Computational Mathematics with SageMath
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Lecture – 25
Partial Derivative with SageMath

Welcome to the 25th lecture on Computational Mathematics with SageMath. In this particular lecture, we will look at partial derivatives and some concepts related to them. Let us get started.

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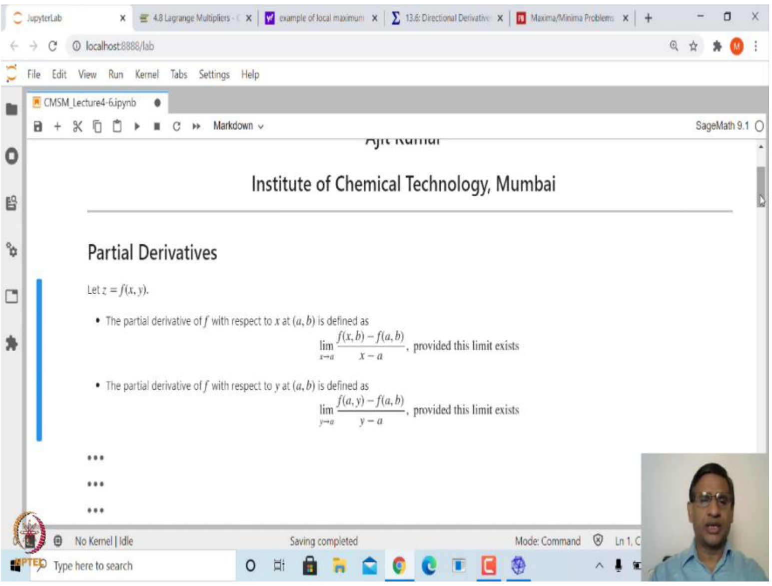

Suppose you have a function z = f(x, y). Let us assume that this function is defined on the entire r2, and then the partial derivative of f w.r.t x at a point (a, b) is defined as $\frac{f(x,b) - f(a,b)}{x - a}$.

The limit of this as x goes to a. In case this limit exists, we say that the partial derivative of f with respect to x at (a, b) exists, and generally, we denote it by the f'(x) at (a, b) or partial derivative that is $\frac{\partial f}{\partial x}$ at (a, b). Similarly, you can define the partial derivative of f with respect to y at (a, b) using $\frac{f(a,y) - f(a,b)}{y - b}$ as y goes to b.

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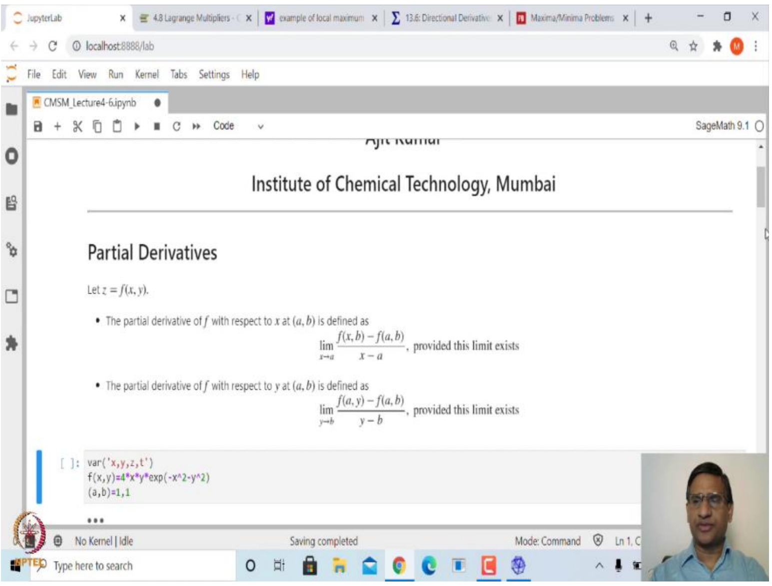

This is how you define the partial derivative of a function. This is defined using a limit. In these two cases limits exist, we can say that f has first-order partial derivatives at (a, b) with respect to x and y.

The sage can find partial derivatives of any function. Of course, provided it exists using the same function, diff or differentiate. Suppose we have a function $f(x, y) = 4xye^{-x^2-y^2}$.

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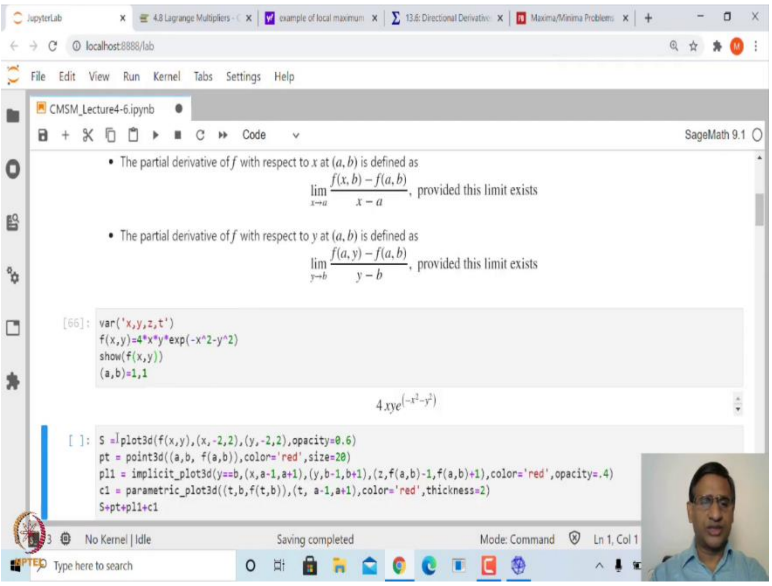

Let me just show you what this function f(x, y) is. Let us take a point (a, b) = (1, 1). We want to find a partial derivative of this function that is $4xye^{-x^2-y^2}$ at (a, b) = (1, 1). How do we do that? We can simply find out using diff, but before that let us look at what is the meaning of partial derivatives.

Suppose we work with this function; suppose we plot a graph of this function and look at the definition. What does it say? Here you are substituting y equal to b. If you look at the plane y equal to b this will lie in the x z plane parallel to the y-axis, it takes the intersection of that surface with y equal to the b plane that intersection will be a curve. You are looking at the rate of change of the function at x here, or along the x-axis and when you intersect that surface by y equal to the b plane.

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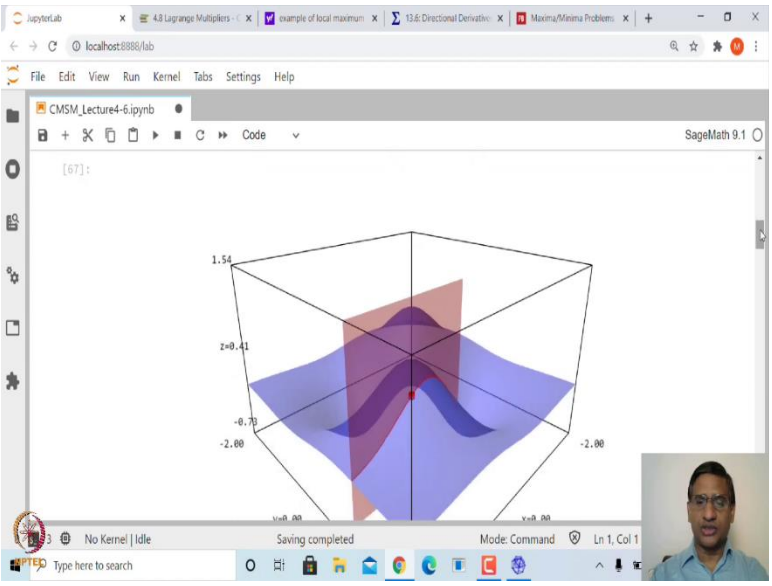

That is what you can look at from this graph. This is the surface that I just rotate.

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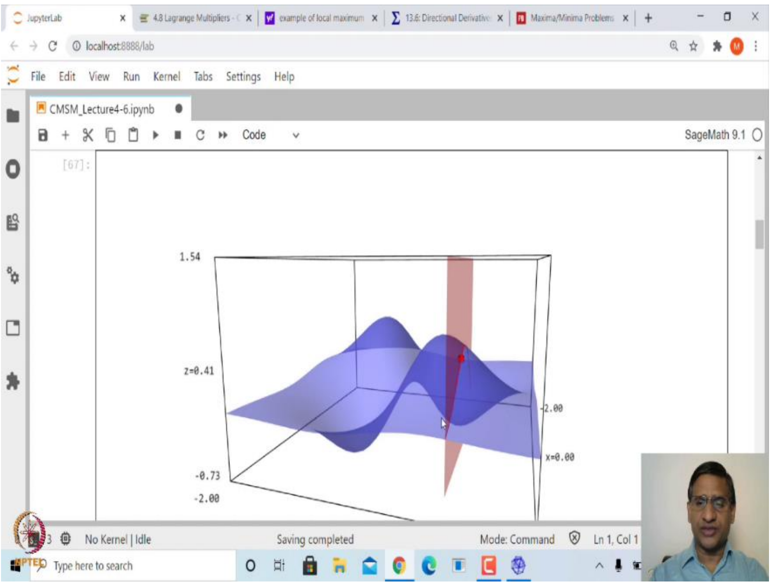

This is the surface, the blue colour. The red point is the point at which you want to define the partial derivative. If you draw the tangent this plane y is equal to b plane and the intersection will be a curve, so at this point, you look at the rate of change of this function along this curve. That is the meaning of the partial derivative of f with respect to x.

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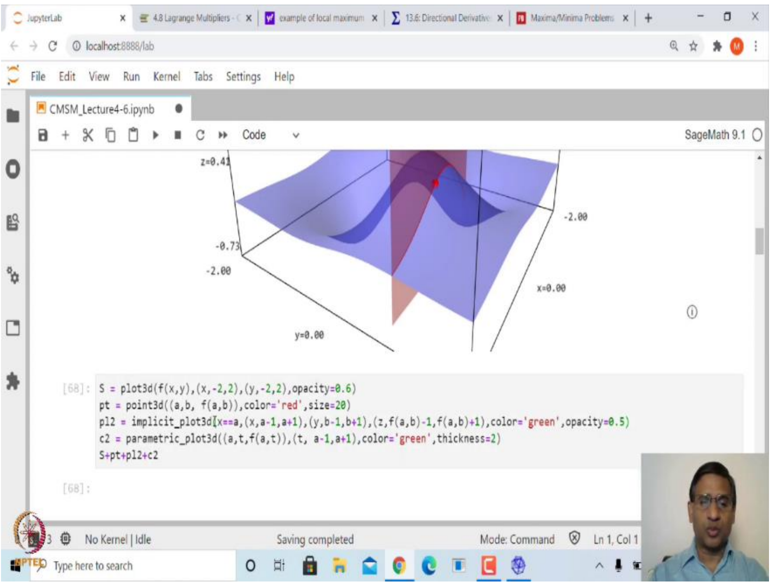

You can try to look at the partial derivative of f with respect to y at a point (a, b).

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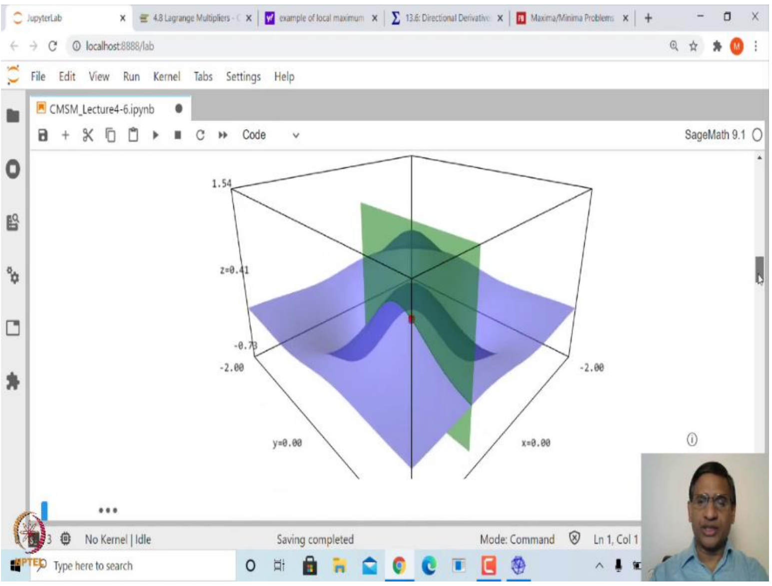

In that case, you need to take a plane x equal to a that is what you can see here and intersect with this surface, and then the intersection will be a curve, so that is what you can see here. And then at that point, you look at the rate of change of this curve.

