Computational Mathematics with SageMath
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Eigenvalues and Eigenvectors
Lecture – 35
Eigenvalues and Eigenvectors Part 1 with SageMath

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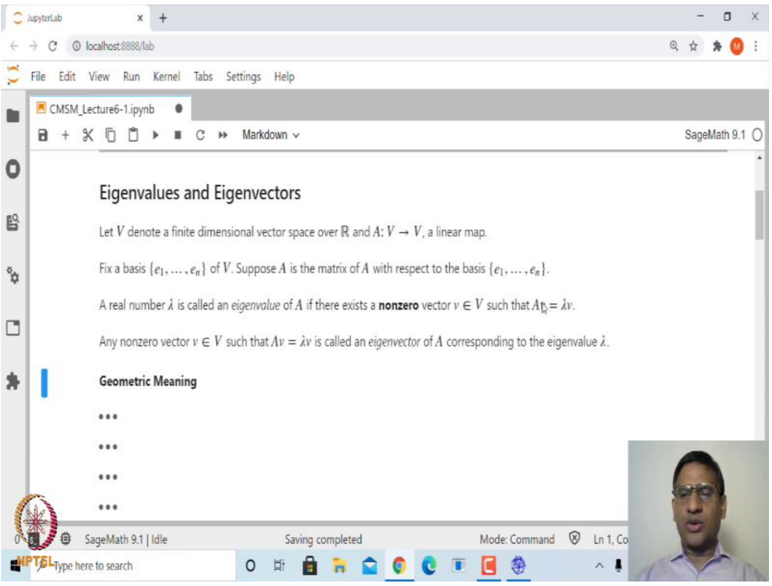

Welcome to the 35th lecture on Computational Mathematics with SageMath. In this lecture, we will explore eigenvalues and eigenvectors of a linear transformation or a matrix using SageMath. Eigenvalues and eigenvectors have numerous applications in mathematics and also in other sciences and engineering including data science. So, let us get started.

First let us define eigenvalues, eigenvectors and then we will see how it can be computed in SageMath. Suppose you have a finite dimensional vector space V over R and a linear map A from V to V. Fix a basis e1, e2, en of V, then you can find the matrix of A with respect to this basis.

Let us denote the matrix of A by A itself. We know that any linear map from finite dimensional vector space to a finite dimensional vector space has a matrix representation with respect to a basis. So, let us call this matrix of A as A itself. Now, a real number lambda is called an eigenvalue of A, if we can find a non-zero vector v, such that A times v is lambda v, that is the definition.

So, what we are saying ? Av is parallel to v, that is the definition, but v should be non-zero, that is important. Because if v is 0, A times 0 is always 0 and 0 can be written as lambda times 0. That would mean that, whatever, whatever lambda you take will turn out to be eigenvalue. That cannot be the definition. The vector v for which Av is lambda v, non-zero vector v is called eigenvector corresponding to eigenvalue lambda of A.

Next let us look at how, we can compute eigenvalues and eigenvectors using SageMath. But before that, let us look at what is meaning of eigenvalue and eigenvector geometrically. As I said Av is equal to lambda v simply means that, v is parallel to Av.

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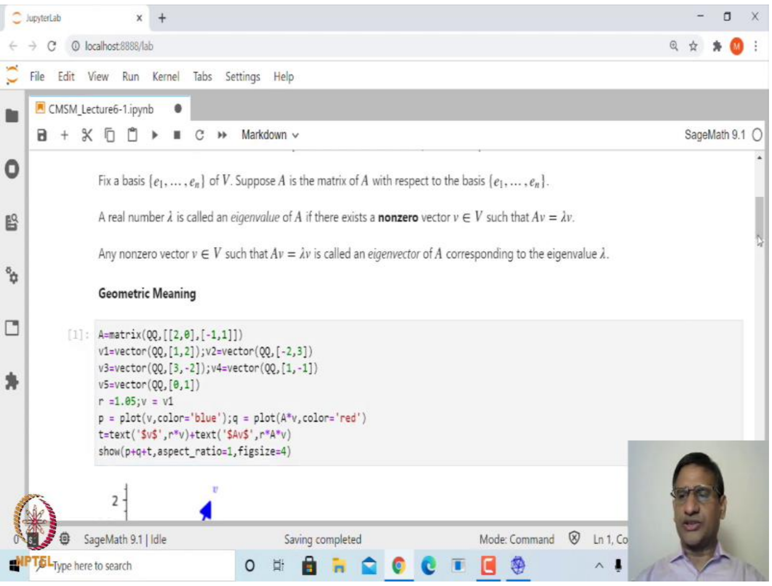

So, what we are actually looking at? You are looking at all those vectors for which Av is parallel to lambda v. Let us just look at with this figure. We are taking a matrix which is 2, 0; minus 1, 1. Let us plot v and Av for some vectors.

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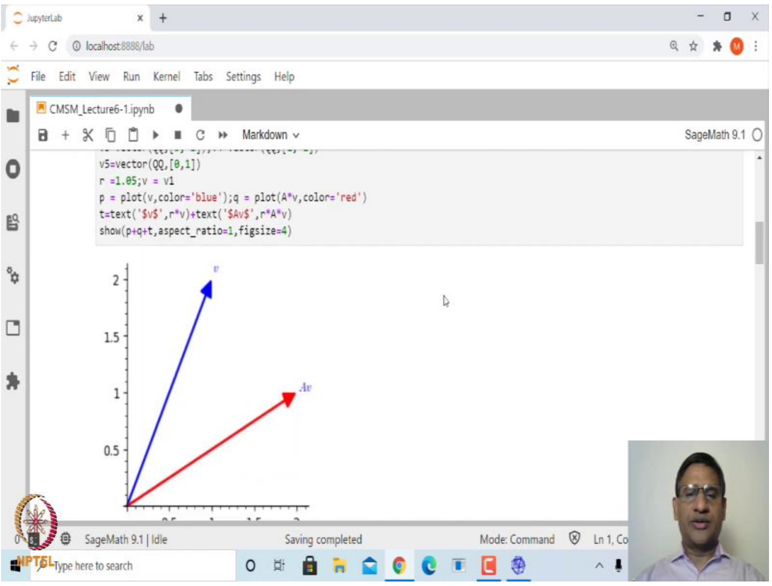

Here we have v and Av is here. So, they are not parallel; that means this v cannot eigenvector of A. In this case v, was 1 comma 2.

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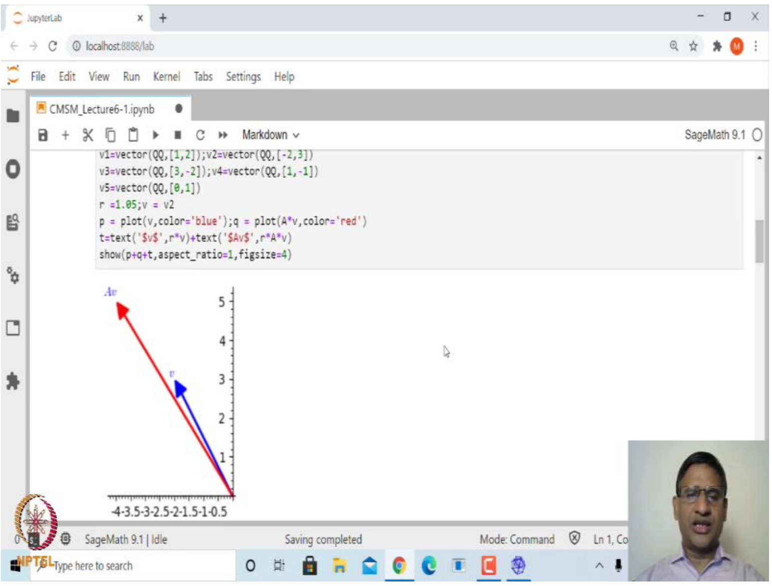

Suppose I take another v, which is actually minus 2, comma 3 and then plot this graph. Then again you can see here, this is v and Av, they are not parallel to each other.

