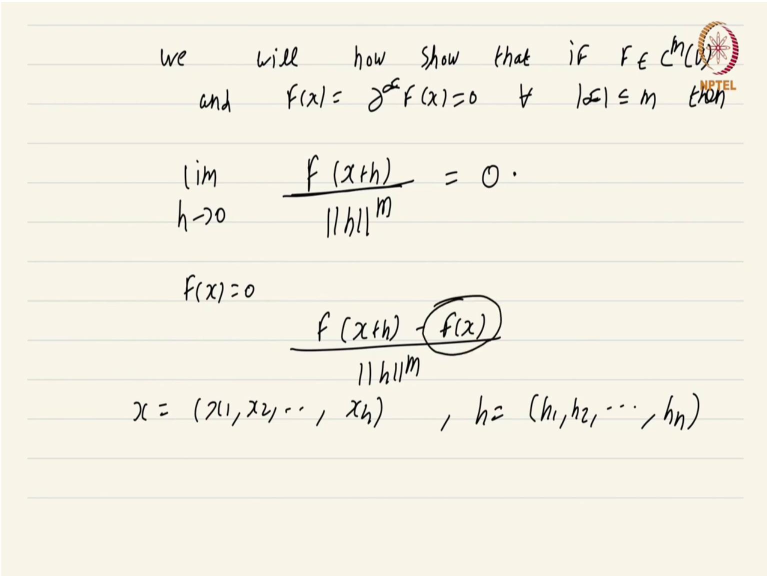

We will now show; we will now show that if F is C m smooth and F of x equal to del alpha F at x is equal to 0 for all mod alpha less than or equal to m. Then, now our F of x plus h by norm h power m is equal to 0.

So, our hypothesis involves F of x equal to 0. So, the numerator can be written as F of x plus h minus F of x by norm h power m. Now what we are going to do is we are going to write x as x 1, x 2, dot dot dot, x n and h as h 1, h 2, dot dot dot, h n ok. The numerator is just then F of x 1, plus h 1 dot dot dot, x n plus h n minus F of x 1 dot dot dot x n.

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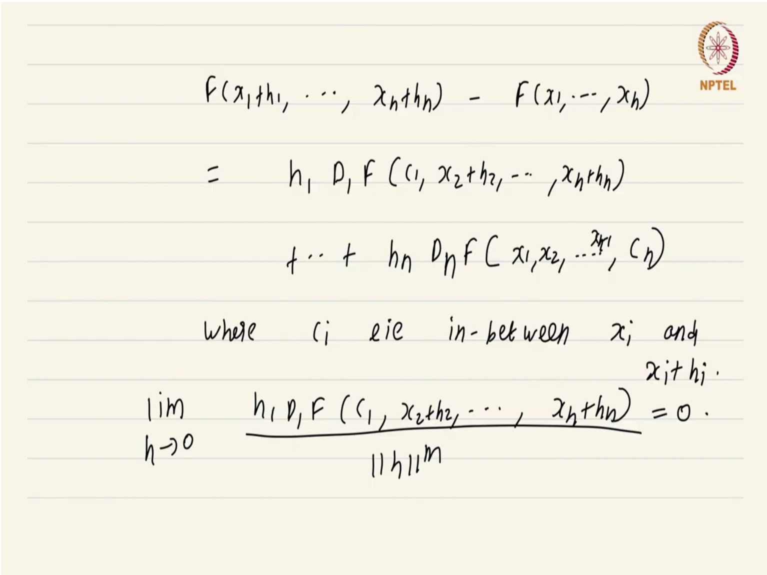

Now, some time ago we established that a function scalar valued function of a vector variable is differentiable if the partial derivatives exist and or continuous. During the course of that proof we had actually considered this difference and we know that this difference is equal to h 1, D 1 of F of C 1 comma x 2 plus h 2 dot dot dot x n plus h n plus dot dot dot h n D 1 of sorry, D n of F of x 1, x 2, dot dot dot c n, x n minus 1 c n where c i lie in between x i and x i plus h i ok.

So, this was established as a part of the proof of the fact that if partial derivatives exist and are continuous then the function, the difference F of x 1 plus h 1 comma dot dot dot x n plus h n minus F of x 1 to x n can be written in this manner. So, how does this help us? Well, we are going to show that limit h going to 0, h 1 D 1 F C 1, x 2 plus h 2 comma dot dot dot, x n plus h n divided by norm h power m is equal to 0 ok.