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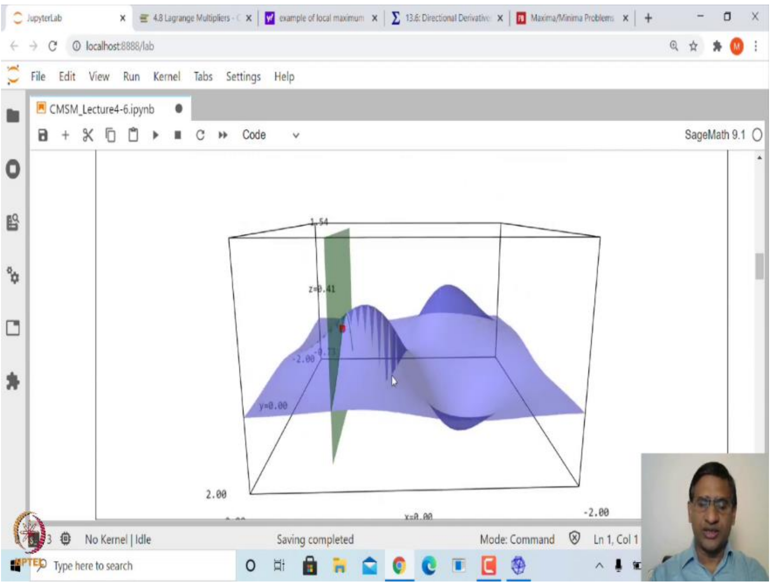

If you look at it in terms of geometrically the tangent to this curve at this point, will be the partial derivative with respect to x and in this case, it will be partial derivative with respect to y, that is the geometric meaning of a partial derivative of a function at a point (a, b) with respect to x and y.

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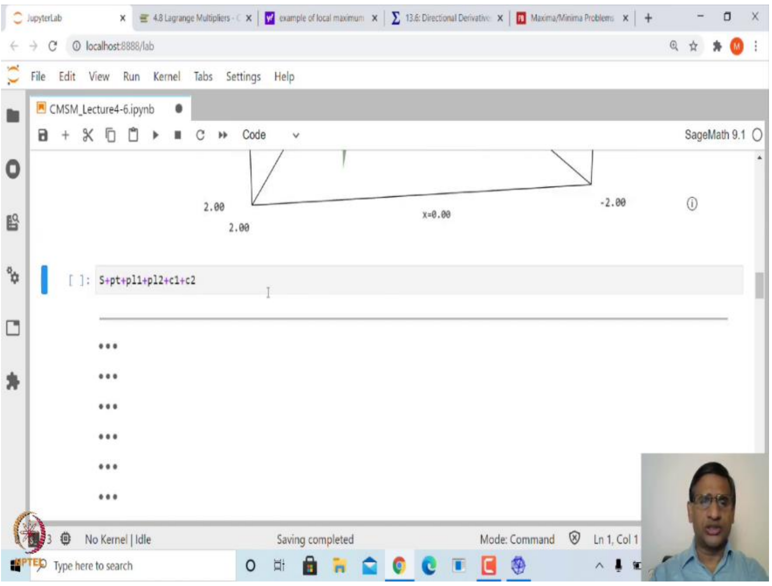

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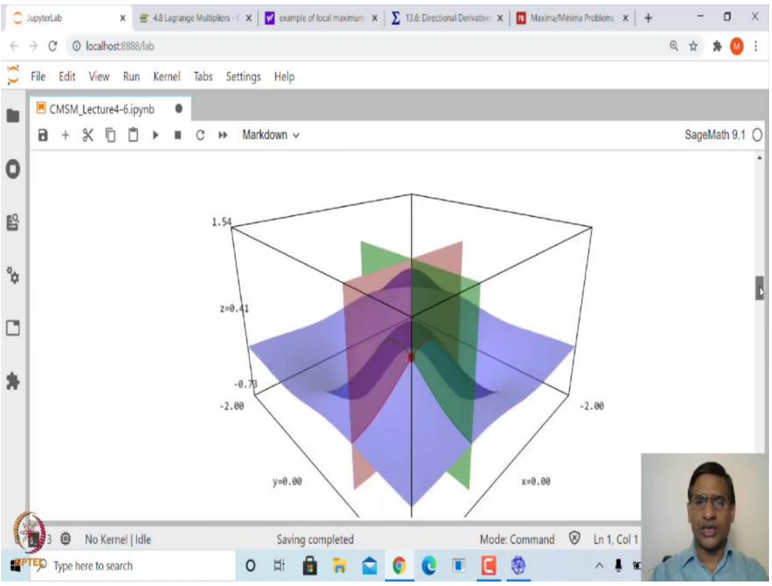

You can plot all these things together. Both x equal to a plane and y equal to b plane together and this is what you see.

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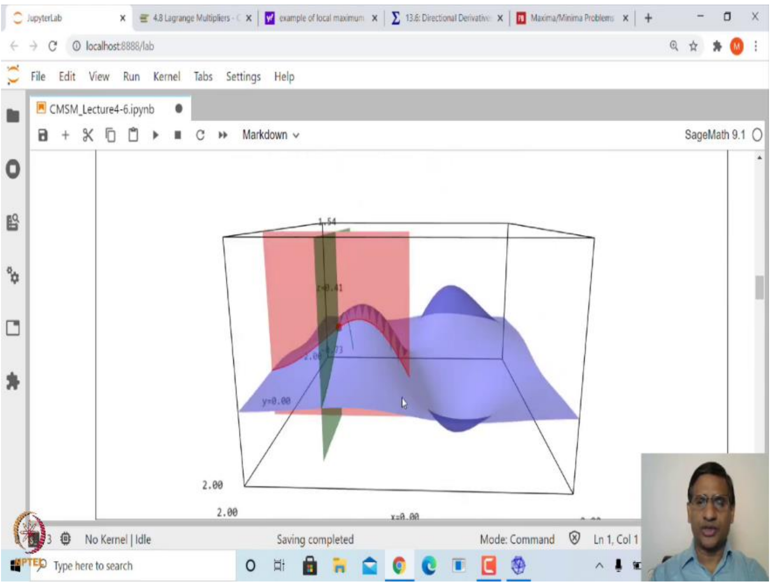

This is the geometric meaning of partial derivatives.

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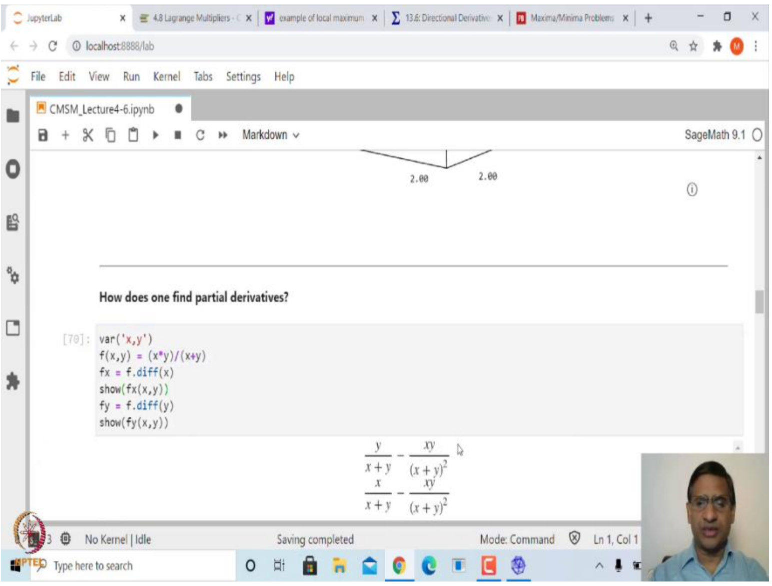

How does one find partial derivatives? To find the partial derivative of a function with respect to x and y, you can just differentiate the function with respect to x and keep y constant. Similarly, when you want to differentiate with respect to y and keep x constant, you will get the partial derivative.

For example, look at the function $f(x, y) = \frac{xy}{x+y}$. We can find the partial derivative of f with respect to x and y using f dot diff with respect to x and f dot diff with respect to y.

This is the partial derivative of f with respect to x and the partial derivative of f with respect to y.

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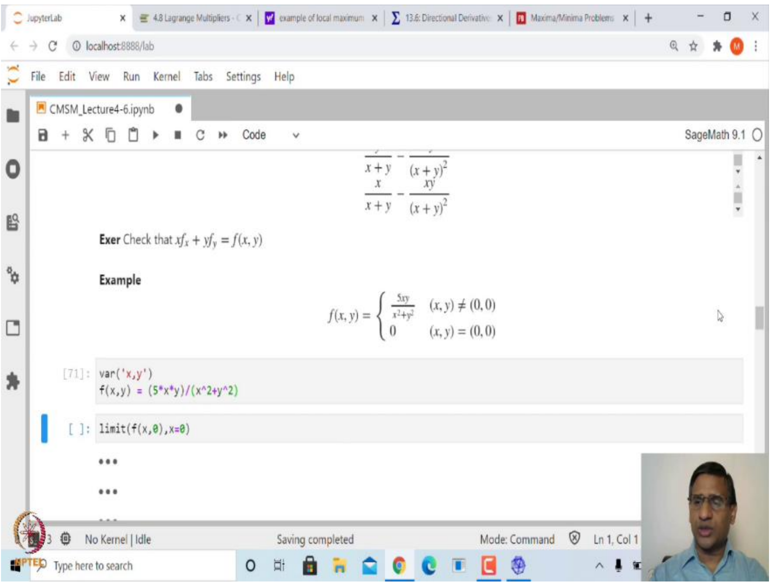

Suppose for this function you can try to verify that x into f(x) that is the partial derivative of f with respect to x plus y into the partial derivative of f with respect to y, this is equal to f(x, y). This is what is called Euler's formula. We are verifying this Euler formula because this function f (x, y) is a homogeneous function of degree 1.

Let us look at, for example, this function $f(x, y) = \frac{5xy}{x^2+y^2}$ and equal to 0 when (x, y) equal to (0,0). if you look at this and try to find the partial derivative of f(x, y) at (0, 0). If you want to find out the first limit of f(x, 0) that is the partial derivative of f with respect to x at (0, 0). You get the answer 0.

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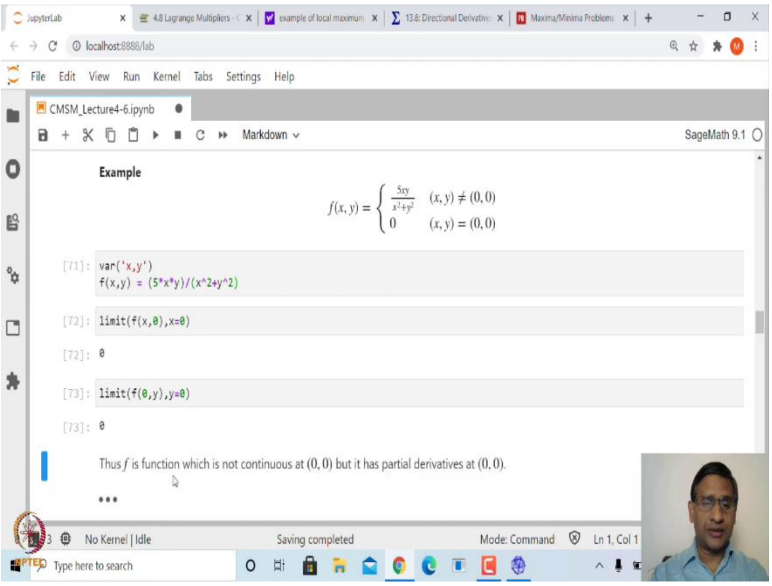

Whereas, if you do the same thing if you try to do the limit of f(x, y) at (0, 0) with respect to y; that means, the partial derivative of f with respect to y at (0, 0); in this case, the limit is again 0. In both cases, this limit is 0.

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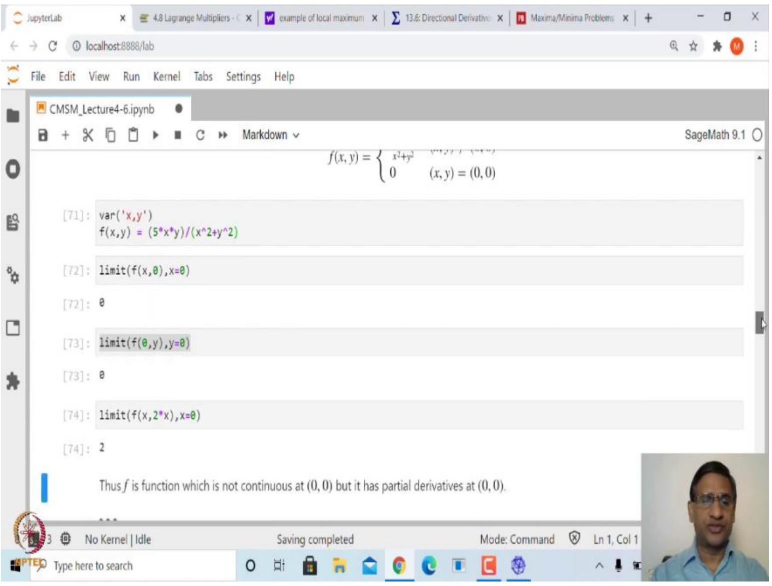

However, if you try to find out its limit with respect to some other curve. For example, let us say the limit with respect to x and y = 2*x and at x equal to 0, this limit is 2. The limit along the x-axis and y-axis both are 0 whereas, the limit along y equal to 2x curve is 2, so that means, this function does not have a limit at (x, y) equal to (0, 0). In particular, this function is not continuous at (0, 0).