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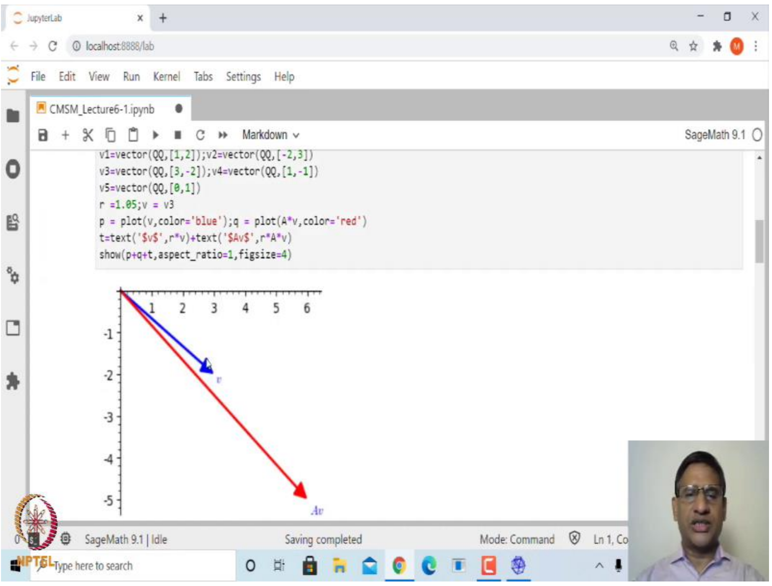

Let us take another vector, let me call this as v3. v3 is3 comma minus 2 and when you plot its graph, again this is not a eigenvector.

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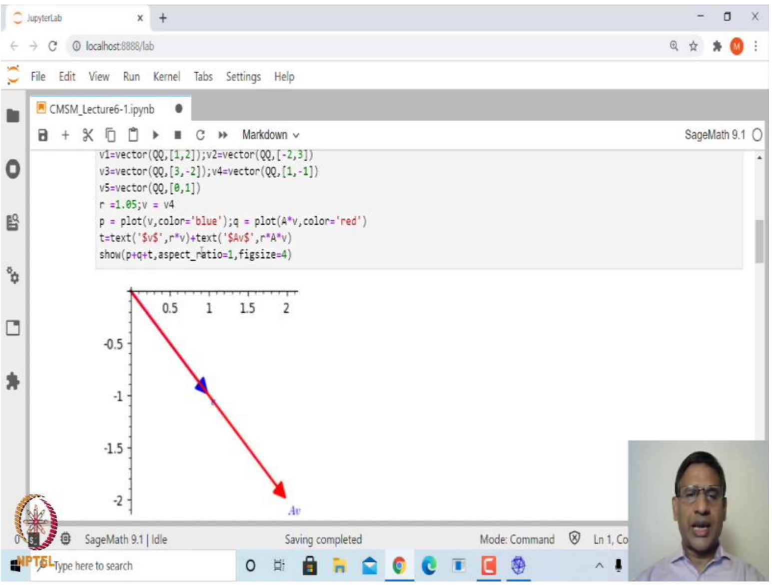

Let us take v4; v4 is actually 1 comma minus 1. And in this case v and A v, they are parallel to each other and it looks like that, Av is twice of v. So if Av is twice of v; that means 2 will be eigenvalue and v is an eigenvector with respect to eigenvalue 2. We do not know exactly how much, but one can find out. Geometrically it looks like twice.

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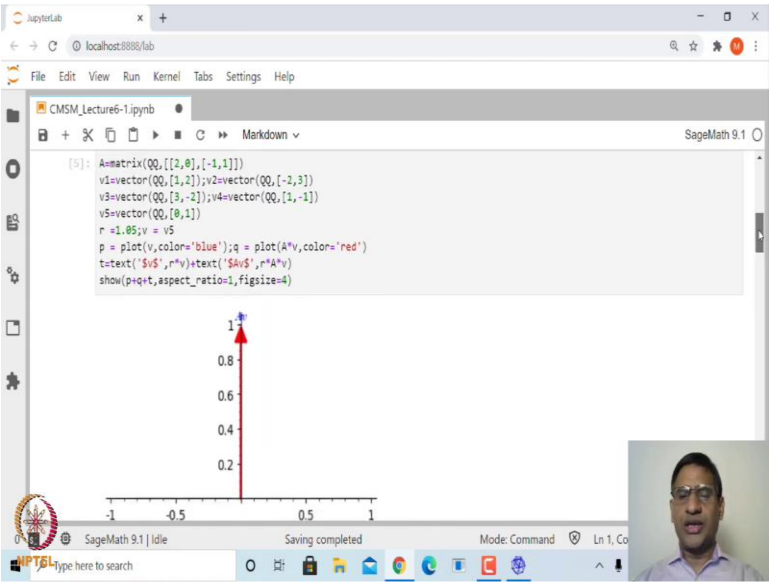

Similarly if I take another vector, let us say v5, which is 0 comma 1, an unit vector. then you can see here both Av and v are same, that simply means that Av is equal to lambda times v in particular lambda is equal to 1 is an eigenvalue and 1 0 is eigenvector.

So, finding eigenvector is nothing, but find a vector v, such that Av and v are parallel to each other, of course non-zero vector v.

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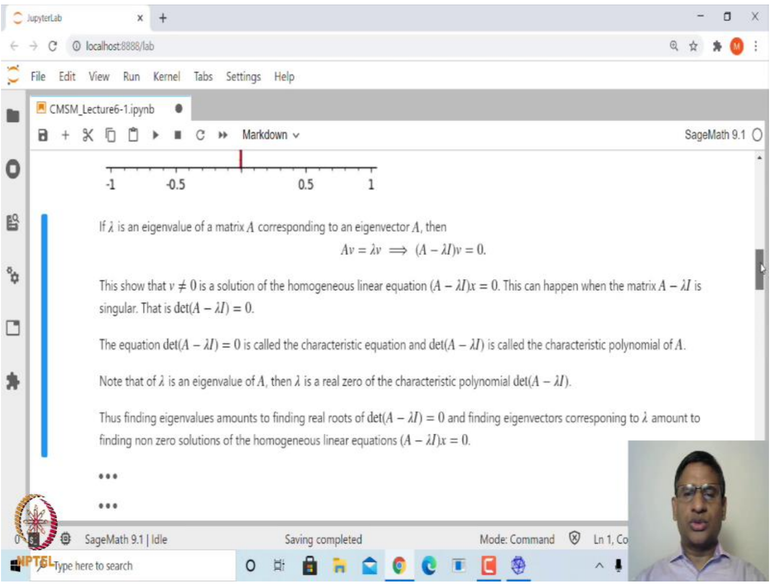

Now, let us look at what it means. So, if you look at this vector v which is eigenvector with respect to eigenvalue lambda, then we have Av is equal to lambda v. This implies, if you bring this lambda v to the left hand side, A minus lambda times identity is equal to 0, where identity is this is n cross n identity matrix. So v is what? v is a non-zero vector. Thus if you look at this equation, this is a homogeneous equation, and it says that, v is a non-zero solution to this homogeneous equation. When do you have a non-zero solution to a homogeneous equation Ax equal to 0. This happens only when A is singular matrix, that is same as saying determinant of A is 0. This means that, if v is an eigenvector of A with respect to eigenvalue lambda, then determinant of A minus lambda I must be 0. This determinant of A minus lambda I is known as characteristic polynomial of A. I you find the determinant of A minus lambda I, it will be a polynomial in lambda which is known as characteristic polynomial. Whereas, this equation, determinant of A minus lambda I equal to 0 is known as characteristic equation of A.

So, lambda is an eigenvalue of A is same as saying lambda is a real solution of the characteristic polynomial, real zero of the characteristic polynomial, determinant of A minus lambda I. And therefore, finding eigenvalue, real eigenvalue amounts to finding real roots of this characteristic equation. Then finding eigenvectors with respect to eigenvalue lambda, amounts to finding non-zero solution of this homogeneous equations. Thus finding eigenvalues, eigenvectors actually boils down to working with this characteristic equation and this homogeneous system of linear equations.