We will just tackle the first term in an analogous way I leave it to you to tackle the other terms. This will in fact show the claim that we want because ultimately our aim is to show that this F of x plus h by norm h power m goes to 0. So, of course, I have forgotten limit h going to 0 here sorry about that.

So, our aim is to show that limit h going to 0 F of x plus h by norm h power m equal to 0 there is an F of x that I have tackled on here because that is just 0 and we have evaluated the numerator using this long expression. So, if you can show that each one of these terms divided by norm h n goes to 0 then we are actually done, ok.

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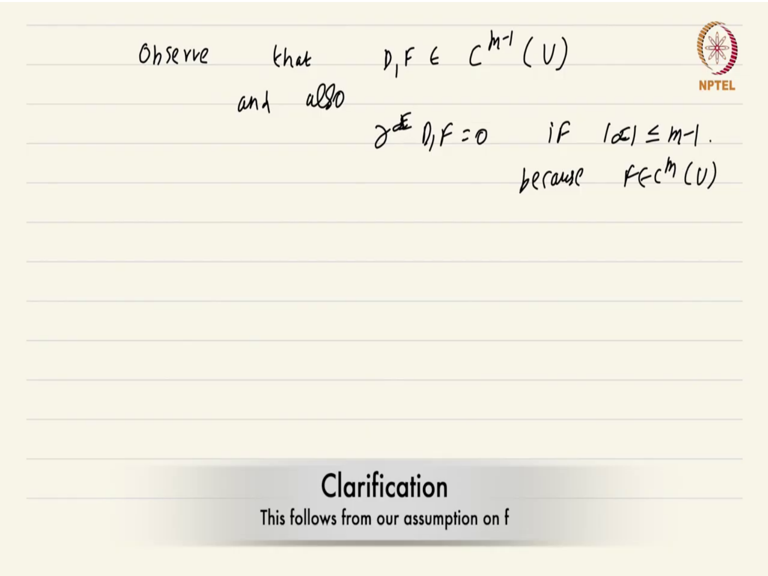

Now, observe that D 1 F is in fact, an element of c m minus 1 of U. This is just because I am just taking one derivative and all derivatives up to order m exists. So, D 1 F all derivatives up to order m minus 1 will exist. And also d alpha of D 1 F is equal to 0 if mod alpha is less than or equal to m minus 1. This just follows from the induction hypothesis, not induction hypothesis this just follows from the fact that we are starting with F in c m 1; because F is in c m of U ok.

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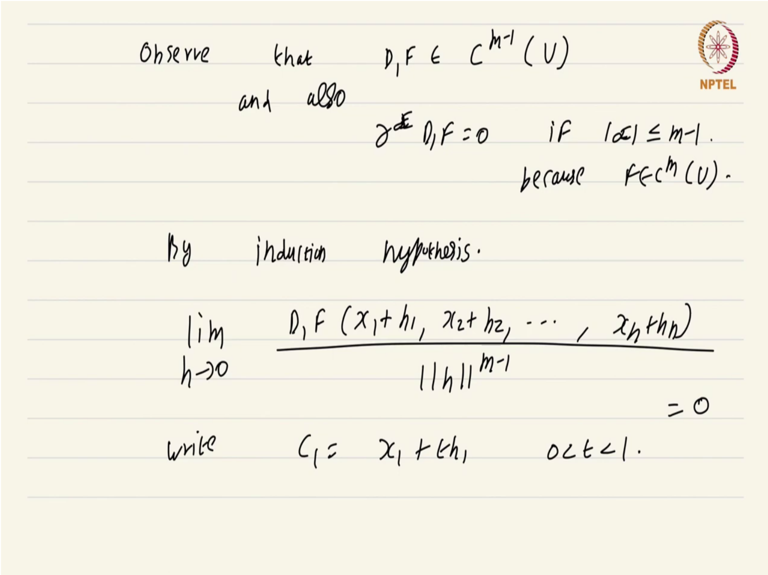

So, now by the induction hypothesis; by induction hypothesis what we have is D 1 of F of x 1 plus h 1, x 2 plus h 2 dot dot dot, x n plus h n divided by norm h power m minus 1 limit h going to 0 of this is equal to 0. This is just the induction hypothesis applied to the function D 1 of F which is actually an element of c m minus 1 of U, ok.

Now the term that we have is somewhat different, the term we have does not have x 1 plus h 1 rather it has this pesky C 1 ok. Now, write C 1 as x 1 plus t h 1 0 less than t less than 1 we can do this because C 1 lies between x 1 and x 1 plus h 1, ok.