However, for these two things, the partial derivative of f with respect to x at (0, 0); this is the partial derivative of f with respect to y at (0, 0); both these things exist whereas the function is not even continuous. You can have a function that is not continuous at some point, but still, it has partial derivatives.

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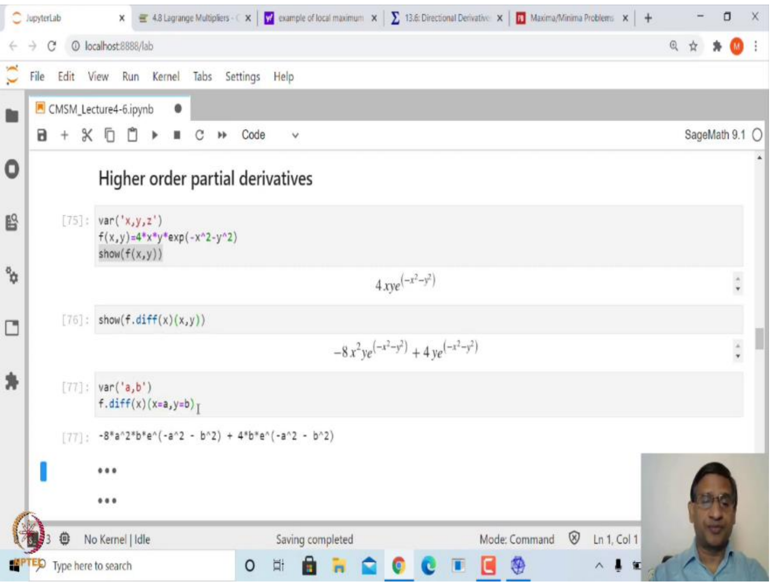

Now, if you have noticed a partial derivative of a function with respect to x and y they are a function of (x, y). you can again talk about the partial derivative of f that they are called higher-order partial derivatives.

For example, let us take again the same function f(x, y) =$4xye^{-x^2-y^2}$. And suppose we want to find the partial derivative of f with respect to x at a point (x, y), then we can use f dot diff with respect to x and at (x, y). That is the partial derivative.

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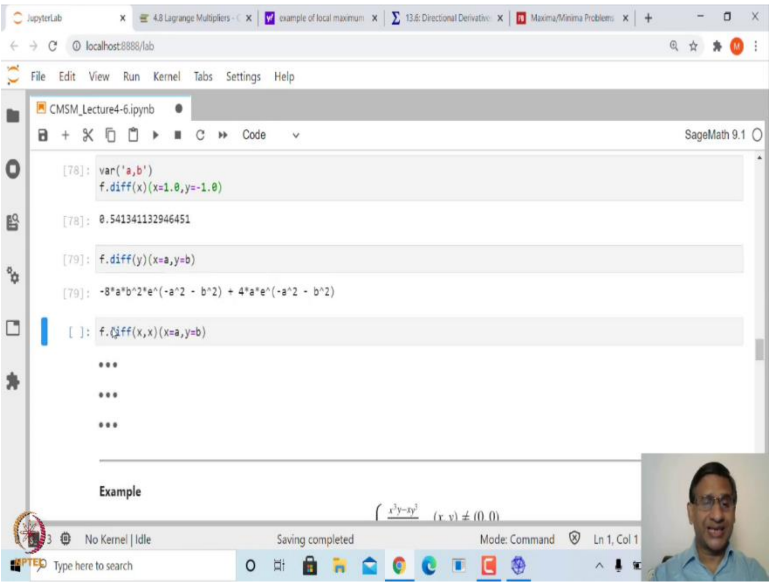

You can find out the partial derivative of f with respect to x at any point (a, b). If I substitute a = 1.0 b = -1.0 you will get the decimal value. Similarly, you can find a partial derivative of f with respect to y at (x = a, y = b).

If you want to find a second-order partial derivative let us say first with respect to x, and then again with respect to x that is called second-order partial derivative with respect to x. You can find it by writing 2. The partial derivative of f with respect to x twice.

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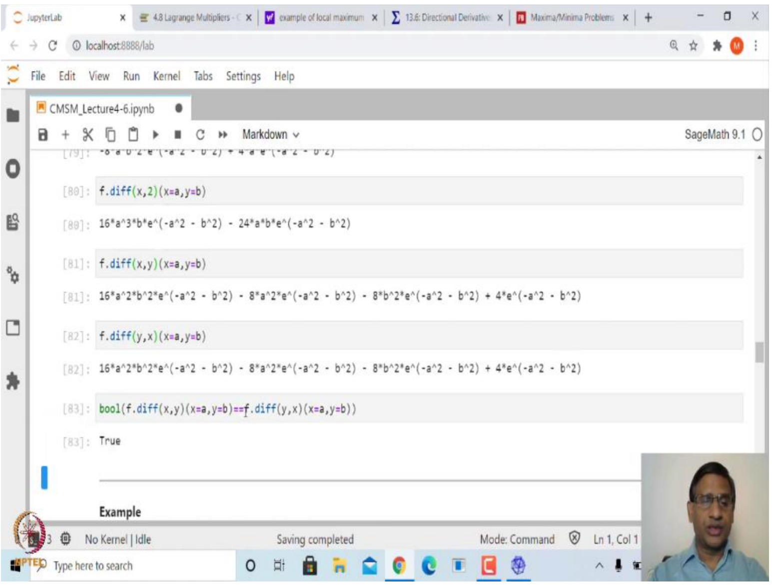

You can find the partial derivative of f with respect to x first and then with respect to y using this f dot diff (x, y) at (a, b). Similarly, you can find out the partial derivative of f with respect to y first and then with respect to x this is what you will get. And maybe for this function suppose you want to check whether these two partial derivatives. These are called mixed second-order partial derivatives at any point (a, b) whether they are the same.

You can look at the partial derivative of f with respect to (x, y) at (a, b) and the partial derivative of f with respect to (y, x) at (a, b). And since these two are expressed and you want to check whether they are equal or not you have to use bool, which stands for Boolean. If I say bool I get the answer True; that means, these mixed second-order partial derivatives are the same.

In this case, you have seen that the second-order mixed partial derivatives of this function at any point (a, b) are the same. However, that is not the case in general. For example, if you look at this function $f(x,y) = \frac{x^3y-xy^3}{x^2+y^2}$. One can show that this function has first-order partial derivatives at (0, 0).

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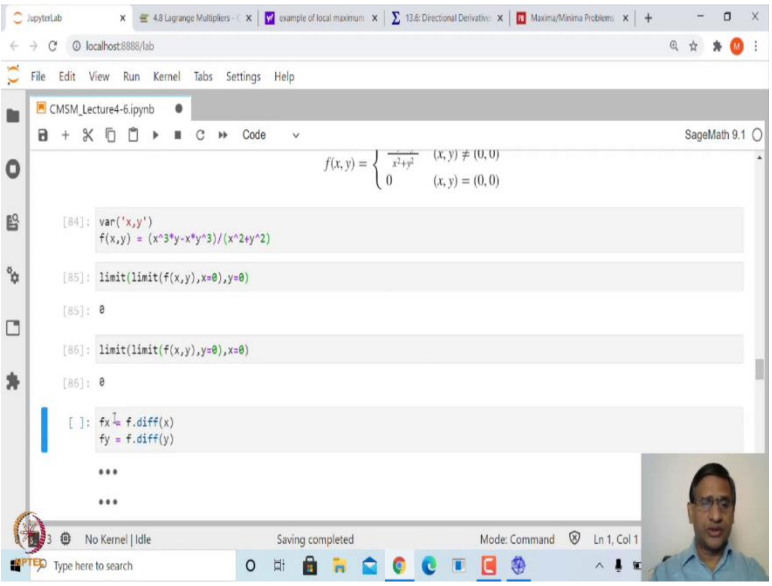

Let us find the partial derivative; first, let us find the limit of f(x, y) at x = 0 and then y = 0. It is an iterative limit that is equal to 0 and you can reverse the order this is also 0.

Now, suppose we want to find fx is equal to f dot diff x and fy equal to f dot diff y. That is the first-order partial derivative of f with respect to x, similarly, this is the first-order partial derivative of f with respect to y. These are functions of x and y.

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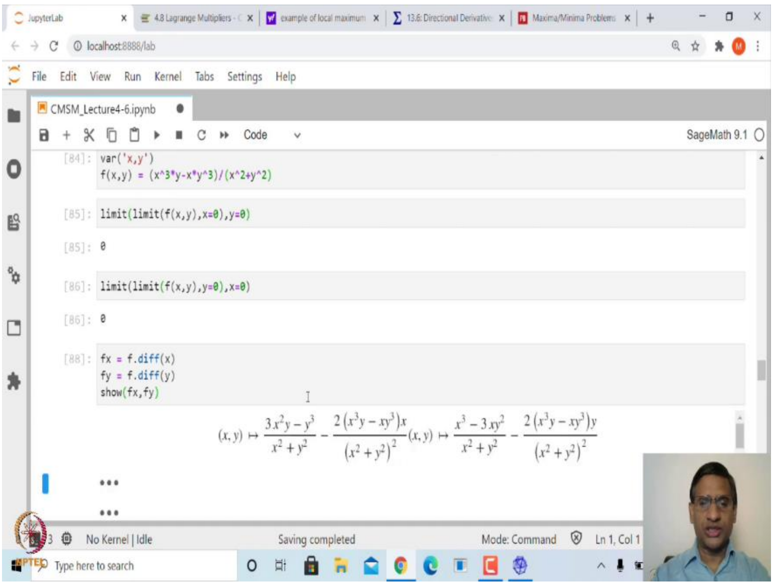

Suppose, now if you want to look how this functions looks. I will say show (fx, fy).

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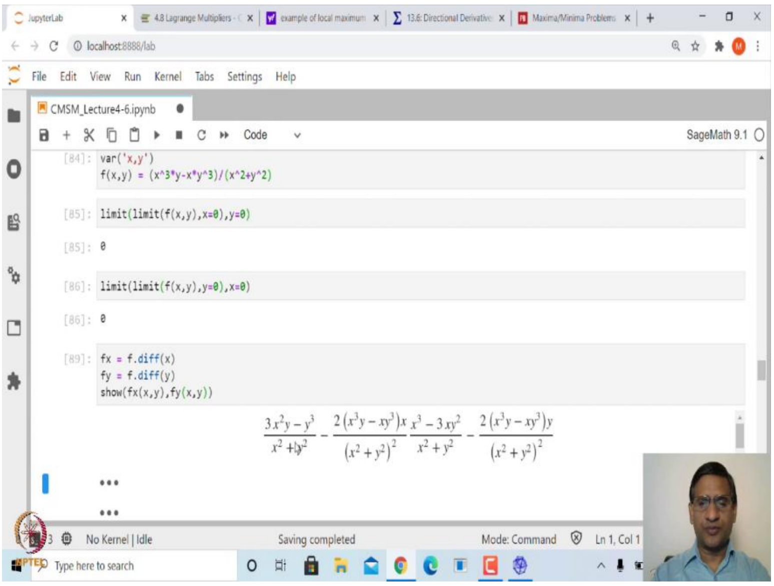

If you want I will put at any point (x, y), here also at any point (x, y). These are the two partial derivatives. Let me use them here to show two different things separately. These are the two partial derivatives.

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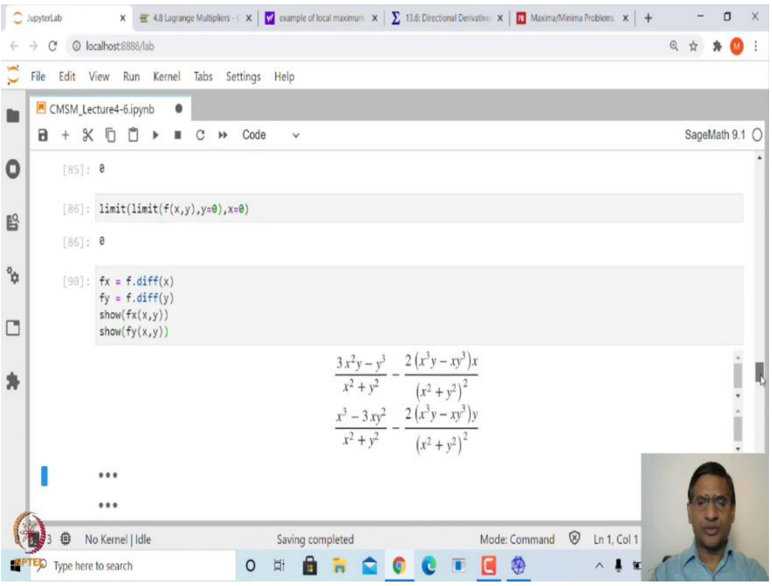

Now, suppose we want to check whether you can find a partial derivative of this with respect to x and y.

