Real Analysis II

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Lecture - 1.3
Examples of Metric Spaces

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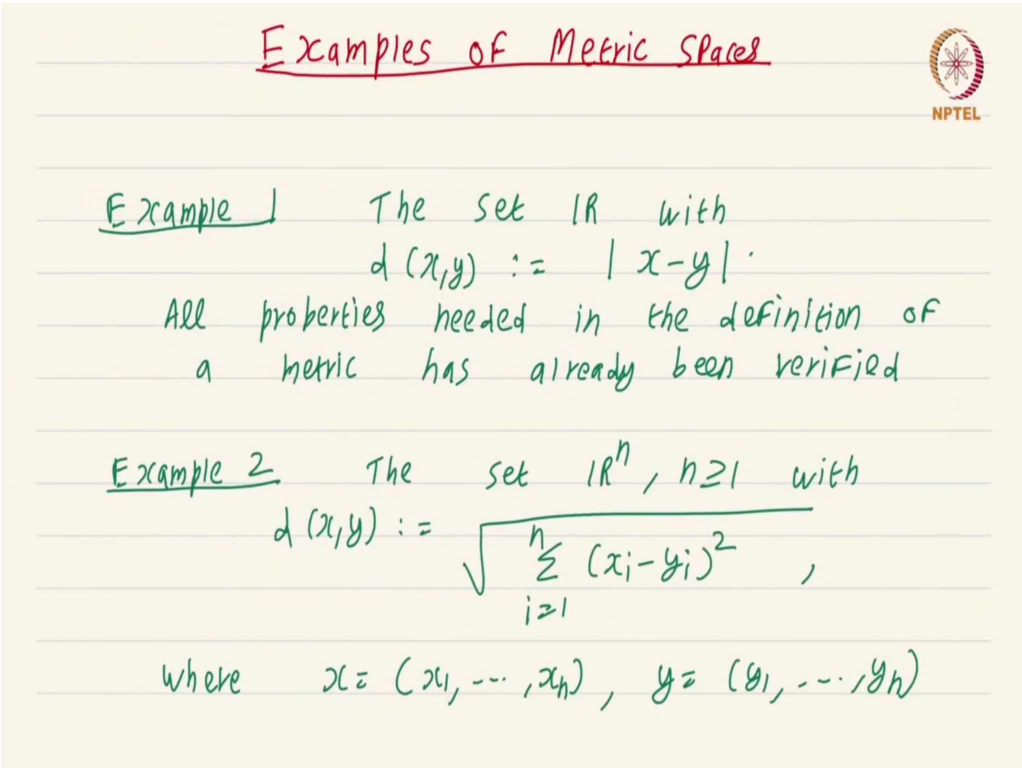

In this module, we are going to see several examples of metric spaces. Please keep these examples in the back of your mind in the future modules, where we will see a number of concepts defined. First example is the example you are most familiar with because we spent a considerable amount of time in Real Analysis I studying this example.

Example 1: The set $\mathbb{R}$ of real numbers with $d(x,y) = |x - y|$.

Now, we have already verified all properties needed in the definition of a metric. In the chapter where we defined or instead took an axiomatic approach to the real numbers, we defined this absolute value function and checked all these properties. Now, example 2 is one of the most important examples and the most pertinent one when it comes to this course on multi-variable real analysis.

Example 2: The set $\mathbb{R}^n$, $n \geq 1$ with $d(x,y)$ defined to be the usual distance between two points in $\mathbb{R}^n$ i.e.

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2},$$

where $x = (x_1, x_{2,} \dots x_n)$ and $y = (y_1, y_{2,} \dots y_n)$. So, the set $\mathbb{R}^n$ along with this metric is called Euclidean space. We have to check that definition satisfies all the three properties of a metric space. The only tricky thing is the last property. So, let me just remark that the last property, or the triangle inequality, will be checked in the future.

