Real Analysis II
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Lecture - 1.2
Metric Spaces

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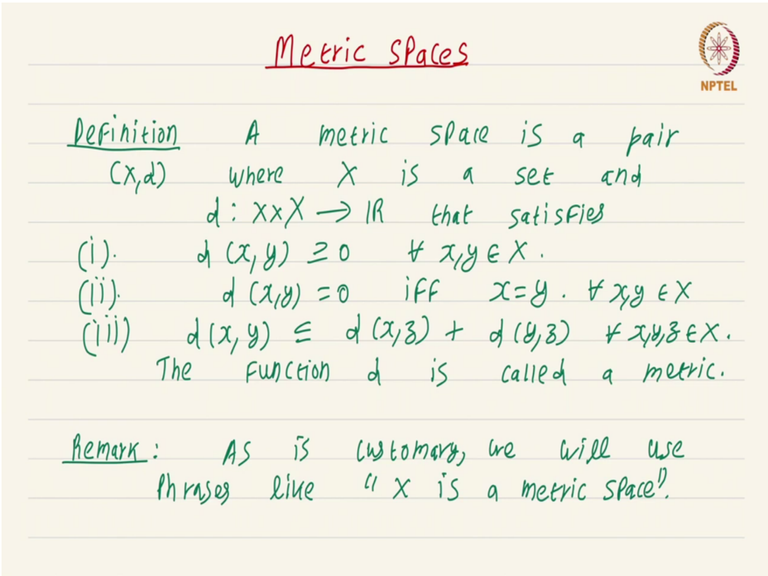

Metric spaces are spaces in which we can measure the distance between two elements. However, the notion of an angle is absent in the theory of metric spaces. The motivation for studying metric spaces comes from analysis and not from geometry.

Metric spaces are the most convenient setting for studying convergence. Most theorems of classical analysis can be unified and given elegant proofs in the context of metric spaces. However, we must mention that this holds true only for those aspects of analysis that do not involve the concept of rate of change or the notion of derivate.

To emphasize the synergy between analysis and topology, our treatment of metric spaces will prioritize sequences. As much as possible, I shall try to define everything in terms of sequences. Without further ado, let me give the definition of a metric space.

Definition: A metric space is a pair (X, d), where X is a set and $d: X \times X \to R$, that satisfies

(i) $d(x, y) \geq 0$ ∀ x, y ∈ X.

(ii) $d(x, y) = 0$ if and only if $x = y$, and

(iii) $d(x,y) \leq d(x,z) + d(y,z) \; \forall \; x,y,z \in X.$

These three properties that are required of a metric space or rather the function d associated with the matrix space are the most natural and basic ones expected of a space in which you can measure distance. Please note that these three properties are clearly satisfied for the usual distance between two points on the real line.

The function d is called a metric. We will use phrases like "X" is a metric space. What this essentially means is that we are going to forget that a matrix space is actually a set X and a function d and just read the set X itself as the space along with the function. We will do this whenever the underlying function d is clear from the context, and there is no scope for ambiguity.

In situations where there could be multiple metrics on the same set, we shall be more precise by saying let (X, d_1) be one metric space and (X, d_2) be the second metric space. In the rest of this course, we shall be studying sets equipped with a metric.

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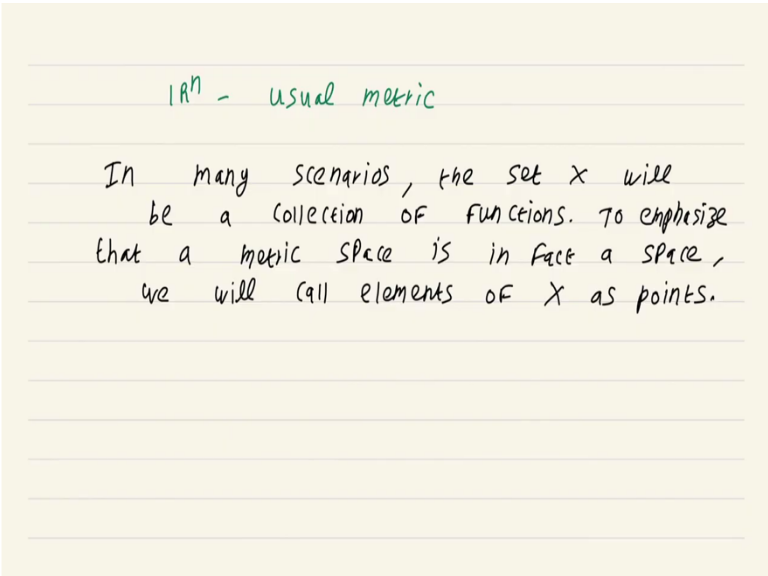

The most common one would be the Euclidean space R^n equipped with the usual metric. We will define this in a moment when we come to examples, but there can be more complicated metric spaces as well. The fact that this R^n with the usual metric is a metric space is easy to see, and the fact that it can be visualized as a space is also easy to see. We usually visualize R^2

and R^3 by drawing the usual pictures of the Cartesian coordinates; we can visualize this space in an easy way.

However, one of the most powerful aspects of this theory is that it unifies several disparate phenomenon. For instance, in many scenarios, the set X will be a collection of functions. So, what will happen is it is not so easy to visualize a collection of functions as it is to visualize R^n, which is just a collection of points in a higher dimensional space. Once you are used to visualizing R^2 and R^3, it is not so difficult to have a vague picture of what R^n looks like.

So, this metric structure when you put it on this collection of functions that we are going to study, it still behaves somewhat like R^n. That is the power of this theory. So, to emphasize this, that a metric space is in fact some sort of space, what we will do is we will call elements of X as points. So, the set X, we will always call a space and the elements of X, we will call points.

This geometric terminology serves both as a psychological aid in digesting several abstract concepts as well as an encouragement for you to draw pictures. However, primitive and inaccurate they might be. So, these pictures will allow you to visualize the concepts.

Now, I accept that it is very difficult to draw a collection of functions, but you just draw rough pictures treating each function as just a point. Now, in the next module, I am going to give a long list of definitions. But before I do that, I want you to recap what you have learnt in Real Analysis I on the chapter on Topology; A Taste of Topology.

Please go through that chapter and watch those lectures. What we are going to do in this week and the next week will be more or less just changing the notation in the chapter on Taste of Topology. That is also illustrates how powerful this theory of metric spaces is.

We have already proved several facts in the chapter on topology about real numbers. For instance, we proved the Heine-Borel theorem, we also saw characterization of compactness in terms of open covers. All of these will more or less translate into this more general setting of metric spaces without much additional effort; that is the power of abstraction. Later, when we study differential calculus in several variables, we will see that all the tools that we developed in this chapter will make our life a lot easier.

Now, before I give the definitions, let me give some examples so that when you see these definitions, it is all grounded in concrete examples for you to visualize. This will be the content of the next module. This is a course on Real Analysis, and you have just watched the module on the definition of a Metric Space.