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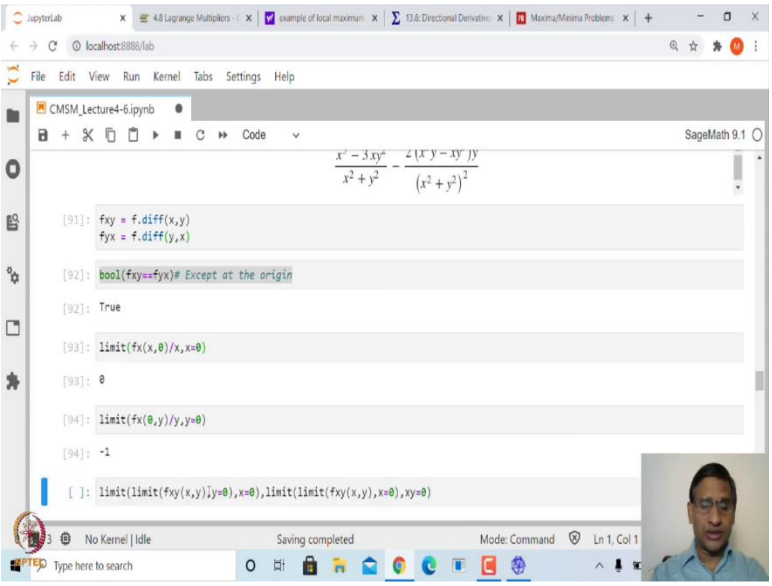

When we look at the second-order partial derivative, we can find f dot diff (x, y) and f dot diff (y,x). These are the mixed second-order partial derivatives. these two exist at any point (x, y) other than the origin. But these two are going to be the same.

However, at the origin, if you want to find, what do you need to do? You need to find the limit of fx at y equal to 0 about x at x equal to 0 then the limit of fx at (x, 0) upon x that is a limit of the partial derivative of fx with respect to x at x equal to (0, 0). This is 0. Whereas, the partial derivative of f x with respect to y at (0, 0) is equal to -1.

It suggests that at the origin these two limits are different. In particular, what we conclude is that the two mixed second-order partial derivatives at the origin are not the same. However, it is the same at any other point other than the origin.

This is an example of a function for which second-order partial derivatives, though they exist, and this mixed second-order partial derivative at the origin are not the same. The mixed second-order partial derivatives at any point are equal provided the second-order partial derivatives are continuous.

The limits of last two which we have computed are different. That is the same as saying that the second-order partial derivatives are not continuous. That is why the second-order partial derivatives are not the same.

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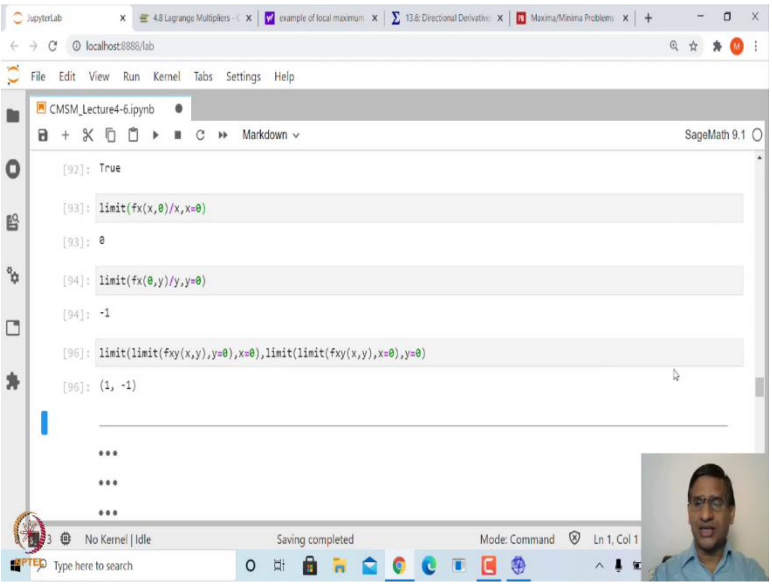

Of course, you can compute this limit or check the continuity of this mixed second-order partial derivatives f (x, y). And you can see the limit when we take about y equal to 0 first, then with respect to x = 0 and first with respect to x and then with respect to y. They are different ones.

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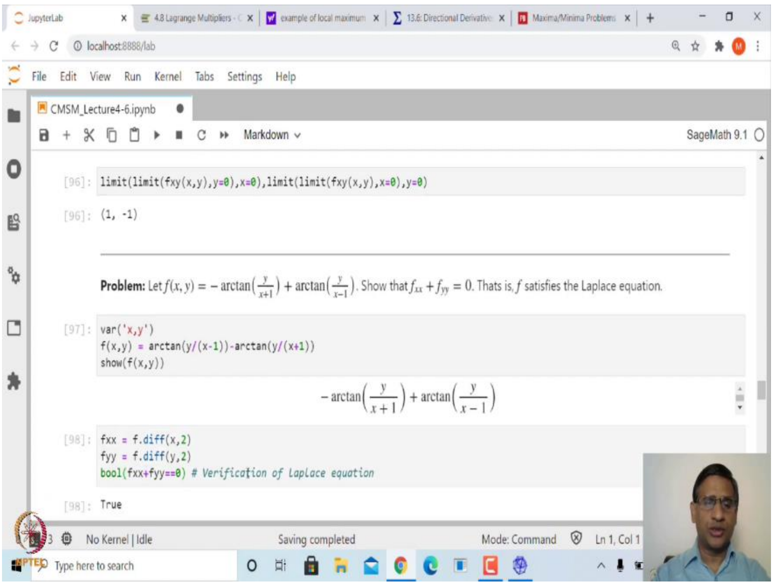

Again, these mixed second-order partial derivatives are not continuous. And that is the reason why the mixed second-order partial derivatives are not equal. In case mixed second-order partial derivatives are equal then we say the function is continuous.

Let us take one smaller example to consider a function f(x, y) = $\tan^{-1}\left(\frac{y}{x+y}\right)$ + $\tan^{-1}\left(\frac{y}{x-y}\right)$, and try to show that fxx is the second-order partial derivative of f with respect to x plus f yy is equal to 0 i.e. fxx+fyy=0. This equation is called the Laplace equation. This particular function satisfies the Laplace equation.

You can think of f(x, y) as a solution to this Laplace equation. This is a second-order partial differential equation, in fact, a homogeneous second-order partial differential equation. This is a solution to the given Laplace equation.

First, define the function and then define the second-order partial derivative of f with respect to x and y, and then add these two and equate it to 0 and say bool(fxx+fyy==0) which gives us True.

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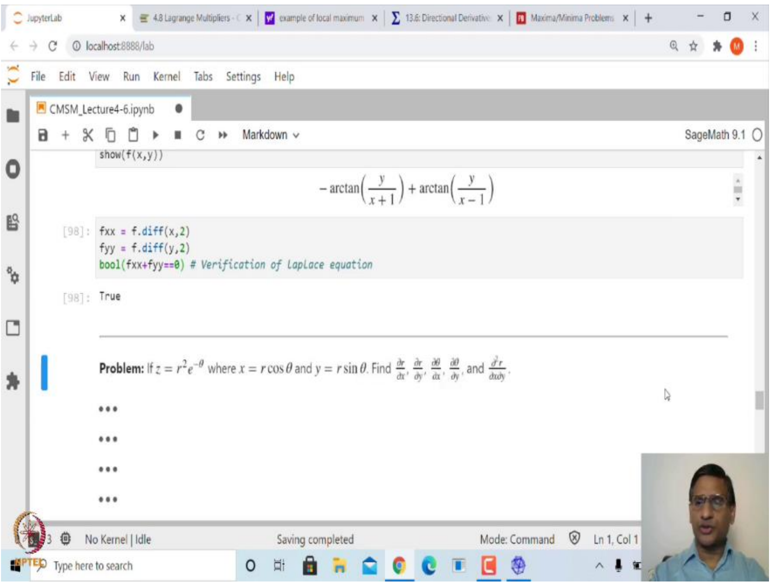

Next, suppose you are given a function in let us say z which is in terms of r and θ which are the polar coordinates, and suppose we substitute $x = r\cos(\theta)$ *and* $y = r\sin(\theta)$. In this case, x is a function of r and θ, y is also a function of r and θ, z is a function of r and θ and in a particular function of x and y.

Suppose we want to find out the partial derivative of r with respect to x, $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$, $\frac{\partial \theta}{\partial x}$, $\frac{\partial \theta}{\partial y}$ *and* $\frac{\partial^2 z}{\partial x \partial y}$. We can use the chain rule for this and let us see how we can make use of SageMath for computing this?

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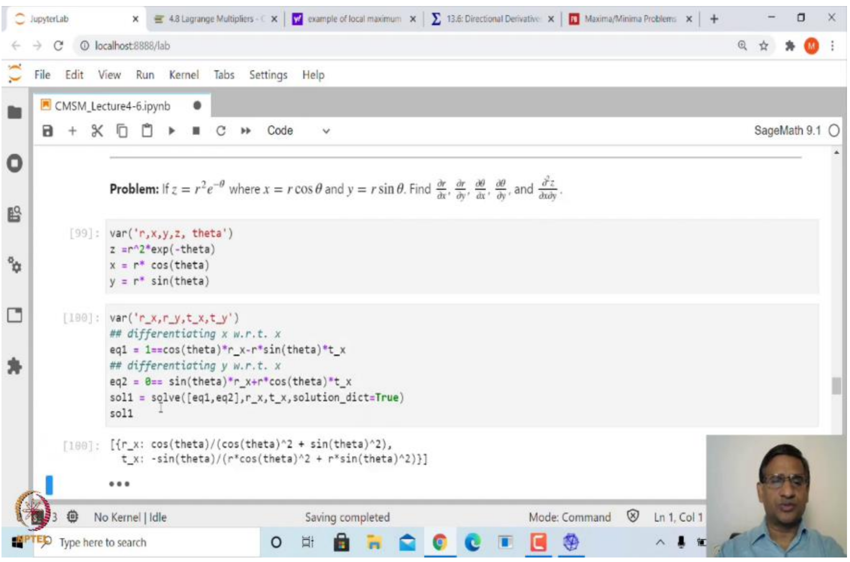

Let us define z, x and y; $z = r^2 e^{-\theta}$, $x = rcos(\theta)$ and $y =$ $rsin(\theta)$. And then the partial derivative of r with respect to x we will denote by r_x. Similarly, r_y is the partial derivative of r with respect to y and is the partial derivative of θ with respect to x, this is t_y a partial derivative of θ with respect to y.

First, let us declare these variables. Then look at that is a function of r and θ, so let us differentiate both sides with respect to x. What will you get? The first equation is a derivative of this with respect to x. Here left-hand side will be 1, right-hand side will be cos(θ) into derivative of r with respect to x that is r_x plus the derivative of x with respect to θ; that means, the $-rsin(\theta)$ and then multiply by $\frac{\partial\theta}{\partial x}$.

Similarly, you differentiate the second equation with respect to x. The left-hand side will be 0 and the right-hand side will be $sin(\theta)$ $\frac{\partial r}{\partial x}$ plus $rcos(\theta)$ $into$ $\frac{\partial\theta}{\partial x}$.

You look at these two equations, equation 1 and equation 2, and you can solve these two equations for r_x , t_x which is the partial derivative of r with respect to x and the partial derivative of θ with respect to x. We have obtained the solution.

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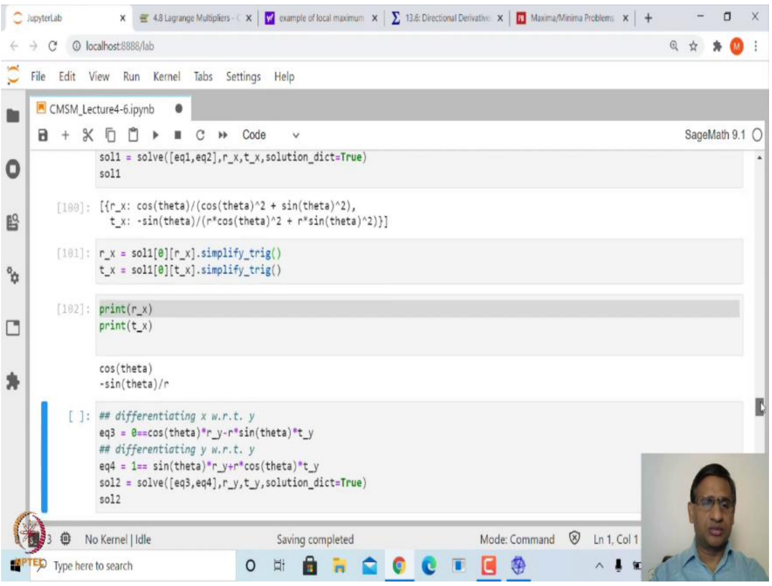

Now, let us store this in r_x and t_x as the value of key r_x , t_x and simplify, simplify_trig() is going to simplify will write-in the simpler form. We can ask it to show what these things are. The partial derivative of r with respect to x is $cos(\theta)$, the partial derivative of θ with respect to x is $\frac{-sin(\theta)}{r}$.