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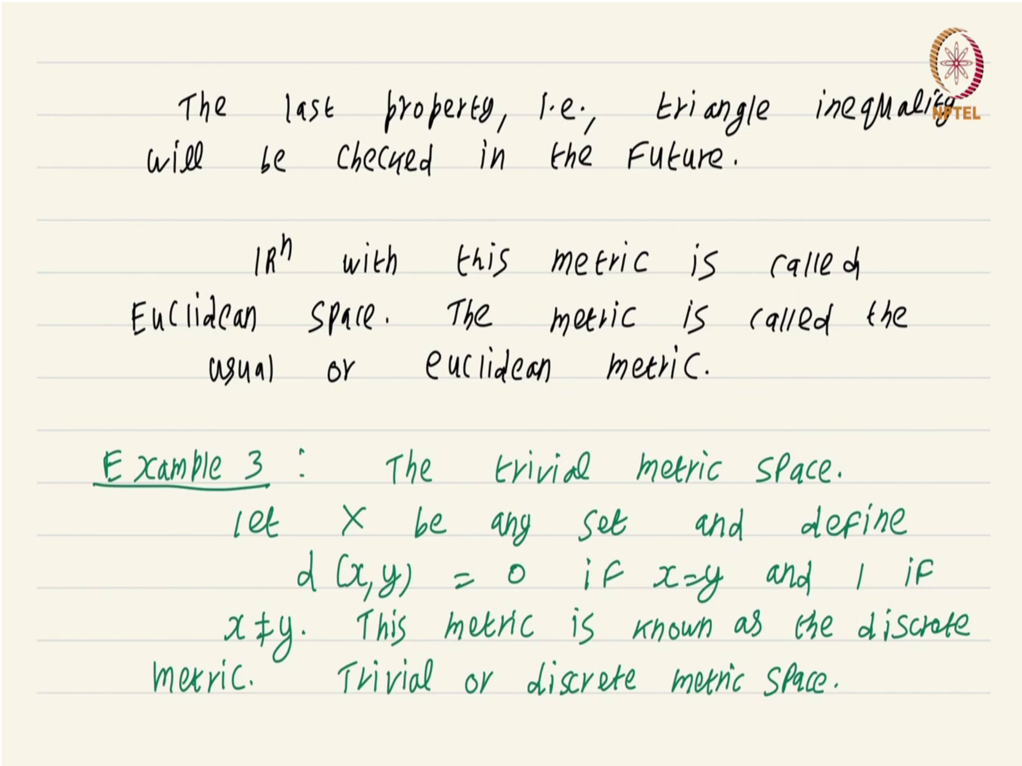

We will study one general class of metric spaces that come from a norm or instead come from an inner product, and we will prove the famous Cauchy Schwarz inequality and this fact that the triangle inequality is satisfied for the metric defined in Example 2 will follow immediately from the Cauchy Schwarz inequality. We will do that at a later stage. So, with that being said, $\mathbb{R}^n$ with the metric d defined in Example 2 is called Euclidean space, and d is called the usual metric.

Mathematicians, as we have seen, are not very creative when it comes to terminology.

There are other natural and valuable metrics that we can define on the space $\mathbb{R}^n$. We shall see that in this module on normed vector spaces, which will come down the line. So, Example 2, as I said, is the most important example when it comes to our course on multi-variable real analysis.

Now, I had remarked that the definition $d(x, y)$ defined in Example 2 is the distance between x and y, the usual distance. Can you figure out why this is the case? At least can you figure this out in $\mathbb{R}^2$ and $\mathbb{R}^3$? Please do that as an exercise; it is a useful exercise that will ground this theory in concrete.

Now, let us move on with more examples. The next example is trivial, but it is very useful when you have some conjecture in mind and you want to get a quick counter-example for this conjecture. So, this is called the trivial metric space. This metric space is defined as follows.

Example 3: Let X be any set and define

$$d(x,y) = \begin{cases} 0, if\ x = y \\ 1, if\ x \neq y. \end{cases}$$

This metric is known as the discrete metric.

This is a very trivial scenario, but it is very useful to get a quick counter-example to some naive conjecture that you make. Now, let us move on away from this trivial example to the most important example in most of the analysis, which will be a collection of functions.