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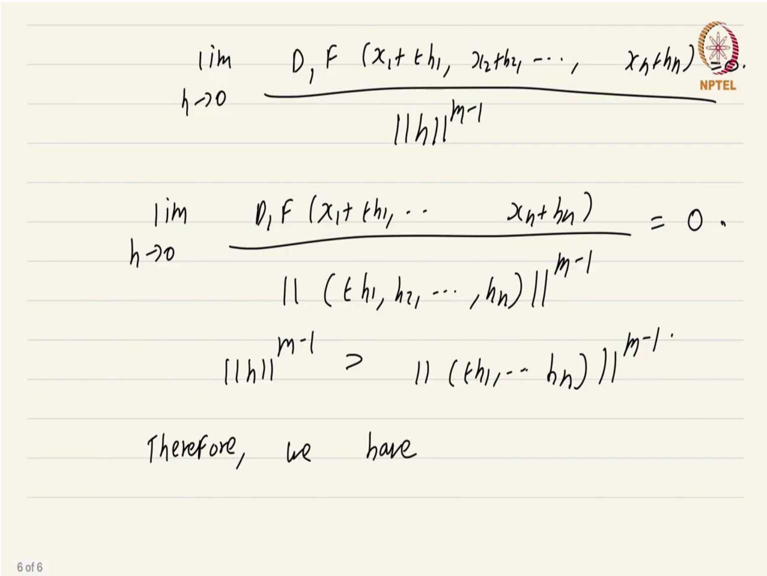

So, we are reduced to showing limit h going to 0 D 1 of F of x 1 plus t h 1, x 2 plus h 2 comma dot dot dot x n plus h n divided by norm h power m minus 1 equal to 0 ok. So, this is what we have to show.

Now, again by induction hypothesis we know that limit h going to 0 D 1 of F of x 1 plus t h 1 comma dot dot dot x n plus h n; I am abbreviating because I am getting bored writing the same thing again and again, but in the denominator you have t h 1, h 2 dot dot dot h n power m minus 1 equal to 0. This just follows by induction hypothesis, but norm h power m minus 1 is greater than norm t h 1 dot dot dot h n, no power m minus 1 this is a larger quantity because 0 is less than t is less than 1.

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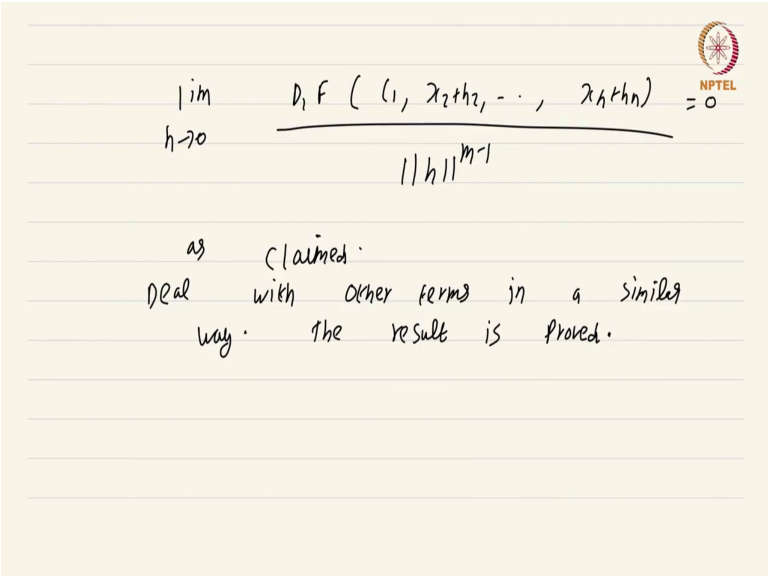

So, therefore we have limit h going to 0 D 1 F C 1, x 2 plus h 2 comma dot dot dot, x n plus h n by norm h power m minus 1 equal to 0 as claimed, ok. Now, deal with the other terms in a similar way; deal with other terms what do I mean by other terms I mean the other terms in this long expansion these terms; deal with these other terms in this in a similar way, in a similar way the result is proved ok.

So, this was a very very very nice result that says that if a function has a lot of vanishing partial derivatives then the function goes to 0 really fast. So, now we can quickly dispose of Taylor's theorem because all the hard work has been done in this proposition.