Similarly, you can differentiate x with respect to y and then y with respect to x and call these equations as equation 3 and equation 4. And then solve equations 3 and 4, this will give me a partial derivative of r with respect to y and a partial derivative of θ with respect to y.

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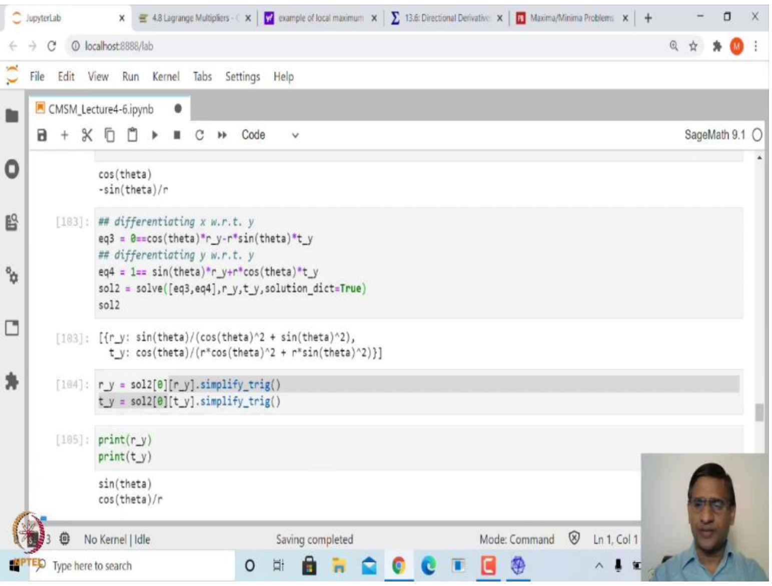

We will reassign this in r_y and t_y, as the value of key r_y and t_y, and then ask it to show. These are the partial derivative of r with respect to y and the partial derivative of θ with respect to y.

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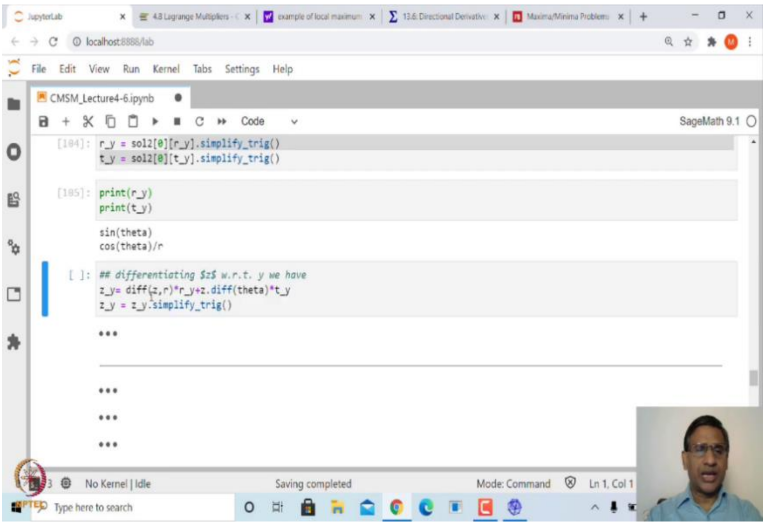

Next, we want to find a partial derivative of z with respect to x and then followed by again a partial derivative of that with respect to y; that means, $\frac{\partial^2 z}{\partial x \partial y}$. How can we achieve this?

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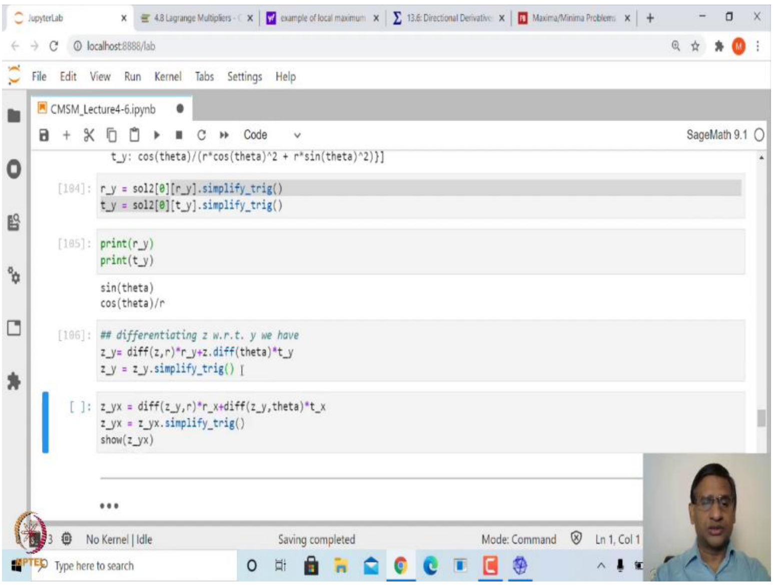

How can we achieve this? We need to differentiate z with respect to y, that is z is a function of r and θ. Using chain rule you will have a derivative of z with respect to r times the partial derivative of r with respect to y plus the partial derivative of z with respect to θ times partial derivative of θ with respect to y and you simplify this using simplify_trig().

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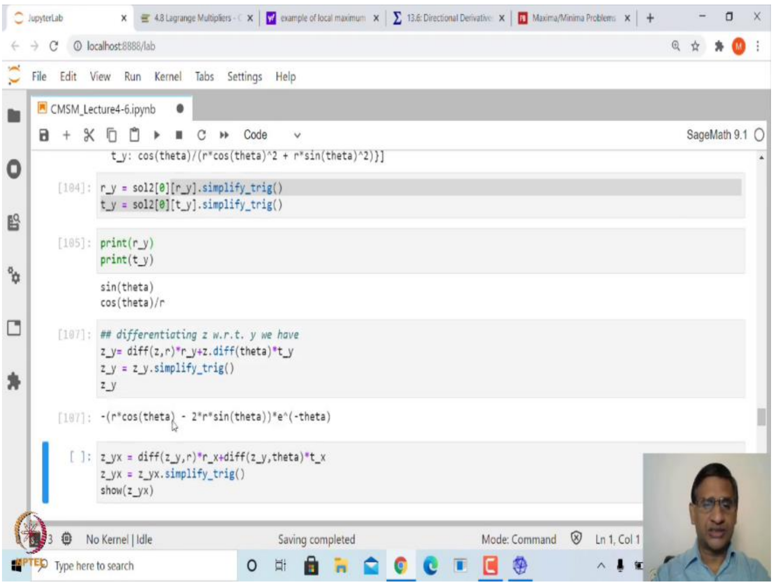

Let me print z_y that is the first-order partial derivative of z with respect to y. Now, find the first partial derivative of z_y with respect to x. You can do this using the chain rule.

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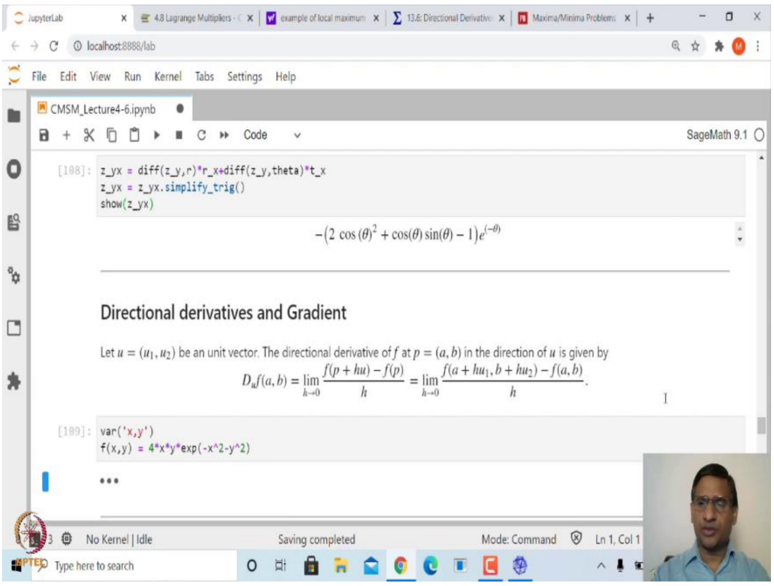

Now, let us define what is the meaning of directional derivative and gradient. If you have a function z is equal to f(x, y) and suppose you take unit vector u=(u1, u2) then the directional derivative of f at a point (a, b) which we will call the point as p, p=(a, b) in the direction of u is given by the limit of $\frac{f(p+hu)-f(p)}{h}$.

what you are looking at is numerator is a change of the function value along the direction u and then this is the rate of change of f at p in the direction of u that is called direction derivative of f with respect to u at (a, b) this you can expand as (u1, u2). This is what you get in terms of coordinates.

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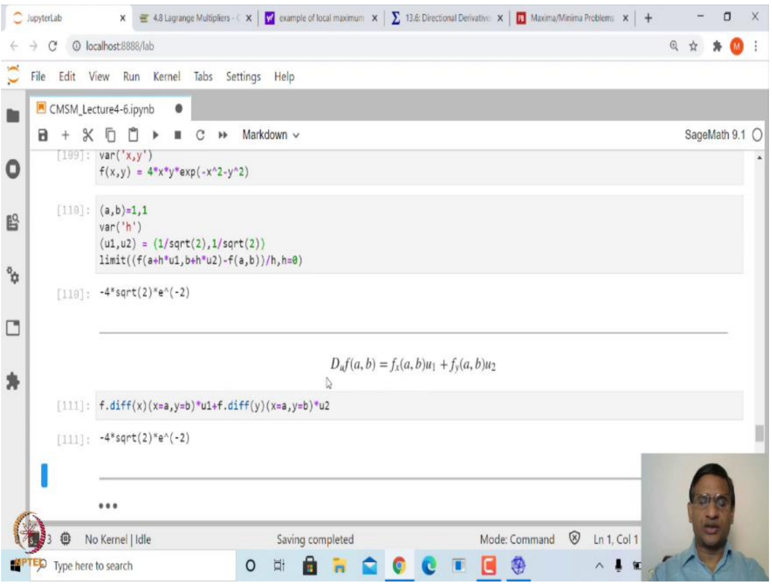

Suppose you have this function $f(x, y) = 4xye^{-x^2 - y^2}$, and you can find out its directional derivative at a point (1, 1) in the direction of letting us say $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ that is a unit vector. The answer is going to be $-4\sqrt{2}e^{-2}$. Now, you can check that the directional derivative of f at (a, b) in the direction of u can be written as a partial derivative of f with respect to x at (a, b) times u1 the first coordinate plus partial derivative of f with respect to y at (a, b) times u2.

One can use $f_x(a, b)u1 + f_y(a, b)u2$ this formula to find the partial directional derivative of f at any point (a, b) and you can see that the answer we got using definition is the same as what we got using this formula.

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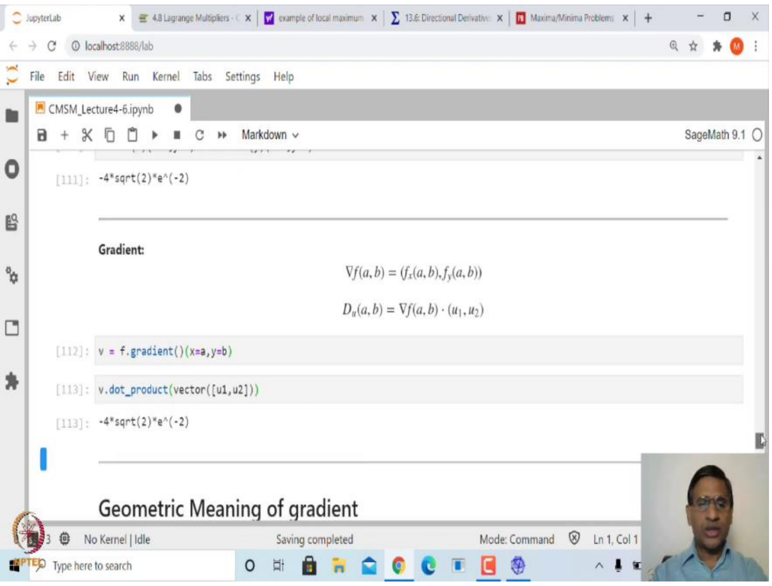

Similarly, you can define the gradient of a function. The gradient of f(a, b), is denoted by $\nabla f(a, b)$. This is read as nabla or del f(a, b). This is the first coordinate partial derivative of f with respect to x at (a, b). The second coordinate is a partial derivative of f with respect to y at (a, b). And then, there is a relationship between the directional derivative and this gradient. Directional derivative of f at (a, b) in direction of u is the dot product of gradient with u.