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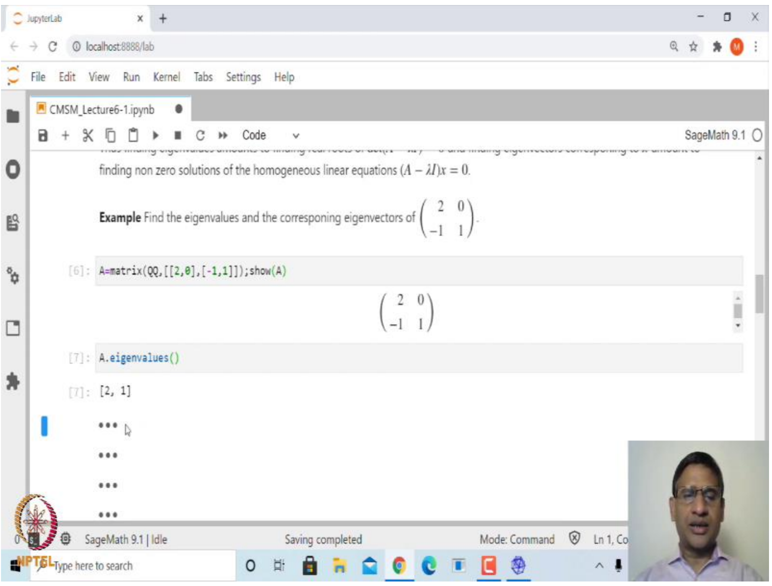

Now, let us look at how to find this using SageMath. Consider this matrix 2, minus 1, 0, 1. This is the same matrix for which we drew this graph. Now, let us find eigenvalues and eigenvectors of this in SageMath. So, first let us define A over QQ, and then the eigenvalues of A can be obtained by A dot eigenvalues. it gives you eigenvalues of A. So, in this case it says that, there are two eigenvalues 2 comma 1 and that is exactly what we observed geometrically.

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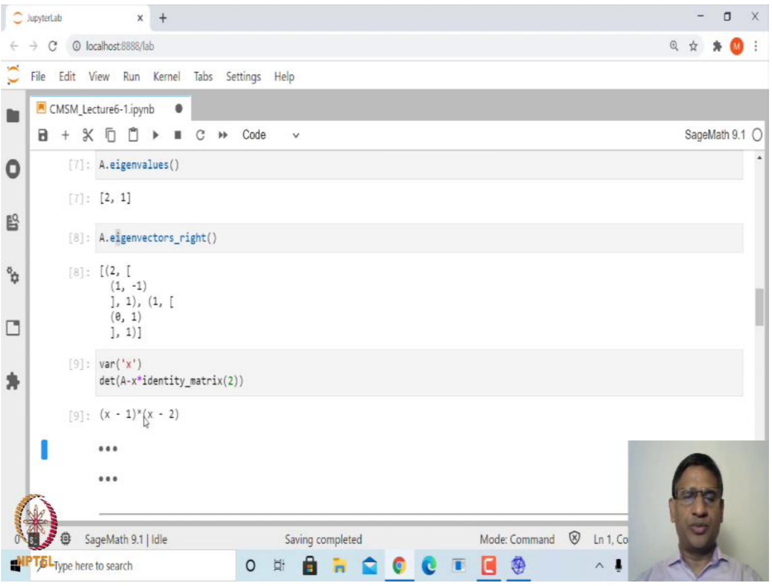

Now, let us find eigenvectors. So, instead of A dot eigenvalues, if you say A dot eigenvectors; but then there is a notion of right and left. Generally we, write Av is equals to lambda v, that means v is multiplied to A on the right hand side. That is why it corresponds to right eigenvectors. So, let us find out A dot eigenvectors underscore right and this gives you this output. Which means that this 2 is eigenvalue and 1 comma minus 1 is an eigenvector.

This says this is 1, which says that how many times this particular eigenvalue has appeared. Here this eigenvalue 2 is appearing only once, this is also known as algebraic multiplicity of eigenvalue 2. Similarly, the second eigenvalue is 1 and the eigenvector corresponding to that is 0 comma 1 and then this also has multiplicity 1. So, that is what exactly we observed from the from the graph.

Now, you can find out the characteristic polynomial of A. What is the characteristic polynomial? Determinant of A minus lambda times I. Instead of lambda, let me write x. So, A minus x times identity matrix 2; this is the 2 cross 2 identity matrix and then find its determinant, of course x you need to declare as variable, Then this is the characteristic polynomial.

And you can see here, the characteristic polynomial is factored as x minus 1 into x minus 2. Therefore, if you equate it to 0; x equal to 0 and x equal to 2 are roots of this characteristic polynomial. And therefore, 1 and 2 are eigenvalues of this matrix A.

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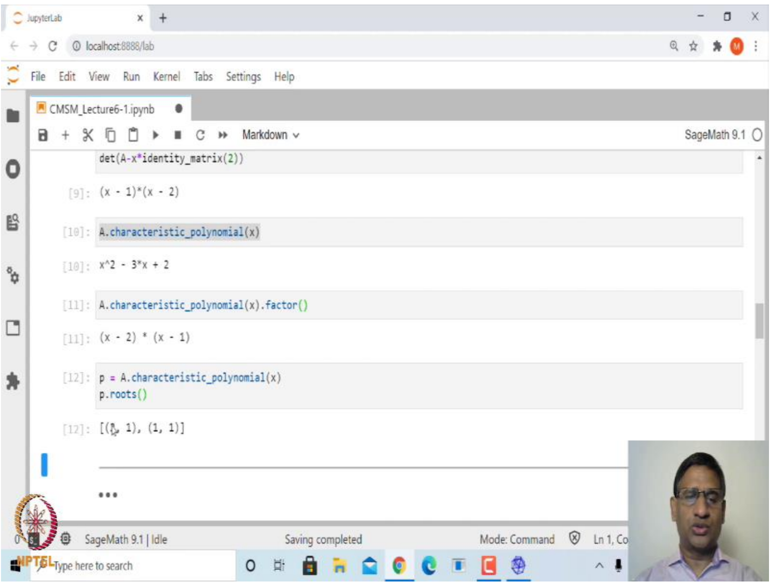

You can find the characteristic polynomial using inbuilt function. A dot characteristic underscore polynomial, in the bracket you write the variable in which you want this characteristic polynomial. So, that will give you the characteristic polynomial of A. This is x square minus 2 x plus 2, which is nothing, but x minus 1 into x minus 2. You can find factor of this.

For example, if I say this dot, factor will give you the factor of this. And then let us find out what are the characteristic roots. That means if I store this characteristic polynomial in p, we can find p dot roots. It says that, it has two roots 1 and 2 with both of them are appearing with multiplicity 1. So, this is how you can compute eigenvalues and eigenvectors using SageMath.

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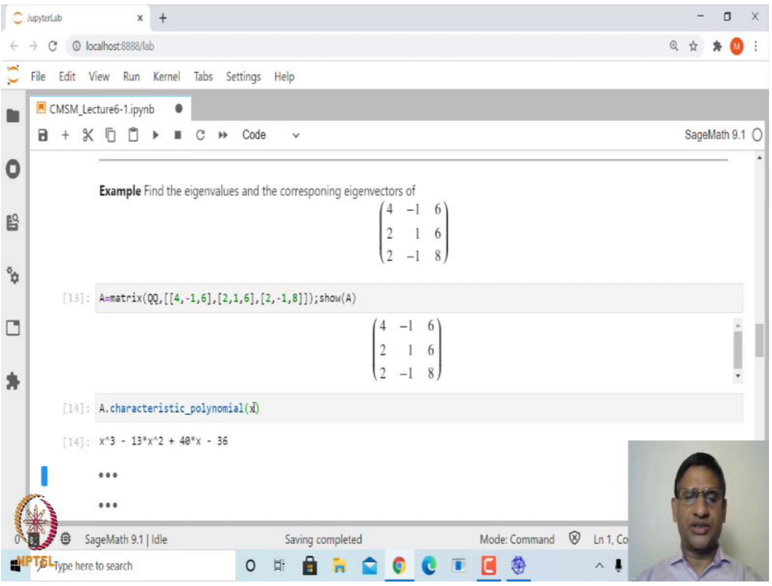

Let us take another, example 3 by 3 example. So, this is 3 cross 3 matrix. we want to find eigenvalues and eigenvectors of this matrix. So, let us first define this matrix A, which is this. How do we find characteristic polynomial of A? A dot characteristic underscore polynomial in x, that is the characteristic polynomial. So, you can see here, A characteristic polynomial of A is a cubic polynomial in x.