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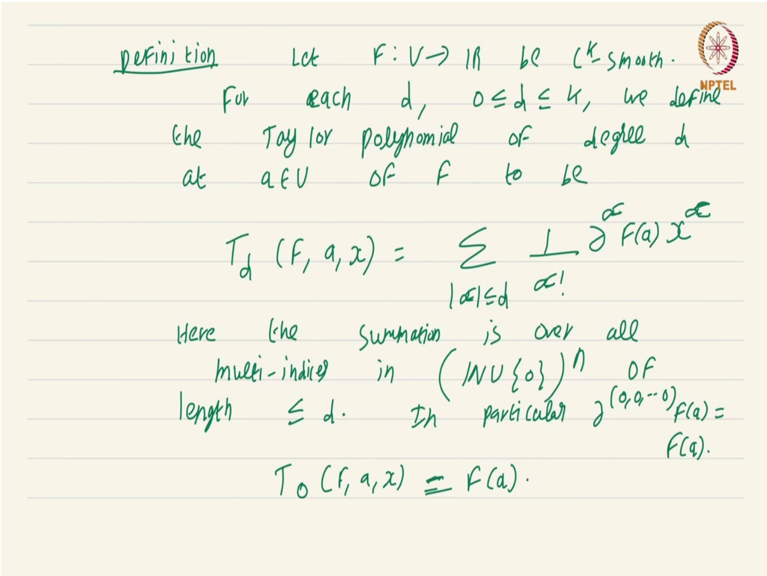

Taylor's theorem we are now going to state the statement before that I need one definition. So, let me first state that definition before I jump to the theorem. Definition; the definition is of our Taylor polynomial.

So, let F from U to R be C K smooth for each for each d 0 less than or equal to d sorry; 0 yeah 0 less than or equal to d less than or equal to K we define the Taylor polynomial; the Taylor polynomial of degree d at a in U of F to be. So, the notation for this is T d F, a, x this is just summation mod alpha less than or equal to d, 1 by alpha factorial del alpha F at the point a into x power alpha.

Now the listener should observe that this form formally is exactly the same as the usual Taylor polynomial in one variable. So, the elegance in this multi index notation is wherever you see a number you can replace it by a multi index and recover the same thing ok.

So, here so I have used this summation that looks complicated, here the summation is over all multi indices in N union 0 power n of length less than or equal to d, in particular d 0, 0, 0 0 0 of F at a is just defined to be F of a. So, taking no partial derivatives at the point a just gives the function. So, more concretely T 0 T 0 of F a x is just is just F of a; is just F a, ok.

So, these Taylor polynomials give the best approximation a polynomial approximation of the given function and that is content of Taylor's theorem.

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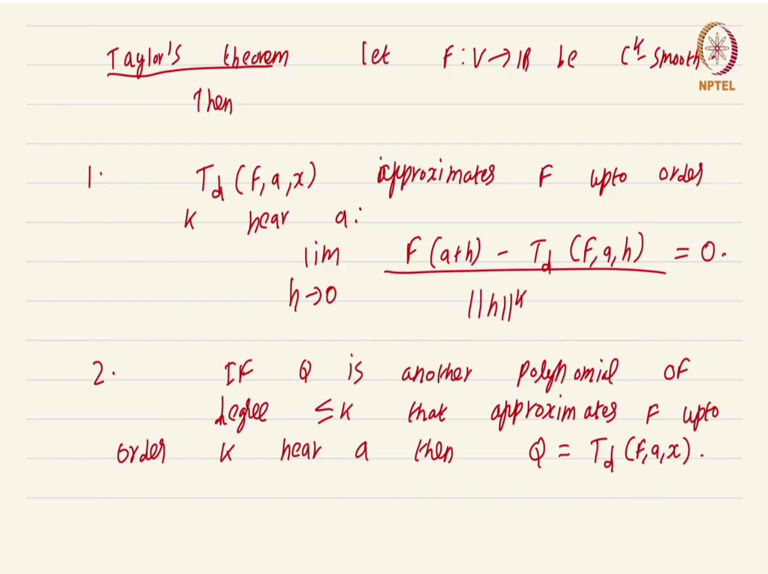

Taylor's theorem: Let F from U to R be C K smooth C K smooth, then number 1 T d F, a, x is approximates F up to order K order k near a. So, this is a somewhat vague sounding statement what this really means is limit h going to 0 of F of a plus h minus T d F, a, h divided by norm h power K is equal to 0 ok.

So, this just says that near the point a you can approximate the function F of a plus h by using this T d F, a, h. Part 2 says if Q is another polynomial of degree less than or equal to K that approximates F up to order K near a then Q is in fact equal to P ok.

So, this polynomial rather let me not say P because P does not make any sense T is equal to T d F, a, x. So, the Taylor polynomial is the unique polynomial that approximates F up to order K near the point 0 sorry near the point a ok. So, this sort of is the unique of course, what is crucial is the polynomial of degree less than or equal to K.