Let us look at the gradient of a function you can find using the f dot gradient and at any point (a, b). This is the gradient and gradient is a vector. You can verify this, using the dot product of v with respect to the vector u. This u is a vector you can define like (u1, u2), they give the coordinates. You should be able to see that this is the directional derivative of f with respect to in the direction of u at (a, b).

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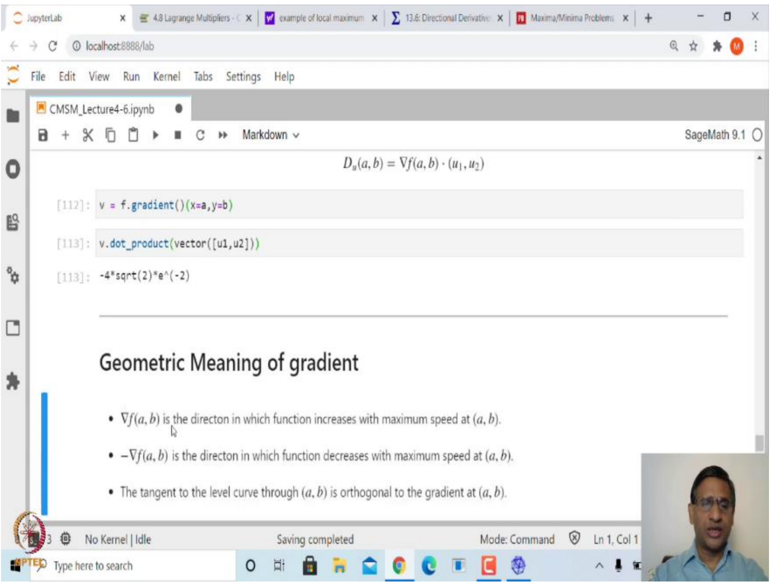

You can look at the geometric meaning of the gradient of a function. the gradient of f at (a, b) is a direction in which the function increases with maximum speed. Similarly, the negative of the gradient is the direction in which the function decreases with maximum speed. That is the geometric meaning of gradient. One can show these two things, it is a very simple problem, one can convert this into one variable optimization problem and one variable problem of maxima, minima.

The second thing is if you look at the tangent to any level curve z is equal to let us say c; that means, you are looking at f(x, y) = c that is the level curve z equal to c. I take any point on the curve let us say (a, b), f(a, b) at that point if you look at the gradient it is orthogonal to the tangent, the meaning is tangent to the level curve this is orthogonal to the gradient at (a, b).

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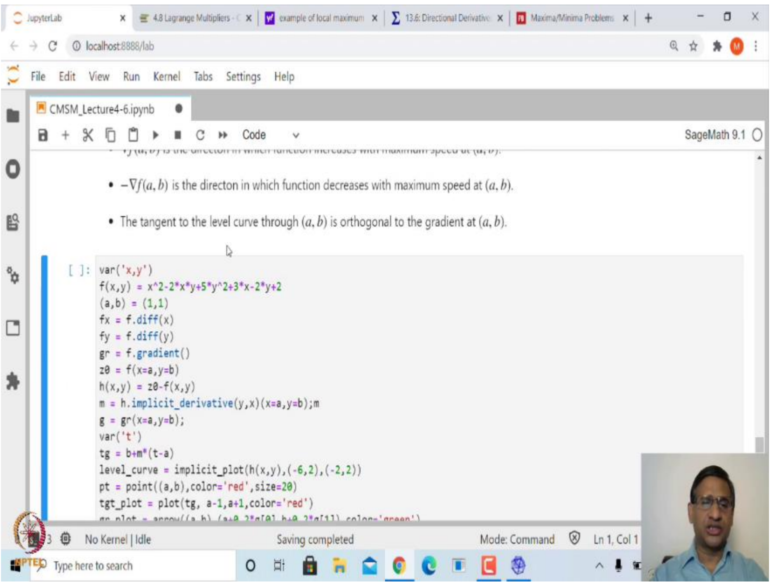

Let me just demonstrate this for function f(x, y)= $x^2 - 2xy + 5y^2 + 3x - 2y + 2$ and take any point let us say (a, b) and at this point let us draw tangent, level curve and gradient.

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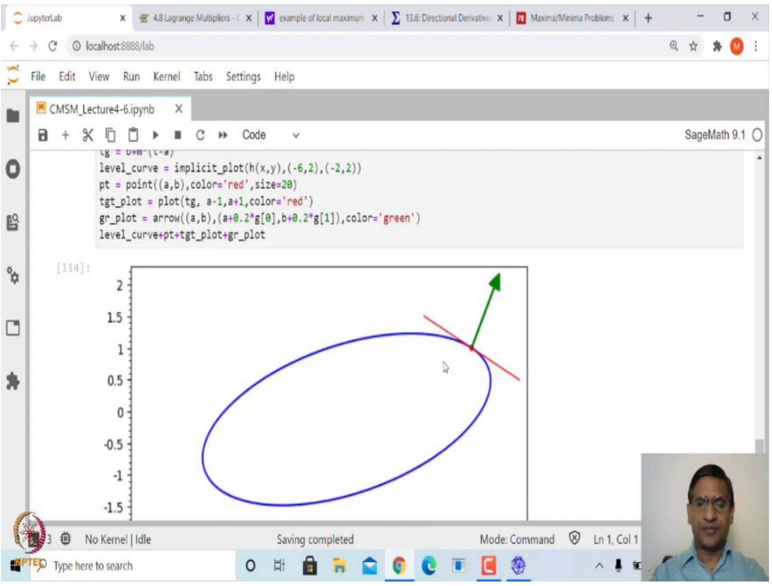

The blue one is the level curve of this function. The red one is the point on a level curve. The red line is the tangent and this is the gradient. You can see they are perpendicular.

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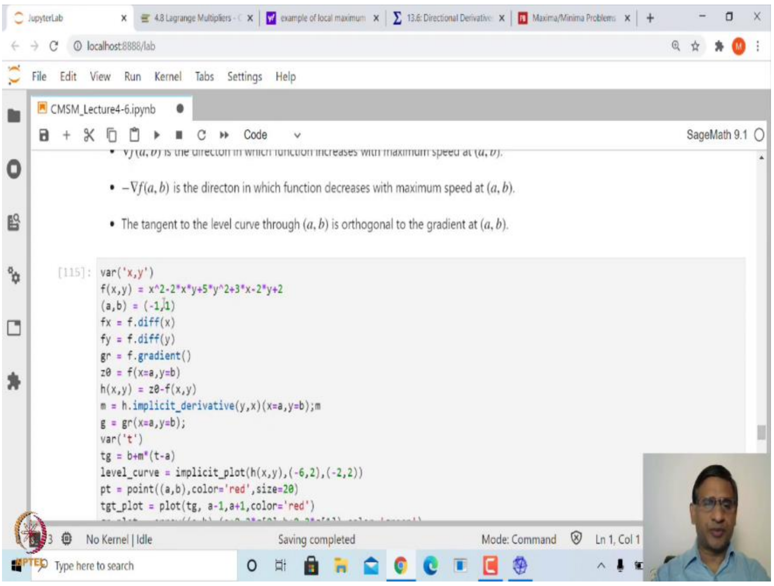

It does not matter, you can choose some other point instead of (1, 1), let us choose (-1,1) and it will be the same thing.

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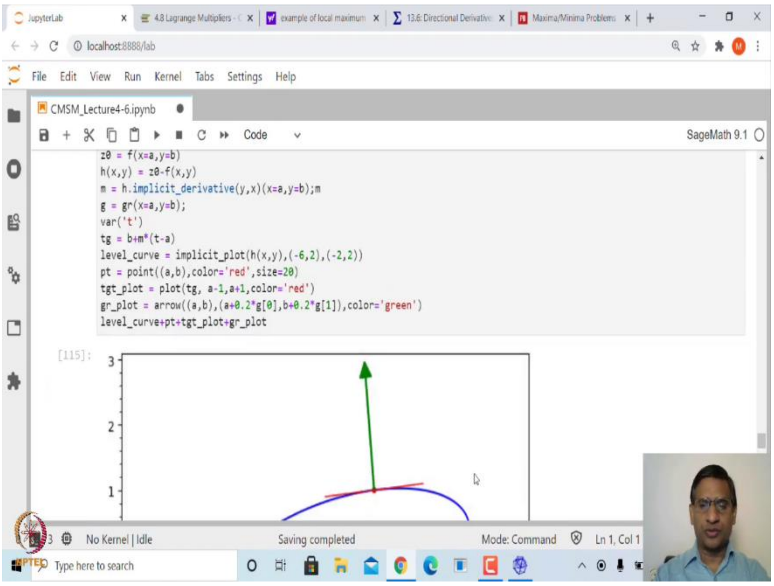

Instead of (-1, 1) let us choose (-1,1/2) again it is the same thing.

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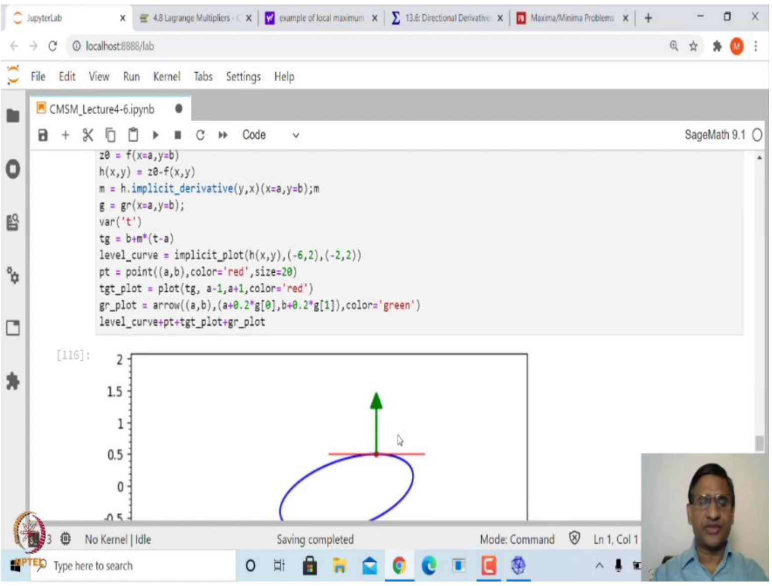

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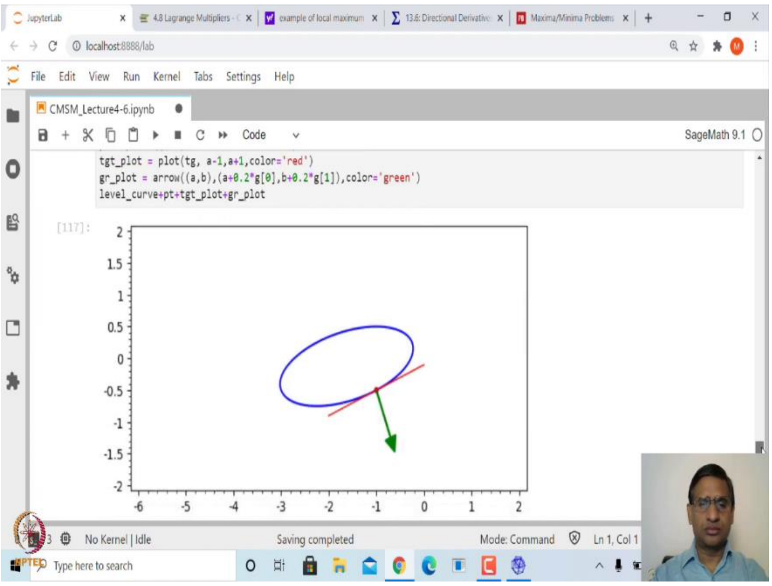

Or you can check (-1, -r), everywhere this is true.