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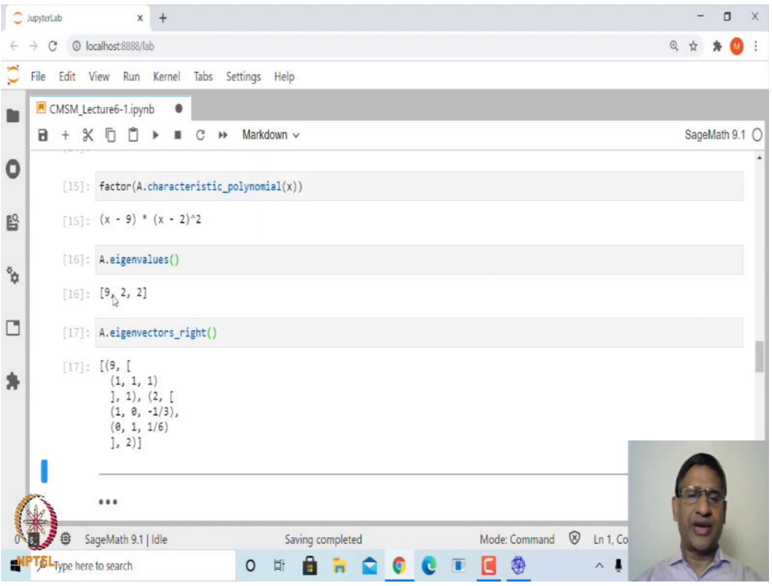

Now, we can factor this characteristic polynomial and it says that, it has factor x minus 9 and x minus 2 square. This means that 2 and 9 are going to be eigenvalues of A. 2 appears with multiplicity 2, whereas 9 appears with multiplicity 1. Let us find out eigenvalues of A. Eigenvalues of A is a list which is 9 comma, 2 comma, 2. That means 2 appears twice.

Similarly, you can find eigenvectors of A. There are only two distinct eigenvalues in this case 9 and 2. So 9 has eigenvector 1, 1, 1.

And then if you look at the eigenvalue 2, then it has two eigenvectors one is 1, 0, minus 1 by 3 and 0, 1, 1 by 6 and 2 has multiplicity 2.

So, eigenvectors right gives you eigenvalues along with the corresponding eigenvectors and its multiplicity. Of course, you could you could find characteristic polynomial and roots manually, as we did in 2 by 2 case. Let us look at some more concepts.

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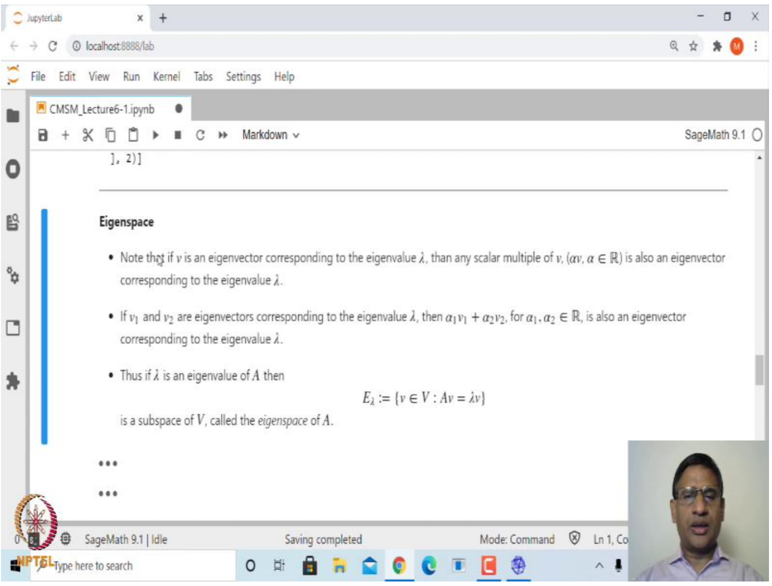

If you have v, an eigenvector corresponding to eigenvalue lambda and then if you take any scalar multiple of v, let us say alpha v and apply A times alpha, v that will give you alpha times Av; but Av means lambda v, therefore A alpha v is nothing, but lambda times Av. Therefore, it means that, any scalar multiple of v is also an eigenvector of A with respect to the same eigenvalue lambda.

Similarly, if you take two eigenvectors v1 and v2 and take any scalar linear combination of v1, v2, such as alpha1 v1 plus alpha2 v2.

And then if you apply A to this; again you will see that, this is nothing, but lambda times alpha1 v1 plus alpha2 v2. In particular this, alpha1 v1 plus alpha2 v2 is also an eigenvector corresponding to the same eigenvalue lambda.

That means that, if I consider set of all eigenvectors of A with respect to eigenvalue lambda, this is a subspace of vector space V, that is known as eigenspace of A. The eigenspace of A with corresponding to eigenvalue lambda.

This is important concept. And so, every vector inside this E lambda, if I apply A to it; it will stay inside E lambda. It also means that E lambda is invariant under A. So, you can define a linear map A, from E lambda to itself just taking v to A v. That is why we say that E lambda is invariant under A.

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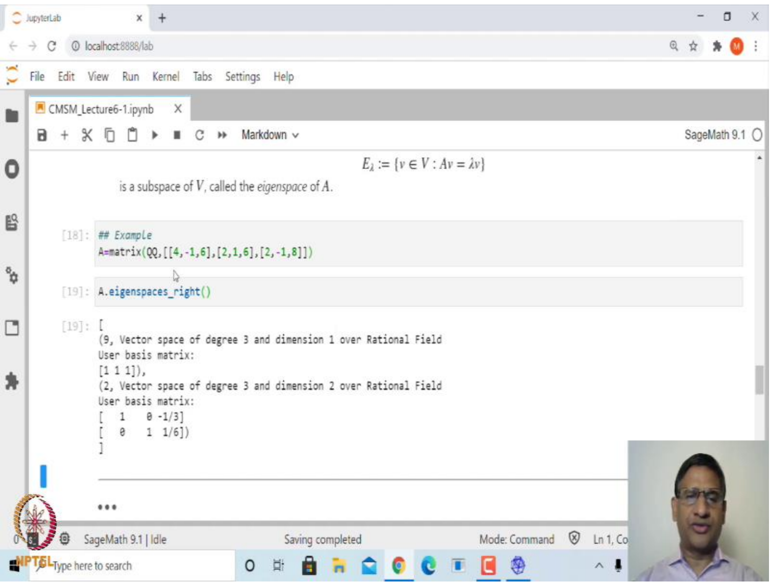

Let us look at again the same 3 by 3 example which we considered in the previous one. So, A is equal to this matrix. Let us find out if there is a function called eigenspaces underscore, right. This will give you eigenspace of A for each eigenvalue lambda. In this case we saw that, that there are two eigenvalues of A, 9 and 2. 9 appears with multiplicity 1, whereas 2 appears with multiplicity 2.

Here corresponding to eigenvalue 9, there is only one eigenvector which is 1, 1, 1 and therefore, the dimension of this eigenspace is 1. Whereas, corresponding to eigenvalue 2; the eigenspace is span by these two vectors, which is called basis of this eigenspace and this has dimension 2. So, E lambda for lambda equal to 2 is 2 dimensional, whereas E lambda for lambda equal to 9 is 1 dimensional.