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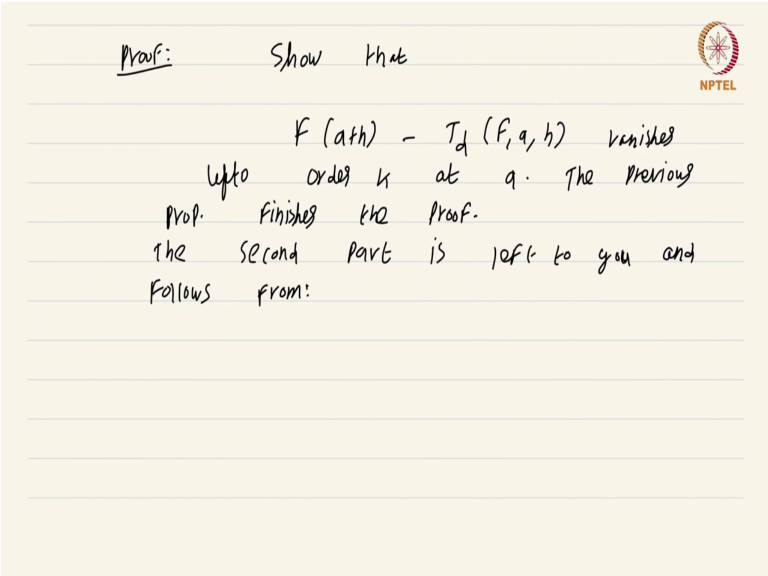

So, let us prove this and there is nothing much to prove. Show that F of a plus h minus T d F, a, h vanishes up to order K at a. In fact, the entire Taylor polynomial was set up so that this happens, when you take the partial derivatives number of cancellations will occur between the partial derivative of F and the partial derivatives of T d F, a, h and you will get that everything cancels up to order K. So, the previous proposition finishes the proof; previous proposition finishes the proof. So that was a very quick short proof.

Now the second part; the second part is also left as an exercise for you. So, the question remains what is it that I am doing, the second part is left to you left to you and follows from; and follows from the next exercise.

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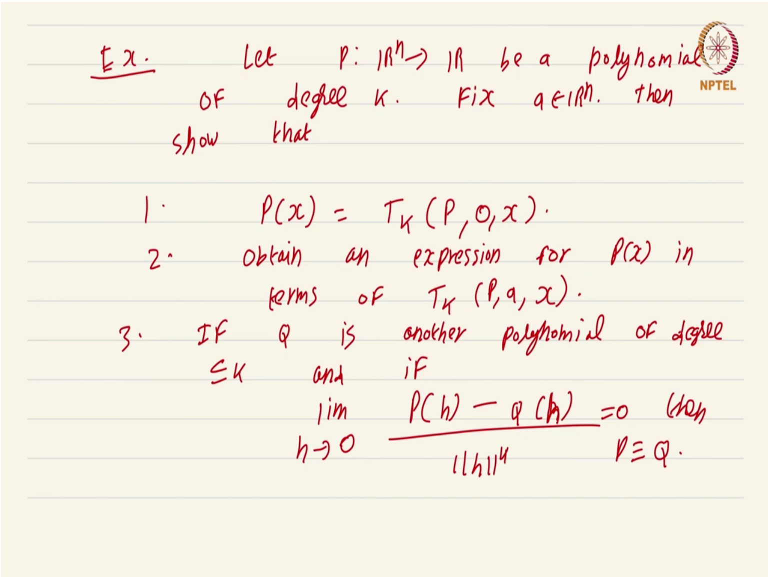

Let P from R n to R and yeah be a polynomial be a polynomial of degree K. Fix a in R n then show that number 1 P of x P of x is actually equal to T K of P, 0, x. Number 2 obtain an expression; obtain an expression for P of x in terms of T K P, a, x. Can you write down what P of x is in terms of T K P, a, x.

3, if Q is another degree K polynomial another polynomial actually I do not require degree K polynomial of degree less than or equal to K less than or equal to K then and if limit h going to 0 of P of h minus Q of h divided by norm h power K equal to 0 then P is the same as Q. There can be only one polynomial with this of degree less than or equal to K with this property, ok.

So, this exercise solve this exercise then Part 2 of the previous theorem will become very clear once you solve this exercise in detail, this is just a bit of routine computation. This is a course on Real Analysis and you have just watched the video on Taylor's Theorem.