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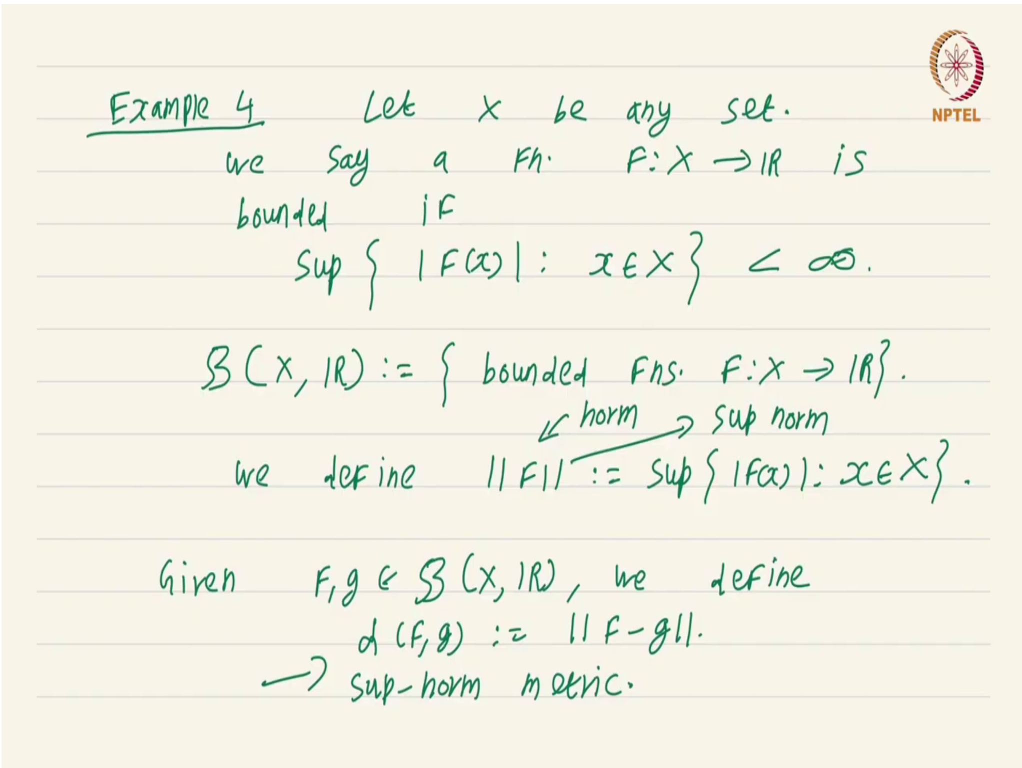

Example 4: Let X be any set. We are going to define a collection of functions defined on X. We say a function $f: X \to \mathbb{R}$ is bounded if

$$sup\{|f(x)|: x \in X\} < \infty.$$

Then, we consider

$$B(X, \mathbb{R}) = \{all\ bounded\ functions\ f: X \to \mathbb{R}\}.$$

So, take any set X and consider the collection of all bounded functions defined on this X. We now define norm $\parallel f \parallel$.

$$\parallel f \parallel = sup\{|f(x)|: x \in X\}.$$

The supremum will exist because we are taking only bounded functions. Given $f, g \in B(X, \mathbb{R})$, we define

$$d(f, g) \ = \parallel f - g \parallel.$$

Because f and g are bounded functions, $f - g$ will also be bounded, and $\parallel f - g \parallel$ makes sense.

Now, check that this example satisfies all the properties needed to make this into a metric space. It is rather easy to check. The only non-difficult thing is the triangle inequality, which is not actually difficult; it is quite straightforward. So, we will spend a considerable amount of time analyzing this particular metric space. It is going to take up most of the entire week of our lectures.

We shall study norm vector spaces, which are vector spaces with an additional structure given by a norm. So, the norm in Example 4 is called the sup-norm, and the metric is called the sup-norm metric.

Again, mathematicians are not very creative when it comes to terminology. This is straightforward terminology, just coming from the fact that there is a supremum involved in the definition of the norm, and that particular norm is involved in the definition of the metric.

So, as I said, what is a norm, and why are these norms important? We will come at a later stage; not so far away, just maybe a couple of modules down the line, you will see norm vector spaces and inner product spaces in quite a detail. They are the most important examples of metric spaces, and this concrete example $B(X, \mathbb{R})$ is by far the most important.

So, we have a small collection of examples at our disposal; more general examples will follow. With these examples at hand, let us move on and define all the basic concepts that you have already seen in the chapter on Taste of Topology and generalize those concepts to the setting of a metric space. This is a course on Real Analysis, and you have just watched the module on examples of Metric Spaces.