Of course, this you can also compute manually, but sage has inbuilt function to find this.

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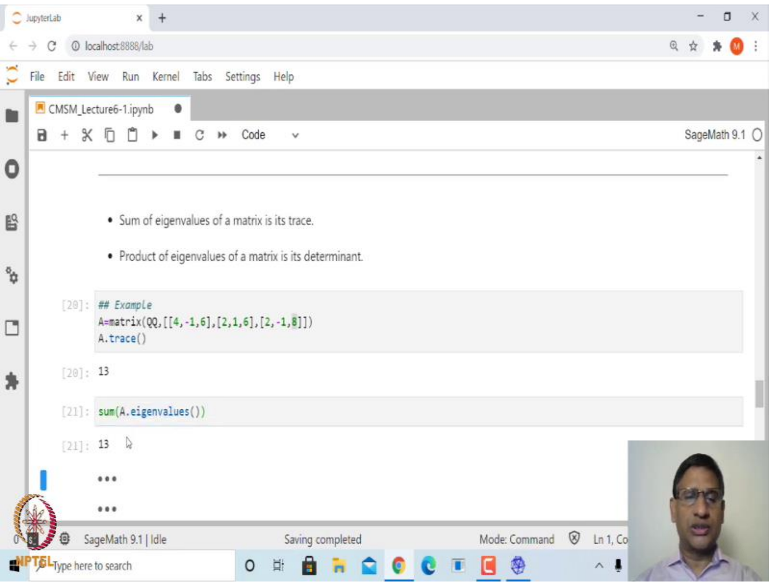

If you look at these eigenvalues, and if you take sum of all eigenvalues of a matrix, it turns out that that is a trace of the matrix, that is sum of diagonal entries of a matrix. Similarly, if you take product of all the eigenvalues, that turns out to be the determinant of a matrix. As a result, if the determinant is 0, that means one of the eigenvalues has to be 0 and vice versa. In case eigen, one of the eigenvalue is 0, then determinant of A must be also 0.

Let us just verify this with the same example, which we considered earlier. A is equal to this and we can find trace of A, the trace of A in this case is 13, which is 4 plus 1, 5 plus 8, 13. And if you look at the eigenvalues of A and take the sum, that should also turn out to be 13.

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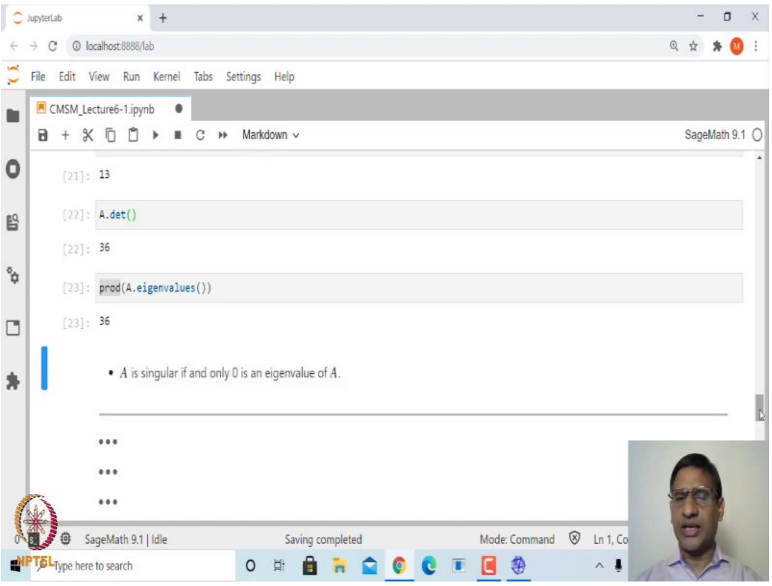

Similarly, if I take determinant of A, the determinant of A in this case is 36, whereas if I take the product of all the eigenvalues, finds the product of a list of real numbers. That also gives you 36, in particular we have verified for this example that, sum of eigenvalues is the trace of the matrix and product of eigenvalues is nothing but the determinant of this matrix.

Of course, you can take any arbitrary matrix and verify this. It requires proof and if you look at any standard book on linear algebra, it will give you proof. Next let us look at, I already mentioned this, in case 0 is an eigenvalue of A, then determinant must be 0, in particular the matrix must be singular and vice versa.

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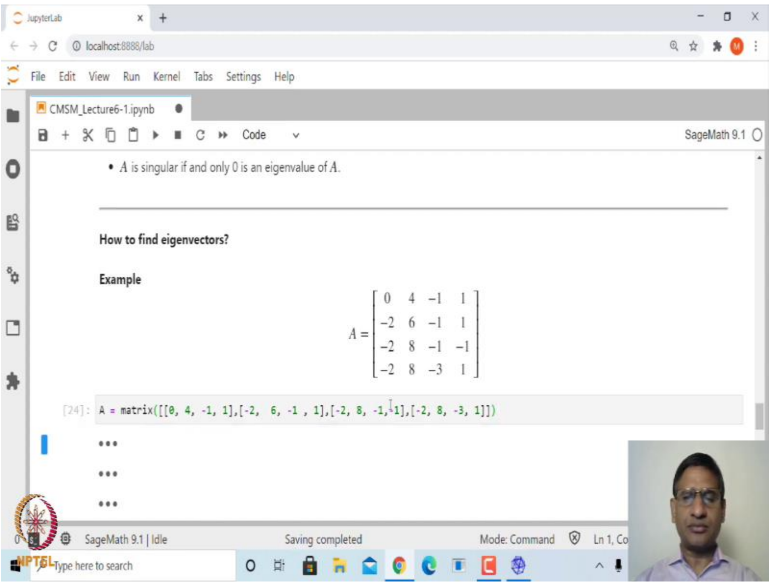

Let us look at how can we find eigenvectors manually. Sage has inbuilt functions to find eigenvectors. But if you have to find manually, how do we do that ? If v is eigenvector of A with respect to eigenvalue lambda, then all you need to do is solve A minus lambda times identity x is equal to 0, this is homogeneous system of linear equations. So, let us again take an example. This is A, which is a 4 cross 4 matrix. Let us define this matrix, this is A.

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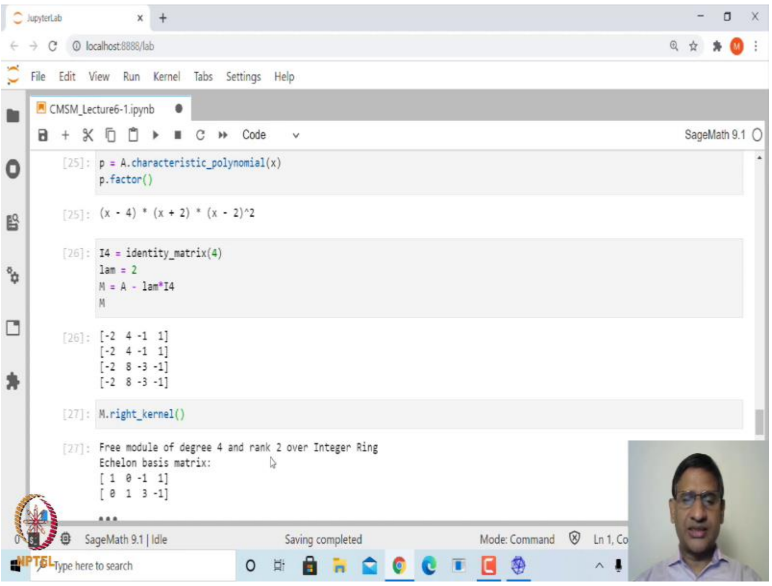

And let us first find characteristic polynomial of A and find its factor. It says that, it has factor x minus 4 x plus 2 and x minus 2 the whole square. That means 4, minus 2, and 2 are going to be eigenvalues of A. 4 and minus 2 have multiplicity 1, whereas 2 has multiplicity 2. This multiplicity is also known as algebraic multiplicity.