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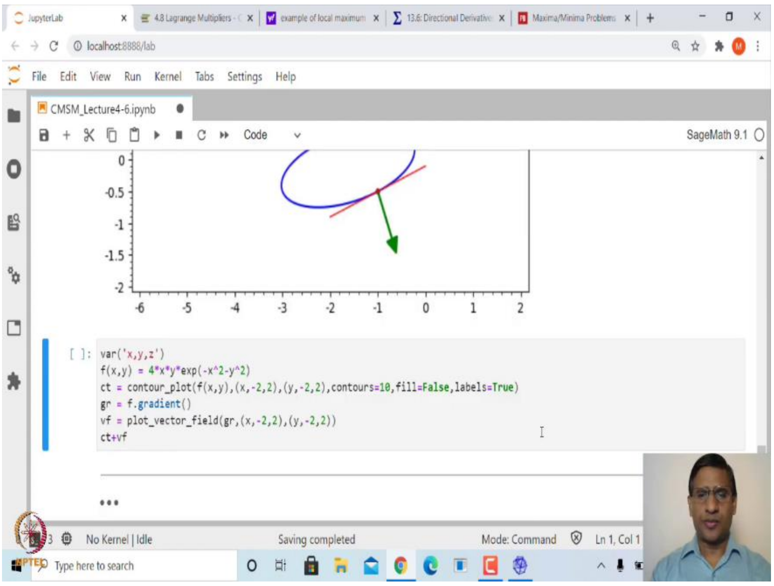

Similarly, you can draw the contour of the function and plot the gradient as a vector field.

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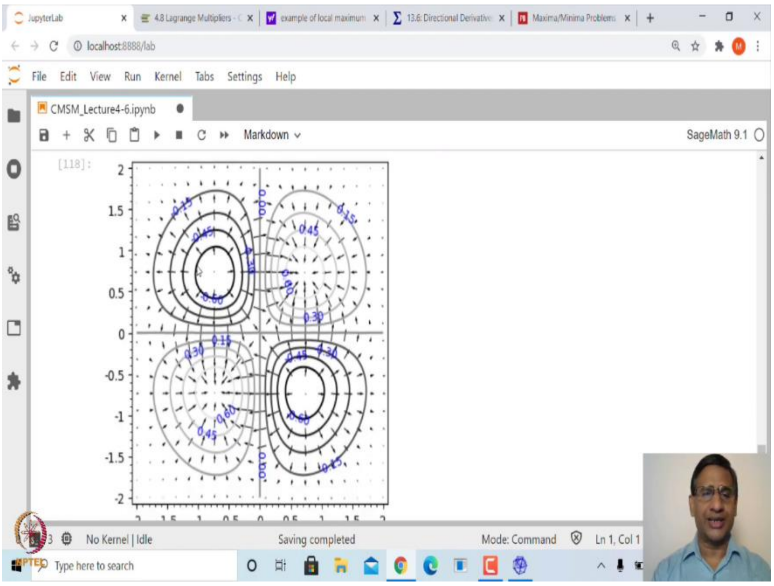

When you plot this. These black curves are the gradients. And for example, if you look at these curves are the level curves at z and these arrows are the gradient. You can also see here gradient is perpendicular to the tangent to this particular curve. That is another way to explain the gradient is perpendicular to the level curve.

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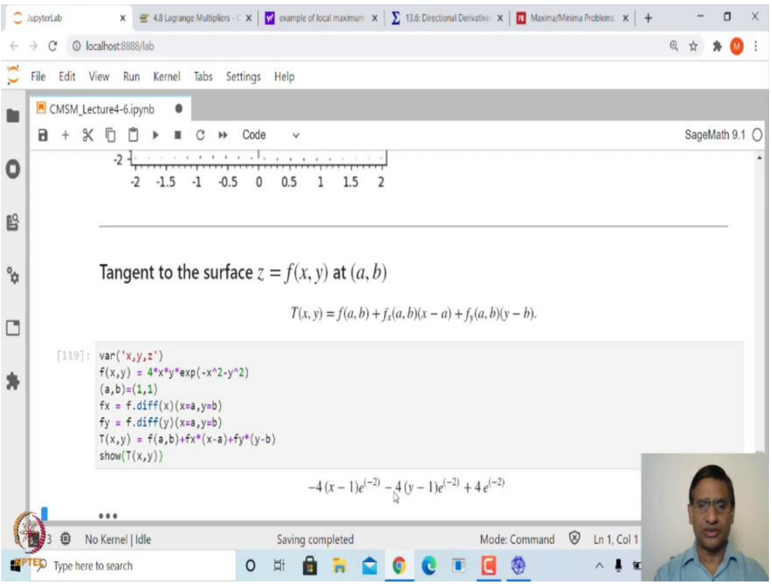

Next, let us look at how to define tangent to surface z = f (x, y). The equation of the tangent plane at any point (a, b) is given by T (x, y) = f (a, b) +fx (a, b) (x-a) +fy (a, b) (y-b).

Let us again take the same surface and try to plot the tangent. $f(x, y) = 4xye^{-x^2 - y^2}$ (a, b) is again (1, 1). Find out fx and fy at (a, b) and then define the tangent and then let us see what is the tangent plane. The tangent plane equation is $-4(x-1)e^{-2} - 4(y-1)e^{-2} + 4e^{-2}$, that is z is equal to the equation of the tangent plane.

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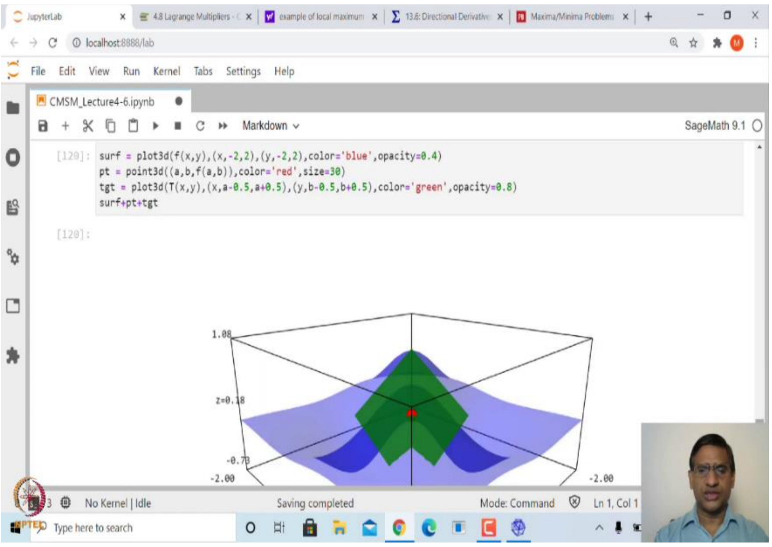

One can try to plot the graph of the surface along with the tangent plane.

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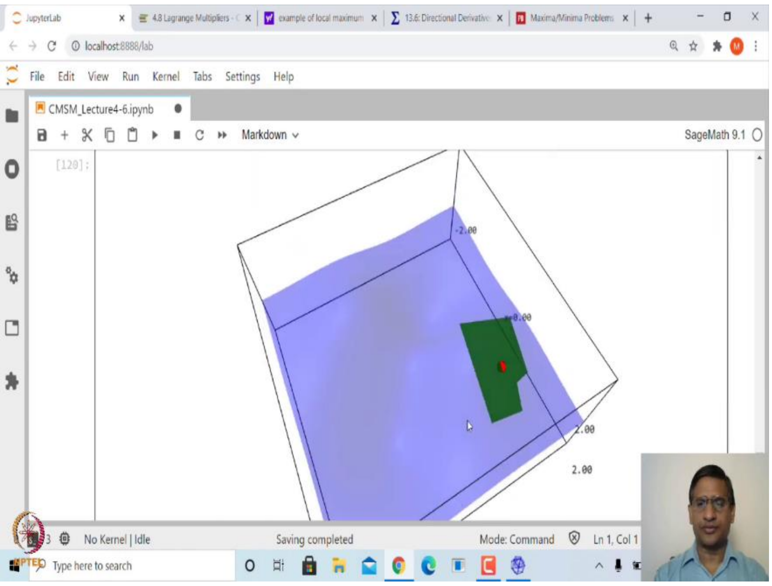
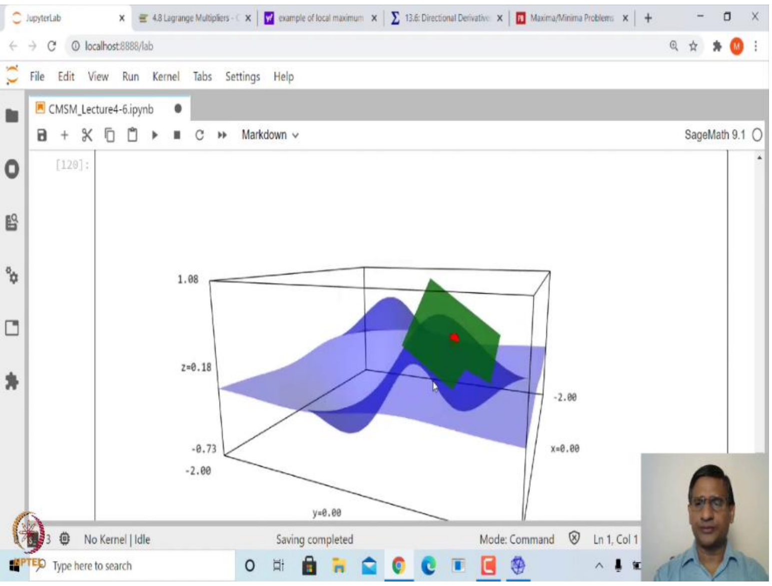


The above one is the graph of the tangent plane at this particular point (Refer Slide Time: 29:37)



Once you have defined the tangent plot then you just use plot3d to plot the surface point3d to plot the point and again plot3d to plot the tangent plane.

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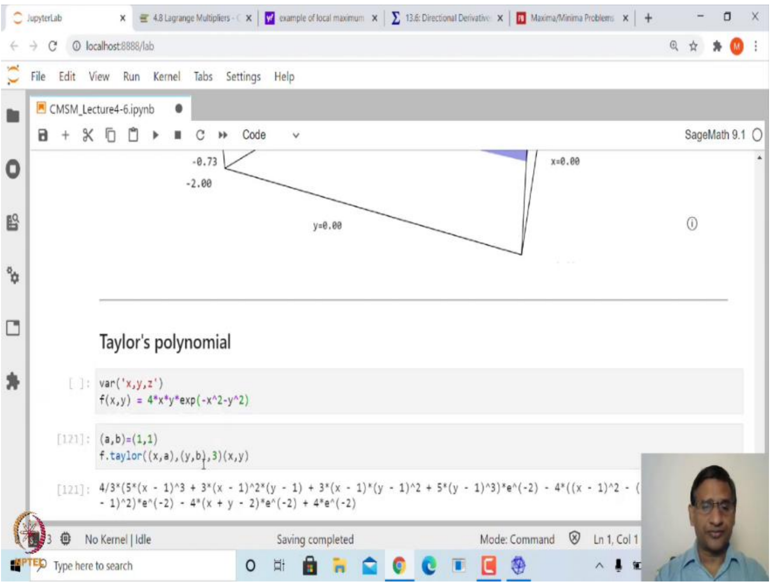

Next, let us look at how we can define Taylor's polynomial, we have already looked at how to find Taylor's polynomial of various degrees at some point for one variable case. The same function can be used for more than one variable.

If I have the same function $f(x, y) = 4xye^{-x^2 - y^2}$ and if you want to find Taylor's polynomial of f at (a, b) of degree 3, then what you need to do is f dot taylor (x, a) that is at x at a and this (y, b) that is y at b and mention the degree.

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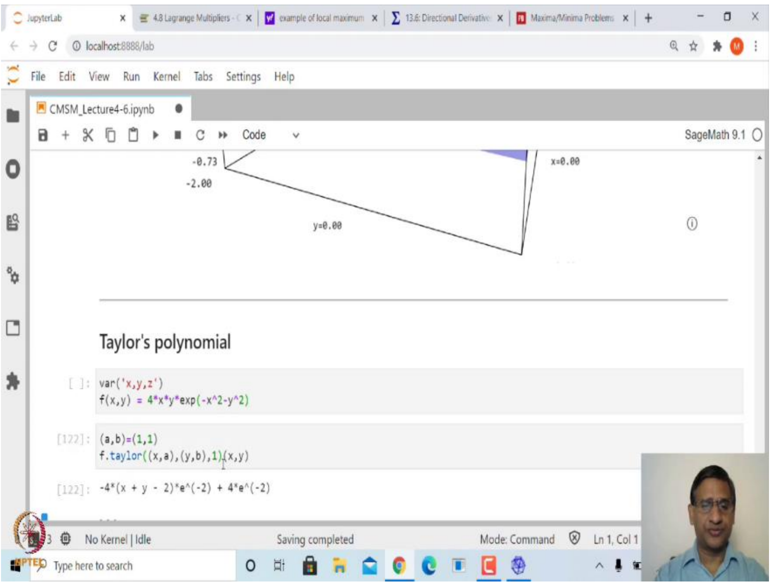

For example, if I look at 1, then this is a first-order Taylor polynomial; this will be nothing, but the tangent plane.