Next let us define identity matrix I4, 4 cross 4 identity matrix and let us take one of the eigenvalue. Let us say for example lambda equal to 2. Let us define M is equal to A minus lambda times I4. That is your A minus lambda times I4. Now, we need to find non zero solution of M x is equal to 0.

We need to find non zero solution of M x equal to 0; M x is nothing, but A minus lambda I times x. So, how do we solve this? Sage has inbuilt function to solve this. All we need to do is, we need to find the null space of M or kernel of this M. That we can find using inbuilt function M dot right kernel.

So, when I say M dot right kernel; it gives you that it is a subspace with two basis 1, 0 minus 1, 1; 0, 1, 3, minus 1, these are the two vectors which forms a basis of this null space, right kernel of M. And we have seen also explicitly how to find this manually, by just applying RREF and then working with that.

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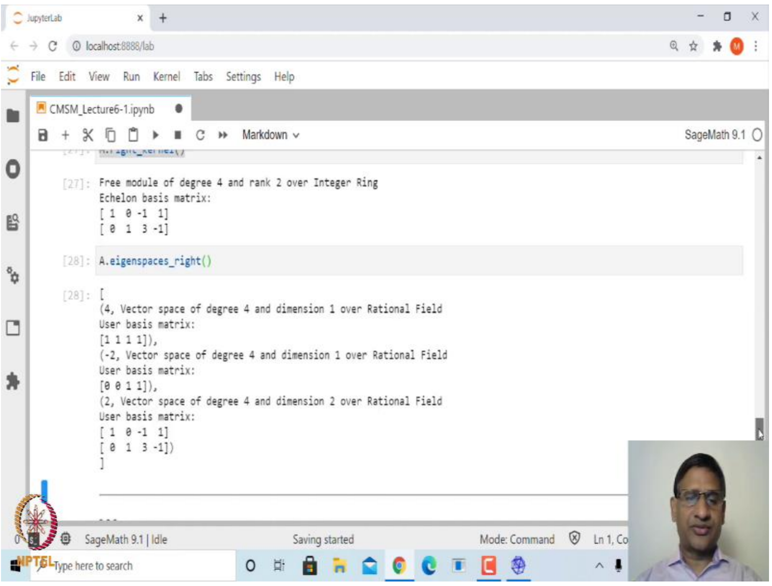

So, I leave this as an exercise for you to also find this manually by using RREF. If you look at what are the eigenspaces of A, in this case. You see that eigenspace with respect to eigenvalue 2, the eigen basis is this which is same as the kernel of A minus lambda I. So, that is how you can obtain or find eigenvectors manually by solving A minus lambda I times x is equal to 0, for non-zero vector x, ok.

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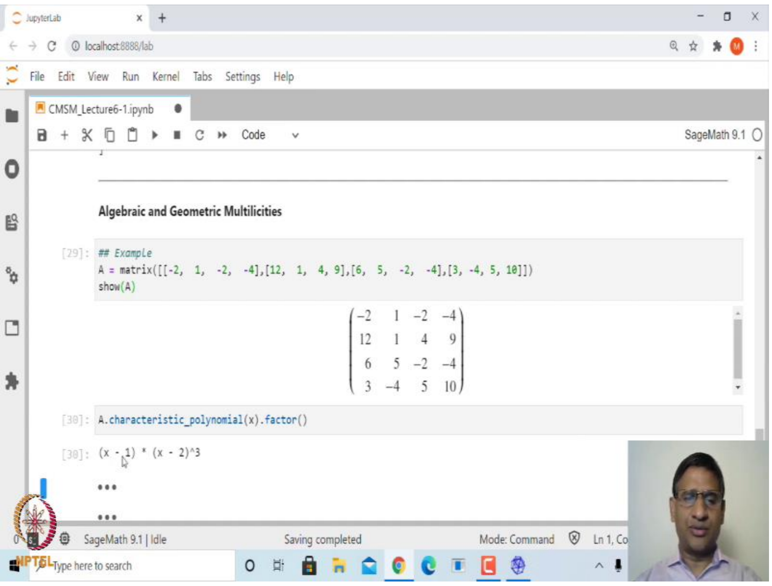

Next let us look at, what is meaning of algebraic and geometric multiplicity ? So, I already explained what is meaning of algebraic multiplicity, and geometric multiplicity actually is nothing, but the dimension of the eigenspace corresponding to that eigenvalue.So, let us take an example, this is a 4 cross 4 matrix. Let us find out what are the eigenvalues of this. So, this has two eigenvalues 1 and 2. 1 appears with multiplicity 1, whereas 2 appears with multiplicity 3.

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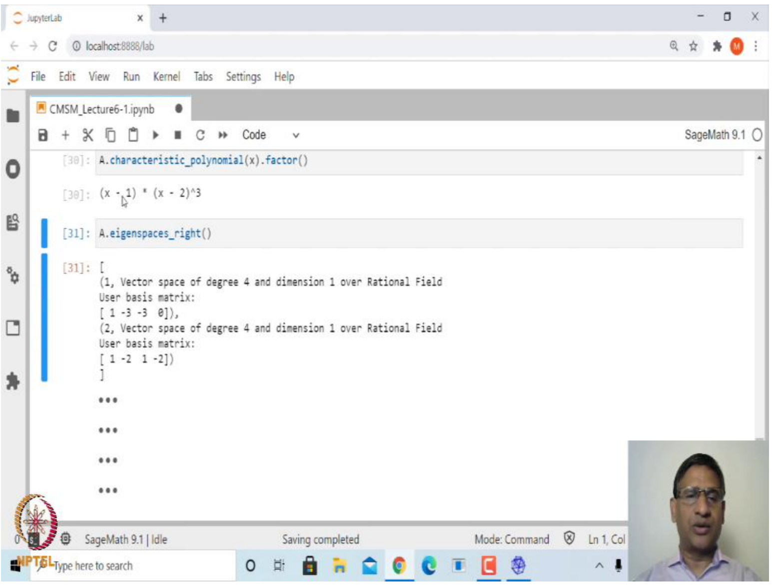

Now, if you find eigenspaces of A, then what we get? The eigenbasis with respect to eigenvalue 1 has only one vector, that means this is dimension is 1, whereas the eigenspace with respect to eigenvalue 2 is also 1 dimensional. So, both eigenspaces with respect to eigenvalue 1 and 2 in this case are 1 dimensional. So, here the dimension of this eigenspace corresponding to eigenvalue is known as geometric multiplicity.

So, in this case what you see, the algebraic multiplicity of eigenvalue 1 is 1 and geometric multiplicity is also 1, whereas for the eigenvalue 2, algebraic multiplicity is 3, whereas geometric multiplicity is 1. In this case algebraic multiplicity and geometric multiplicity of eigenvalue 2 are not the same. However, in some cases they may be same as we saw earlier.

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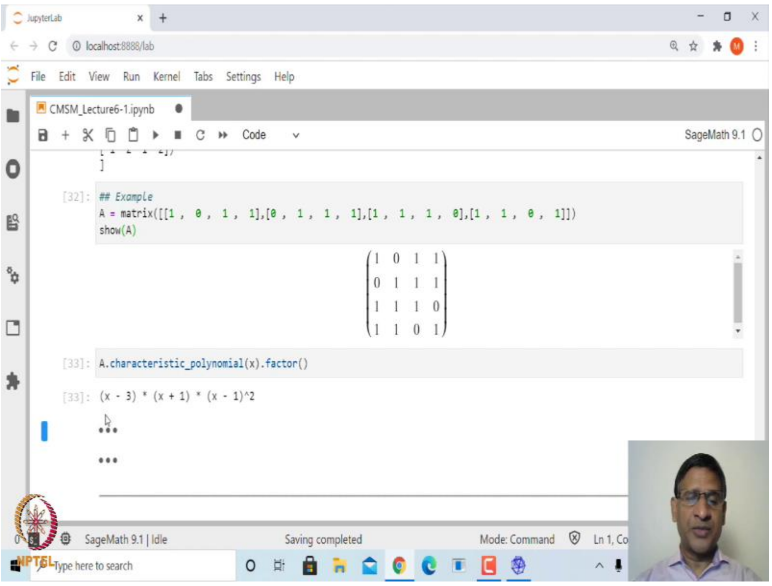

So, for example, if I look at this, this matrix 1 0 1 1; 0 1 1 1; 0 0 1 0; 1 1 0 1 and if you find out the characteristic polynomial of this. It splits as x minus 3, x plus 1, x minus 2 the whole square. So, in this case, the geometric multiplicity we need to find out. Here algebraic multiplicity of 3 is 1, minus 1 is 1, and 2 is and 1 is 2.

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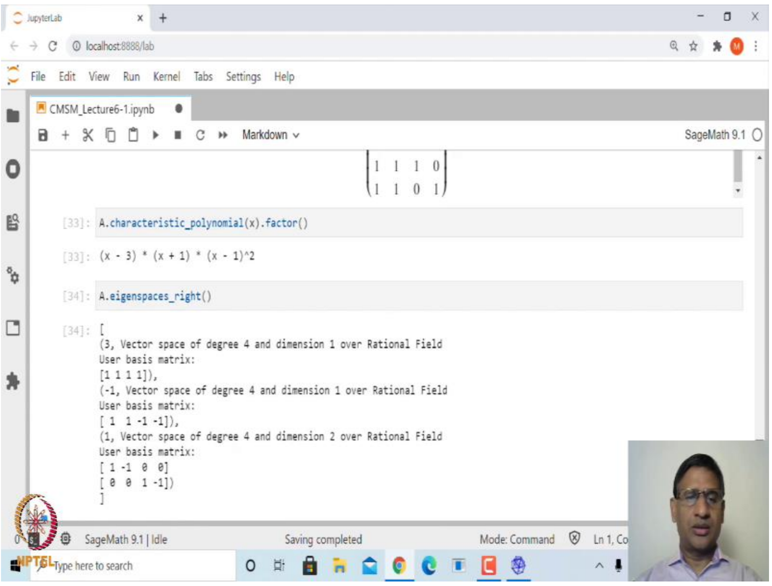

So, if you look at eigenspaces, in this case eigenspace, dimension of eigenspace with respect to eigenvalue 3 is 1, with respect to minus 1 is 1, and with respect to 1 is 2 which is same as geometric multiplicity. So, one can show that, algebraic multiplicity is always bigger than equal to geometric multiplicity and they need not be equal.

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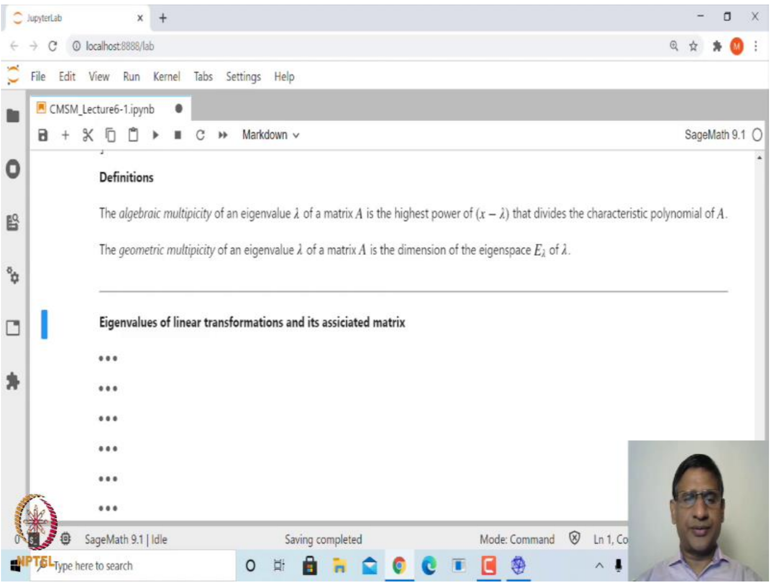

So, that is the definition, algebraic multiplicity of an eigenvalue lambda of a matrix A is the highest power of x minus lambda that divides the characteristic polynomial, A minus, determinant of A minus lambda I. And the geometric multiplicity is the dimension of the eigenspace corresponding to eigenvalue lambda. So, that is the definition of geometric and algebraic multiplicity of eigenvalues.

Now, let us look at. If you look at the definition of eigenvalue, eigenvector, the way we started, we started with a linear map and we defined matrix of that, as same as A and then we defined eigenvalues, eigenvectors of A. So, one can define eigenvalue, eigenvector of the linear map A itself, exactly the same definition. So, let us look at. Suppose we define a linear transformation. Can we find its eigenvalues and eigenvector?

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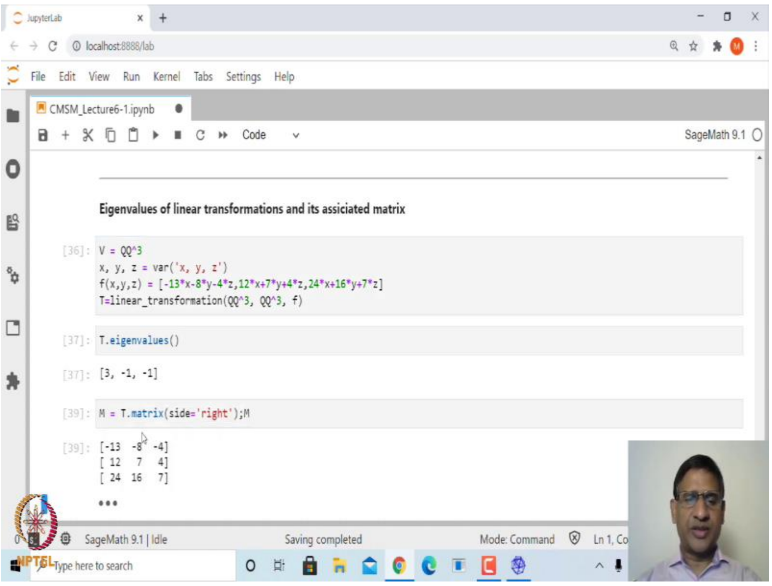

Let us take an example, suppose we have a linear transformation T, which is defined as a linear map from Q3 to Q3 given by T x, y, z is equal to minus 13 x minus 8 y minus 4 z. Second coordinate is 12 x plus 7 y plus 4 z and third coordinate is 24 x plus 16 y plus 7 z.

So, let us define this T. Now you can find eigenvalues of T using the same command. In this case eigenvalues are 3 minus 1 and minus 1. That means algebraic multiplicity of eigenvalue 3 is 1, whereas algebraic multiplicity of eigenvalue minus 1 is 2.

Now let us find matrix of T with respect to standard basis or for that matter with respect to any basis. So let us define M to be the matrix of this linear transformation T, with respect to the standard basis. And if you look at this M, this is the matrix.

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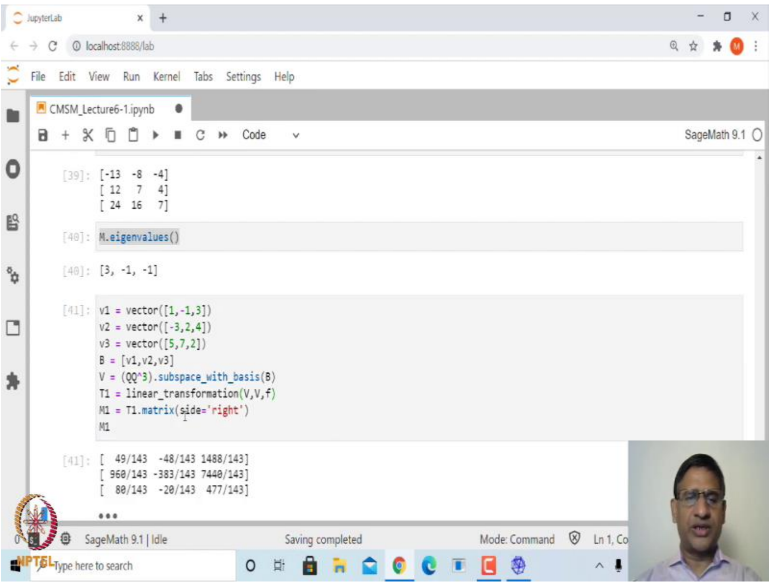

Now, let us find out what are the eigenvalues of M. That is same as the eigenvalues of T. It does not matter whether you find eigenvalues of T or you find eigenvalues of M with respect to standard basis, both are same. However, if you change the basis, what happens. Let us change a basis, let us say v1, v2, v3 are these three vectors. One can check that, these three vectors form a basis of Q3.