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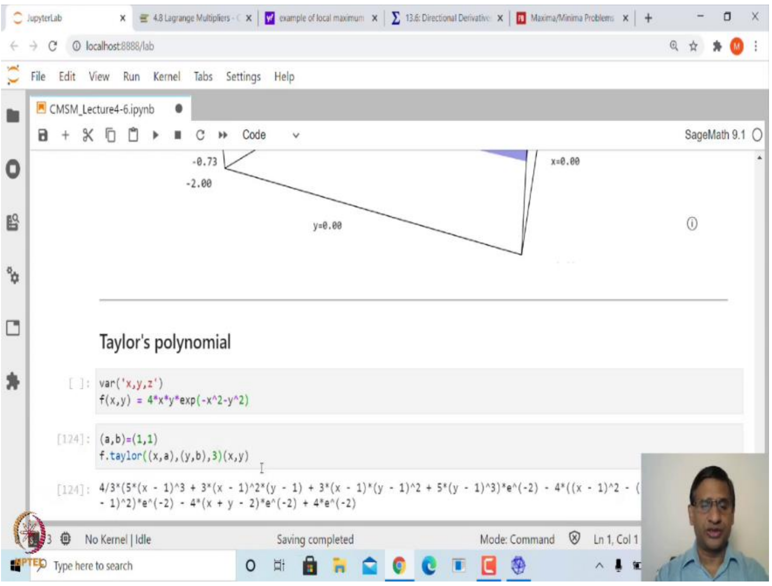

If I mention here 2, then it will give me a second-degree Taylor's polynomial in x, y and 3 will give me a third-degree Taylor's polynomial of f (x, y) at (a, b) equal to (1, 1).

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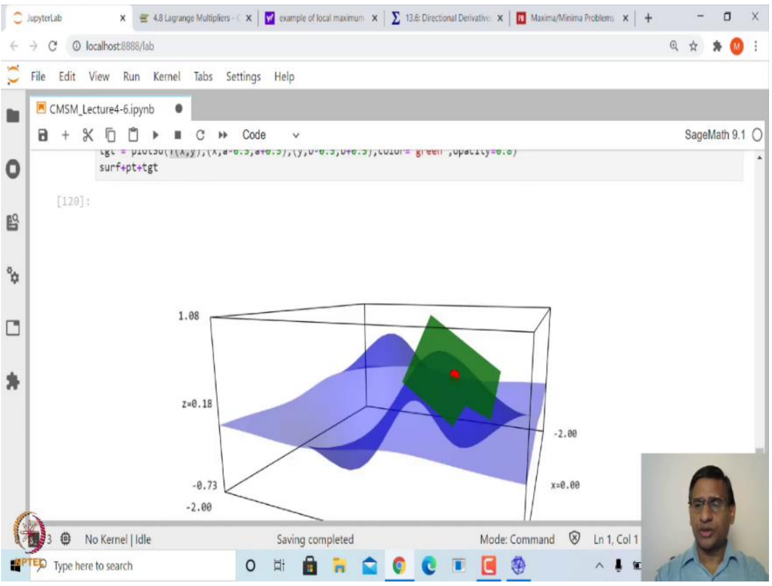

You can even try to plot the graph of the surface along with the various Taylor's polynomials.

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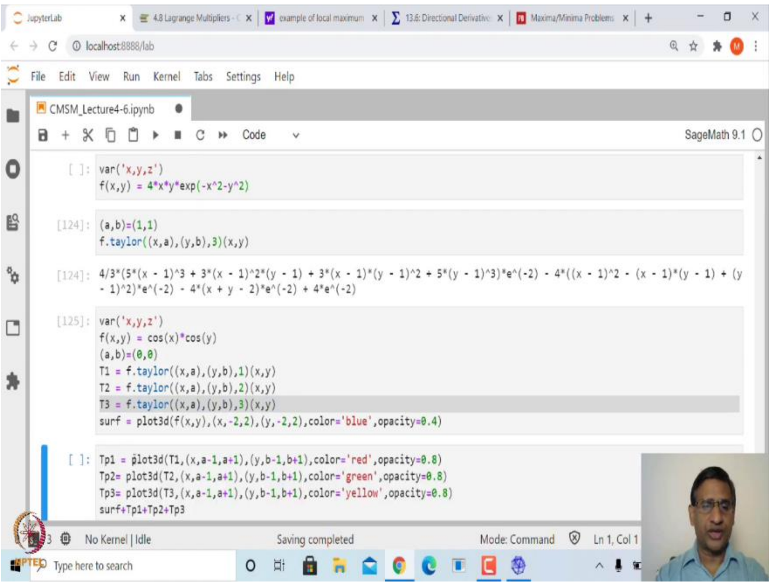

For example, if you look at the function f(x, y) = cos(x)*cos(y) and let us take the point (0, 0) at the origin, let T1 is the first order first degree Taylor's polynomial, T2 is second degree Taylor's polynomial, T3 is third-degree Taylor's polynomial.

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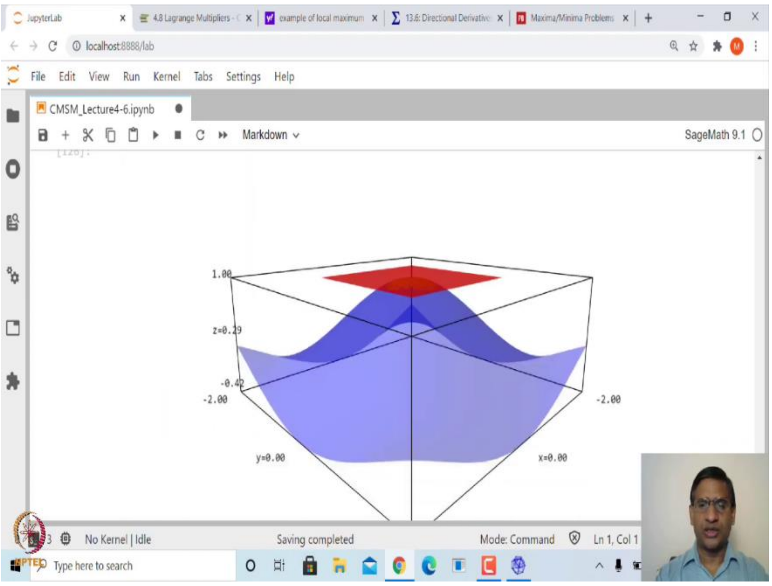

Let us try to plot the surface along with all these Taylor's polynomials and then you can see. The red one is the tangent plane which is first-order Taylor's polynomial. Second-order Taylor's polynomial, in this case, is the local maximum; it will be very close to the actual function and the yellow one is the third-degree Taylor's polynomial.

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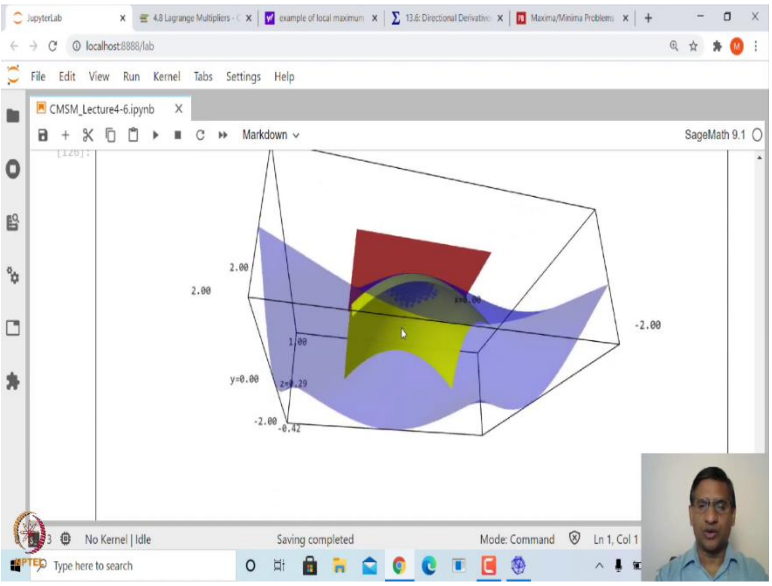

you can take the various functions and try to approximate them. You can go to a higher degree and then you will see that the higher degree Taylor's polynomial at any point will be very close to the graph of the surface.

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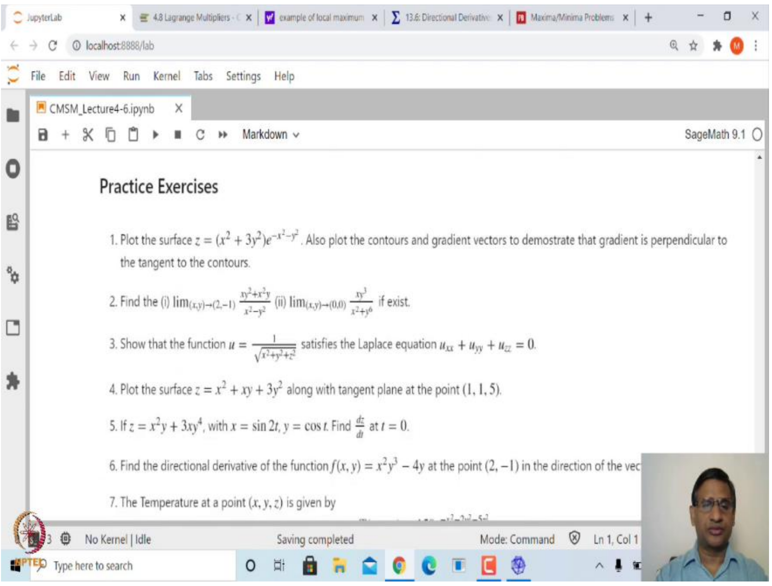

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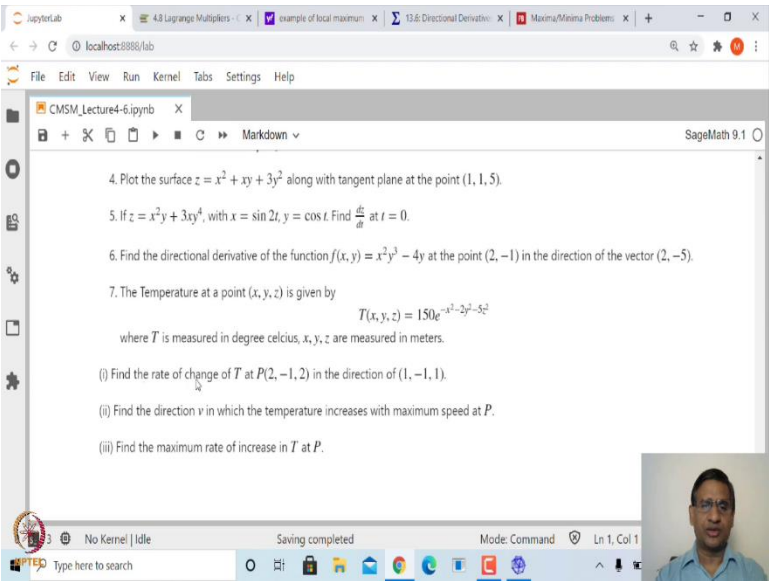

Let me leave you with few simple exercises. The first exercise is to look at this surface $z = (x^2 + 3y^2)e^{-x^2-y^2}$ try to plot the contour and gradient, and then demonstrate that the gradient is perpendicular to the tangent to the contours. The second one is finding the limits of given functions if the limit exists. The third one is to show that the function satisfies the Laplace equation, this is a function of 3 variables.

Plot the surface $z = x^2 + xy + 3y^2$ along with the tangent plane at point(1,1,5). Look at this function $z = x^2y + 3xy^4$ and let us take x = sin(2t), y=cos(2t), find $\frac{dz}{dt}$ that is a chain rule at t equal to 0. The next one is to find the directional derivative of $f(x, y) = x^2y^3 - 4y$ at point (2, -1) in the direction of (2, -5). That means this is the direction you have to convert this into a unit vector. You have to divide this by length.

The next problem which is the last problem is supposed the temperature at some point is given by T (x, y, z)= $150e^{-x^2-2y^2-5z^2}$. T is measured in degree Celsius and x, y, z are in meters. Find the rate of change of T in the direction of (1, -1, 1) at P= (2, -1, 2). This is nothing, but a directional derivative of T at point P in the direction of (1, -1, 1).

Similarly, find the direction of the v in which the temperature increases with maximum speed, which means you have to find the gradient and also find the maximum rate of increase of T at P; that means, you have to find the length of the gradient. These are simple exercises. I am sure it will be quite easy for you to solve.

Thank you very much. Next time we will look at the application of partial derivatives as finding the maximum local minimum of the function of two variables.

Now, let me just make a comment that whatever we have done for two variables finding the limit, checking continuity, finding partial derivatives, and various concepts like directional derivative, gradient, etcetera all these things can be extended to the function of more variables and the same functions of sage can be used.

Thank you very much.