Let us define V to be subspace of Q3 spanned by basis B or subspace of V with basis B and then define a linear map T1 from V to V, V to V with the same image f and find the matrix of T with respect to this particular basis. So, here the same basis we are taking on domain and codomain. So, this is the matrix.

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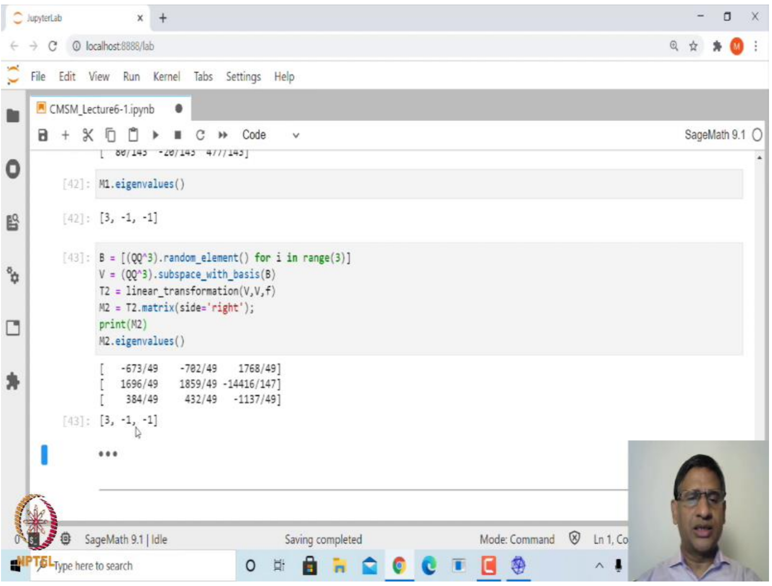

Now, if you find eigenvalues of this M1 ; then what do you see? Again you see that, eigenvalues are the same, right. So, change of basis has not affected the eigenvalues of this M. Similarly, Now let me take any random three vectors.

And usually it will form a basis and let us assume that, it forms a basis and then define the subspace with basis B. And define a linear transformation from V to V with the same image f and define the matrix M2 of T2 with respect to this basis. And then let us find out what will be eigenvalues. Again you can see the eigenvalues are 3, minus 1, minus 1.

So, eigenvalue of a linear transformation are same as eigenvalues of the matrix associated with that linear transformation with respect to any basis. So, what it means and of course, one can prove this.

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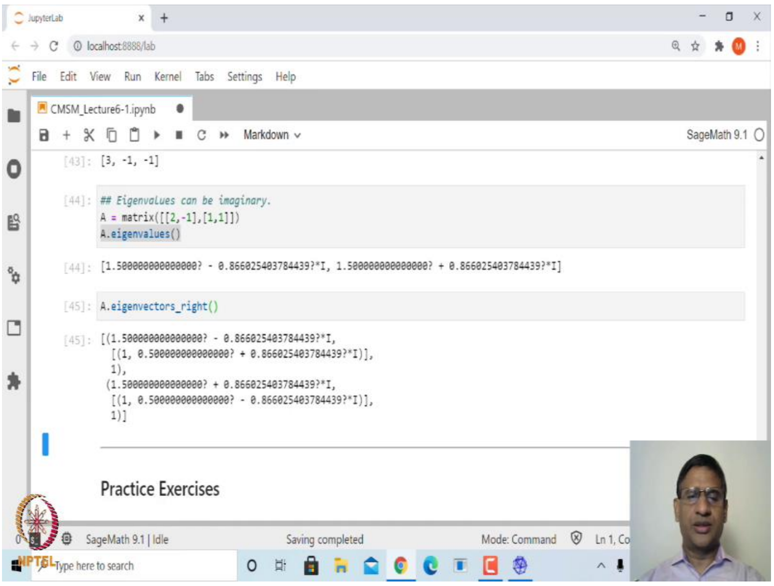

See we have defined eigenvalues as a real number; because we defined A as a linear map from vector space V over R to itself.

So, in this case, eigenvalue has to be real, similarly eigenvector. But if you look at the characteristic polynomial, the characteristic polynomial will be some polynomial in x. But that need not be always real ; roots of a characteristic polynomial can be imaginary.

So, you can have imaginary eigenvalue ok, imaginary eigenvalue. So, in particular in case you define linear map from vector space V over complex field to itself ; it will also have imaginary eigen values and hence imaginary eigenvectors. So, if you look at for example, these 2 cross 2 real matrix 2, minus 1; 1, 1 and then try to find what are the eigenvalues of A, you will see that it gives you imaginary eigenvalue.

So, it here it means that, 1.5 minus 0.866 times I which is imaginary number. And this question mark actually it gives you saying that, this is actually a numerical computation. So, it is not exact, it is numerical computation. If you try to find eigenvectors of A, that also you will get. So, A dot eigenvectors right, again you will see this gives you imaginary eigenvectors. So, it is not necessary that even if, for every real matrix, you will have real eigenvalues; you may not have eigenvalues. So, for example, if I look at let us say rotation matrix; say let us say rotation in R 2, rotate it by let us say 30 degree, no vector is going to be fixed under rotation. So, that means rotation matrix will not have any eigenvector and eigenvalue.

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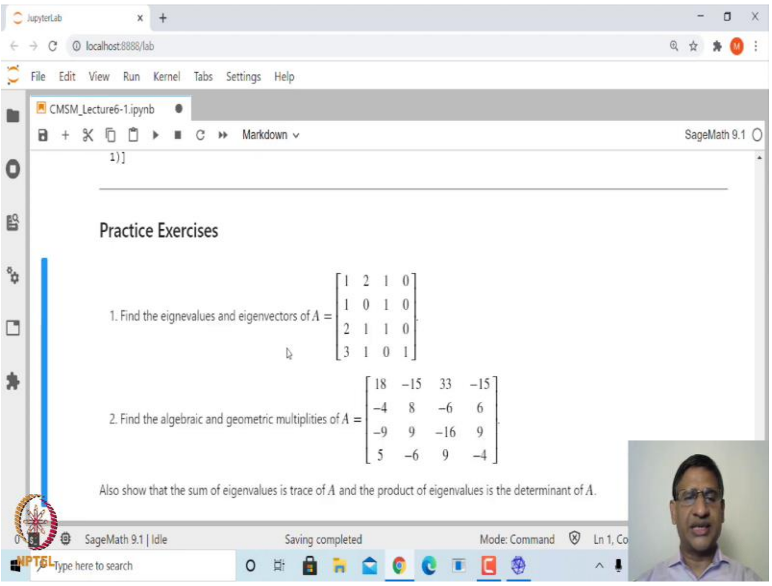

So, let me leave you with few simple exercises; these are straight forward. So, find eigenvalues and eigenvectors of this matrix ; find eigenvalues and eigenvector of this matrix and also find algebraic and geometric multiplicity of A.

And further verify the determinant of A is product of eigenvalues of A and the trace of A is sum of eigenvalues of A. So, these are two very simple exercises.

In the next lecture, we should look at more concepts on eigenvalues and eigenvectors with SageMath.

Thank you very much.